Chromatic Localization

Origin story

Thm (Sullivan, Atiyah-Segal) After p-completion there is an equivalence of infinite loop spaces

\[ BSU \longrightarrow BSU^\otimes \]
\[ BSO \longrightarrow BSO^\otimes \]

Thm (Mahowald, Adams-Priddy) If \( X \) is a connected 2-complete spectrum with \( \Omega^\infty X \sim BSO \) then \( \Sigma X \sim bso \). Similarly if \( X \) is a connected p-complete spectrum with \( \Omega^\infty X \sim BSU \), then \( \Sigma X \sim bsu_p \).

Remark Not true for \( BU \).

\[ BU \sim CP^\infty \times BSU \]
\[ bu \neq \Sigma HZ \times bsu \]

Remark This is not true without p-adic completion.

Thm (Hessle, Smith-Tornehave) After p-completion the map

\[ \Sigma^\infty: [bsu, bu] \longrightarrow [BSU, BSU] \]

is a monomorphism with a describable image.

Bousfield gave an elegant, "non-computational" proof.
Bousfield's proof

1) Factorization

Construct a functor \( \tilde{\Sigma} \) fitting into a factorization

\[
\begin{array}{ccc}
\text{Spaces} & \xrightarrow{\tilde{\Sigma}} & K(i)\text{-local spectra} \\
\downarrow \scriptstyle{\Sigma^\infty} & & \downarrow \scriptstyle{L_{K(m)}} \\
\text{Spectra} & \xrightarrow{\text{ }} & \end{array}
\]

2) Connectivity

Suppose that \( E \) is a spectrum. Show that the fiber of the map

\[
L_{K(m)} \Sigma^{\infty} E \to \Omega^{\infty} L_{K(m)} E
\]

is \( 2 \)-connected, so that

\[
L_{K(m)} \Sigma^{\infty} E \langle 3 \rangle \to \langle \Omega^{\infty} L_{K(m)} E \rangle \langle 3 \rangle
\]

is an equivalence.

Remark: Actually one shows \( \pi_i = 0 \) \( i \geq 2 \)

\( \Sigma E \) is torsion free.

Proof of uniqueness of BSUP:

\[
\begin{array}{c}
\Sigma^{\infty} X \xrightarrow{\sim} \text{BSUP} \\
\xrightarrow{\text{L}(\underline{\text{}})} \end{array}
\]

\[
\text{L}(\underline{\text{}}) X \xrightarrow{\sim} L_{K(m)} \text{bsup}
\]

Connectivity:

\[
\Sigma^{\infty} X = L_{K(m)} \Sigma^{\infty} X = L_{K(m)} \Sigma^{\infty} E \langle 3 \rangle = \Omega^{\infty} L_{K(m)} E \langle 3 \rangle
\]

so the desired spectrum equivalence is

\[
L_{K(m)} X \langle 3 \rangle \xrightarrow{\sim} L_{K(m)} \text{bsup} \langle 3 \rangle
\]
Remark. There are higher chromatic analogues of the above. Here is an example of a much more general result of Jeremy Hahn.

Theorem (Hahn) If $\mathcal{X}$ is a connective, $p$-complete spectrum with $S^\infty\mathcal{X} \sim S^\infty \Sigma^p \Omega^\infty \mathcal{X}$

then $\mathcal{X} \sim BR\mathbb{C}$. 

Not provable by the analogue of Bousfield's argument.
Bousfield localization for spectra

$\mathcal{S}$ is category of spectra.

$E \subset \mathcal{S}$ a spectrum.

$\text{Null}_E = \{ \Sigma \in \mathcal{S} \mid E \wedge \Sigma \sim * \}$

**Definition** $\Sigma \in \mathcal{S}$ is $E$-local if

$\Sigma \in \text{Null}_E \Rightarrow [E, \Sigma] = 0$.

**Theorem (Bousfield)** There exists a functor

$L_E : \mathcal{S} \rightarrow \mathcal{S}$

and a natural transformation

$1_\mathcal{S} \rightarrow L_E$

with the following properties

i) $L_E \Sigma$ is $E$-local.

ii) The map $\Sigma \rightarrow L_E \Sigma$ is an $E$-equivalence.

**Consequences** i) The fiber $H_{L_E \Sigma} \rightarrow \Sigma \rightarrow L_E \Sigma$

is $E$-acyclic, so if $\Sigma$ is $E$-local then the restriction map

$[L_E \Sigma, \Sigma] \rightarrow [\Sigma, \Sigma]$

is an isomorphism, and

$\text{Map}(L_E \Sigma, \Sigma) \rightarrow \text{Map}(\Sigma, \Sigma)$

is a weak equivalence. So $L_E$ can be regarded as a left adjoint to the inclusion $\mathcal{S}_E \subset \mathcal{S}$ of the full subcategory of $E$-local spectra.

ii) $L_E$ is idempotent: the map $L_E \Sigma \rightarrow L_E L_E \Sigma$ induced by $1 \rightarrow L_E$ is a weak equivalence.
Bousfield equivalence

Note that $1 \rightarrow L_E$ depends only on $\text{Null}_E$.

**Definition.** Two spectra $E$ and $F$ are **Bousfield equivalent** if $\text{Null}_E = \text{Null}_F$.

The Bousfield class $\langle E \rangle$ of $E$ is the equivalence class of $E$ under this equivalence relation.

**Relations among Bousfield classes:**

\[
\langle E \rangle \geq \langle F \rangle \iff \text{Null}_E \subseteq \text{Null}_F
\]

\[
\langle E \rangle \cup \langle F \rangle = \langle EF \rangle
\]

\[
\langle E \rangle \cap \langle F \rangle = \langle E \cap F \rangle
\]

There is a cool theorem which we won't need.

**Thin (Okawa).** The collection of Bousfield classes forms a set.

Note that if $\langle E \rangle \geq \langle F \rangle$ then

\[
\Sigma \text{ is } E\text{-local} \Rightarrow \Sigma \text{ is } F\text{-local}
\]

and so there is a natural transformation

\[
L_E \rightarrow L_F
\]

which may be regarded as being derived from $1 \rightarrow L_F$ by applying $L_E$. 
Dror nullification (Bousfield co-localization)

Generalization of Bousfield localization gotten by replacing \( \text{Null}_E \) by something more general. First described by Bousfield in "A Boolean algebra of spectra." Later generalized by Dror/Farjoun.

We will consider a special case.

Suppose \( A \in \mathcal{S} \).

**Definition** Let \( \mathcal{S} \) be \( A \)-null if for all \( t \)

\[
(\Sigma^t A, \Xi) = 0
\]

or equivalently if the function spectrum \( \Xi^A \) is contractible.

Then (Bousfield, Farjoun) There is a functor \( N_A : \text{Spectra} \to \text{Spectra} \) and a natural transformation \( 1 \to N_A \) with the properties

1) \( N_A \) is \( A \)-null

2) For any \( Y \) which is \( A \)-null, the map

\[
[N_A \Xi, Y] \to [\Xi, Y]
\]

is an isomorphism.

**Variation** Given \( f : A \to B \) say \( \Xi \) is "\( f \)-null" if

\[
\Xi^A \to \Xi^B
\]

is a weak equivalence. There is also a functor \( N_f \)

and a natural transformation \( 1 \to N_f \) as above.

**Remark** In spectra \( \Xi \) is \( f \)-null iff \( \Xi \) is \( C \)-null, where \( C = B \cup C_A \).

It follows that \( N_f = N_C \).
Construction

In principle, every assertion about \( L_B \) and \( N_A \) follows from the universal property. Some things, however, are easier to prove using the construction.

The functor \( N_A(\Sigma) \) is constructed (transfinite) inductively as follows. Choose an infinite ordinal \( \kappa \) for which \( A \) is \( \kappa \)-small:

\[
\lim_{\kappa} \text{Map}(A, Y_\kappa) \cong \text{Map}(A, \lim_{\kappa} Y_\kappa)
\]

(for example one can assume \( A \) is a CW spectrum and take \( \kappa \) to be the smallest infinite ordinal of cardinality greater than the number of cells of \( A \)). One constructs \( N_\alpha \Sigma \) inductively, starting with \( \Sigma_0 = \Sigma \) and, having defined \( \Sigma_\alpha \) for all \( \alpha < \beta < \kappa \), defining \( \Sigma_\beta = \lim_{\alpha < \beta} \Sigma_\alpha \), if \( \beta \) is a limit ordinal, and by the pushout diagram

\[
\begin{array}{ccc}
\bigvee t \Sigma^t A & \longrightarrow & \bigvee C \Sigma^t A \\
\downarrow t, \Sigma^t A \rightarrow \Sigma_{\beta-1} & & \downarrow t, \Sigma^t A \rightarrow \Sigma_{\beta-1} \\
\Sigma_{\beta-1} & \longrightarrow & \Sigma_{\beta}.
\end{array}
\]

In our case of interest, \( A \) will be a finite CW complex so we can take \( \Sigma \) to be the ordered set \( \{0 < 1 < 2 < \ldots \} \).
Here is an example of a theorem most easily seen from the construction:

**Theorem** If $E_\Lambda \sim \ast$ then for all $\mathcal{X}$, the map

$$E_\Lambda \mathcal{X} \rightarrow E_\Lambda \nu_\Lambda(\mathcal{X})$$

is an equivalence.

**The example Theorem**

**Examples of Bousfield classes**

$C_0 = p$-localization of finite spectra

$C_n = \{ \mathcal{X} \in C_0 \mid \mathcal{K}(h^{-1}) \mathcal{X} = 0 \text{ for all } h \geq n \}$

finite spectra of type $n$.

Thm (Landweber, Ravenel) $C_n \subset C_{n-1}$

Thm (Mitchell) $C_n \neq C_{n-1}$

Thm (H, Smith) If $C \subset C_0$ is thick then $C = C_{n}$ for some $n$.

Cor (H, Smith) If $\mathcal{X}, \mathcal{Y}$ are finite then

$$\langle \mathcal{X} \rangle \leq \langle \mathcal{Y} \rangle \iff \exists h \mid \mathcal{K}(h^{-1}) \mathcal{X} = 0 \neq \exists h \mid \mathcal{K}(h^{-1}) \mathcal{Y} = 0$$

**Class invariance**
**Def.** Suppose $X \in C_n$. A map $f: \Sigma^d X \to X$ is a $u_n$ self-map if

1. $K(n)_* f$ is nilpotent for $m \geq n$
2. $n \neq 0$ and $H_n(f; \mathbb{Z}) = \text{mult by } d$ for some $d$
   or
3. $n = 0$ and $K(n)_* f$ is an isomorphism.

**Remark.** There was a choice made in this definition. After raising to a power a $u_n$ self-map can be shown to satisfy

i') $K(n)_* f = 0$ for $m \geq n$

ii') $K(n)_* f$ = mult by a power of $u_n$.

**Thm (H, S).**

i) $X$ admits a $u_n$ self-map $\iff X \in C_n$

ii) If $u, u'$ are $u_n$ self maps of $X$ then $u^a = u'^b$ for some $a, b$.

iii) If $X, Y \in C_n$, $f: X \to Y$ then there exist $u_n$ self maps

$$v_X: \Sigma^d X \to X \quad \quad v_Y: \Sigma^d Y \to Y$$

such that

$$\begin{array}{ccc}
\Sigma^d X & \to & \Sigma^d Y \\
\downarrow & & \downarrow \\
X & \to & Y
\end{array}$$

commutes up to homotopy.

**Exercise.** i) If $X \in C_n \setminus C_{n+1}$ and $v: \Sigma^d X \to X$ is a $u_n$ self-map then

$$X u v \Sigma^d X \in C_{n+1} \setminus C_n.$$
Lemma (Ravenel) Suppose \( f : X \to X \) is a self map. Then
\[
\langle X \rangle = \langle f^*X \rangle \vee \langle X \cup C \Sigma X \rangle
\]

**Proof** An enjoyable exercise.

Start with \( X_0 \in C_0 \cdot C, \) (say \( X_0 = S^0. \))
\[
\nu_0 : X_0 \to X_0 \quad (\text{say} \quad \nu_0 = p)
\]

a \( \nu_0 \) self map

Set \( X_1 = \text{cofiber} \nu_0. \)

Having defined \( X_{n-1} \) choose a \( \nu_{n-1} \) self-map
\[
\begin{align*}
\nu_{n-1} : & \Sigma X_{n-1} \to X_{n-1} \\
\end{align*}
\]

and set
\[
X_n = \text{cofiber} \nu_{n-1}. \]

By class invariance \( \langle X_0 \rangle = \langle S^0 \rangle. \) By Ravenel's lemma
\[
\langle S^0 \rangle = \langle \nu_0^* X_0 \rangle \vee \langle X_1 \rangle
\]
\[
= \langle \nu_0^* X_0 \rangle \vee \langle \nu_1^* X_0 \rangle \vee \langle X_2 \rangle
\]
\[
\vdots
\]
\[
= \langle \nu_0^* X_0 \rangle \vee \cdots \vee \langle \nu_{n-1}^* X_0 \rangle \vee \langle X_n \rangle.
\]
Proposition  The above decomposition of $\langle s^0 \rangle$ is orthogonal in the sense that if $i \neq j$ then $v_i^* \mathbb{X}_i \wedge \mathbb{X}_j$ is contractible and hence so is $(v_i^* \mathbb{X}_i) \wedge (v_j^* \mathbb{X}_j)$.

Proof  Some power of

$$v_i \wedge 1 : 1 \cup \mathbb{X}_i \wedge \mathbb{X}_j \to \mathbb{X}_i \wedge \mathbb{X}_j$$

is zero in all Morava K-theories, hence contractible.

For simplicity write

$$E^f_n = v_0^* \mathbb{X}_0 \vee \cdots \vee v_n^* \mathbb{X}_n$$

(it is really the Bousfield class of $E^f_n$ we care about). Note that

$$\langle s^0 \rangle = \langle E^f_n \rangle \vee \langle \mathbb{X}_n \rangle$$

and that

$$E^f_n \wedge \mathbb{X}_n \simeq \ast .$$

We are interested in the localization functors

$$L_n^f = L_{E_n}$$

$$L_T(n) = L_{v_0^* \mathbb{X}_n}$$
Proposition \[ L^n = N_{\Sigma^{n+1}} \]

Bousfield \[ \uparrow \]
Dror \[ \uparrow \]

We need:

Lemma Both \( L_E \) and \( N_A \) commute with finite homotopy (co-)limits. In particular if \( S \) is finite (dualizable) then the natural maps

\[
(L_E \Sigma) \wedge S \to L_E (\Sigma \wedge S)
\]

\[
(N_A \Sigma) \wedge S \to N_A (\Sigma \wedge S)
\]

are weak equivalences.

Proof For example, to show that

\[
(L_E \Sigma) \wedge S \to L_E (\Sigma \wedge S)
\]

is an equivalence it suffices to show that if \( W \) is \( E \)-local then \( W \wedge S \) is \( E \)-local. For this, suppose \( Z \in \text{Null}_{E} \). Then

\[
E \wedge (E \wedge (E \wedge (E \wedge (E \wedge S))) \wedge S) \sim
\]

implies that \( Z \wedge S \in \text{Null}_{E} \). Now observe

\[
[2, \wedge S] = [Z \wedge DS, \omega] = 0.
\]

The other assertions are similar. \( \square \)
proof of the Proposition: Since \( E_n^t \cdot \mathcal{X}_{n+1} \sim \mathcal{X} \), the spectrum \( \mathcal{X}_{n+1} \) is \( E_n^t \)-null and so for all \( \mathcal{X} \)
\[
(L^t \mathcal{X})_{\mathcal{X}_{n+1}} \sim \mathcal{X}.
\]

By the universal property of \( N^{\mathcal{X}}_{\mathcal{X}_{n+1}} \) this means that there is a unique natural transformation
\[
1 \rightarrow N^{\mathcal{X}}_{\mathcal{X}_{n+1}} \rightarrow L^t_n
\]
of functors (over \( \mathcal{X} \)). To show it is an equivalence it suffices to show that for all \( \mathcal{X} \), \( N^{\mathcal{X}}_{\mathcal{X}_{n+1}} \mathcal{X} \) is \( E_n^t \)-local. For this suppose that \( \mathcal{Z} \in \text{Null}_{E_n^t} \). We must show that any map
\[
\mathcal{Z} \rightarrow N^{\mathcal{X}}_{\mathcal{X}_{n+1}} \mathcal{X}
\]
is null. Since \( N^{\mathcal{X}}_{\mathcal{X}_{n+1}} \) is idempotent, it suffices to show that
\[
N^{\mathcal{X}}_{\mathcal{X}_{n+1}} \mathcal{Z}
\]
is contractible. By orthogonality, we know that for \( i \leq n \)
\[
\mathcal{X}_i \wedge \mathcal{X}_{n+1} \sim \mathcal{X}.
\]
The example theorem then implies that for all \( i \leq n \)
\[
(\mathcal{X}_i \wedge \mathcal{Z}) \rightarrow (\mathcal{X}_i \wedge \mathcal{Z}_{\mathcal{X}_{n+1}})
\]
and so
\[
(\mathcal{X}_i \wedge \mathcal{Z}_{\mathcal{X}_{n+1}}) \sim \mathcal{X}.
\]
But we also know, by definition, that

\[ D \mathcal{X}_{n+1} \land \mathcal{X}_{n+1} \sim (\mathcal{X}_{n+1}) \sim \mathcal{X}. \]

By class invariance, \( \langle D \mathcal{X}_{n+1} \rangle = \langle \mathcal{X}_{n+1} \rangle \). So after all of this we learn that

\[ (\mathcal{X}_{n+1}) \land v_1 \mathcal{X} \sim \mathcal{X} \quad i \leq n \]

\[ (\mathcal{X}_{n+1}) \land \mathcal{X}_{n+1} \sim \mathcal{X}. \]

It follows from

\[ \langle S_0 \rangle = \langle v_0 \mathcal{X} \rangle \cup \cdots \cup \langle v_n \mathcal{X} \rangle \cup \langle \mathcal{X}_{n+1} \rangle \]

that

\[ \mathcal{X}_{n+1} \sim S_0 \land \mathcal{X}_{n+1} \sim \mathcal{X}. \]
Localization and nullification of spaces

The theory needs to be set up a little differently in \textit{Spaces}.

\textbf{Def} A map \( A \to B \) is an \( E \)-equivalence if \( E \cdot A \to E \cdot B \) is an iso.

\textbf{Def} A space \( W \) is \( E \)-local if for all \( E \)-equivalences \( A \to B \), the map

\[ W^B \to W^A \]

is a weak equivalence.

Bousfield constructs an \( E \)-localization functor \( L_E : \text{Spaces} \to \text{Spaces} \) and a natural transformation

\[ X \to L_E X \]

characterized by the evident analogue of the properties characterizing the localization of spectra.

Let \( A \) be a space. A space \( X \) is \( A \)-null if the inclusion

\[ X \to X^A \ast \]  \hspace{1cm} \text{to indicate unpointed maps}

of the constant maps is a weak equivalence. Following Bousfield, Farjoun (Dror) constructs an \( A \)-nullification functor (and natural trans)

\[ X \to N_A X \]

characterized by properties analogous to those characterizing the nullification of spectra.
Unstable $L_n^f$ localization (all spaces are pointed, all homology is reduced)

So we have several candidates for $L_n^f$ on spaces.

1) Localization with respect to $E_n^f$

2) Dyer nullification with respect to $Σ_n × pt ← space$ of type $(n+1)$.

Proposition Suppose that $V$ and $V'$ are finite CW complexes whose suspension spectra are in $C_{n+1} \sim C_{n+2}$. Choose an integer $m>0$ with the property that $∃ d, d' ≤ m$ with $H_d(V; Z_p) ≠ 0$ and $H_{d'}(V'; Z_p) ≠ 0$. Then there is a natural weak equivalence

$$N_V I_i <m> \sim N_{V'} I_i <m>$$

(m-i)-connected

cover

Proposition If $V$ and $d$ are as above, and $ω$ is a spectrum then the maps

$$(L_n^f Σ^ω ω) <d> \rightarrow (Σ^ω L_n^f ω) <d>$$

$$(N_V Σ^ω ω) <d> \rightarrow (Ω^ω N_V ω) <d>$$

are weak equivalences.
I will establish one key point in the argument for these results now and return to the rest of the proof in a later talk.

**Proposition** Suppose that $X$ is a space, and $d > 0$ is an integer and

1. $(E^d_\pi)_\pi X = 0$
2. $H_d(X; Z_p) \neq 0$. \(K(Z_p, d)\) Proposition

Then for all $d' > d$,

$$(E^d_\pi)_\pi(K(Z_p, d')) = 0.$$ 

Note that for any non-contractible finite spectrum of type $\pi$, some suspension of $V$ is the suspension spectrum of a space which we might as well call $V$. Since $V$ is not contractible, there is a $d$ for which $H_d(V; Z_p) \neq 0$. Thus $d$ exists with the property that for $d' > d$,

$$L^d_\pi K(Z_p, d') \sim V.$$
The smallest possible $d$ is one greater than the minimum possible connectivity of an $E_{\infty}$-acyclic space. The theorem shows that this is attained on an Eilenberg-MacLane space $K(\mathbb{Z}_p, d)$ (and not on a finite complex). The precise value of $d$ is not known. One does know that

$$d \geq n+1$$

and the conjecture is that $d = n+1$. There is some progress toward proving this.

The proof of the proposition requires:

**Lemma.** Suppose that $E$ is a homology theory. If $X \to B$ is a map having the property that for each $b \in B$ the map

$$F_b \to b$$

is an $E$-equivalence, where $F_b$ is the homotopy fiber over $b$, then

$$X \to B$$

is an $E$-equivalence.

**Proof.** One can use the usual argument inducting over a covering of $B$ over which
the fibration is a product, or, equivalently write

$$X = \holim_B F_b$$

$$B = \holim_B \text{pt}$$

and appeal to the fact that $E$-equivalences are preserved under homotopy colimits.
**Lemma** Suppose $E$ is a spectrum. If $X \to pt$ is an $E$-equivalence then $QX = S^\infty \Sigma X \to pt$ is an $E$-equivalence.

**Proof:** If $E$ is contractible there is nothing to prove. If $E$ is not contractible then $X$ must be connected. In this case we can use the Smith splitting of the Hopf model

$$
\Sigma^\infty QX \sim \vee E\Sigma_n \wedge X^n
$$

to reduce to showing that $E\Sigma_n \wedge X^n$ is $E$-acyclic. But as in the proof of the previous proposition the map

$$
E\Sigma_n \wedge X^n \to E\Sigma_n \wedge pt \sim pt
$$

is an $E$-equivalence.

**Corollary** With $E$ and $X$ as above, if $Z$ is a $(\ell)$-connected spectrum then

$$S^\infty Z \times X$$

is $E$-acyclic.
Proof. Working through a cell decomposition of $\Sigma$ one reduces to the
assertion that if $s^{n-1} \to 2_1 \to Z_1$ is a cofibration sequence and $n \geq 1$ then the map

$$s^{\infty} Z_1 \vee \Sigma \to s^{\infty} \Sigma \vee \Sigma$$

is an $E$-equivalence. But this is a principal fibration with fiber

$$s^{\infty} \Sigma \vee s^{\infty} \Sigma$$

so the result follows from the fibration lemma, and the above lemma with $s^{n-1} \Sigma$ playing the role of $\Sigma$. □

Proof of the $K(\mathbb{Z}_p,d)$ proposition. Since $H_2(\Sigma, \mathbb{Z}_p) \neq 0$ the spectrum $\Sigma^d \mathbb{H}_p$ is a retract of $\mathbb{H}_p \Sigma$, and so $K(\mathbb{Z}_p,d)$ is a retract of $\Sigma^d \mathbb{H}_p \Sigma$. The latter is $E^f$-acyclic by the corollary above. This implies that $K(\mathbb{Z}_p,d)$ is $E^g$-acyclic. Replacing $\Sigma$ with $\Sigma^{d-d'}$ gives the result for $K(\mathbb{Z}_p,d')$. □
Nullification analogue

The above results hold for the nullification $N_\alpha$ with respect to a finite space $V$ of type $\alpha$.

We start with some general facts. First

**Definition** A space $\Sigma$ is $A$-null if the inclusion of the constant maps $\Sigma \to \Sigma^{A^+}$ is an equivalence.

A space $\Sigma$ is $A$-periodic (or co-null) if for all $A$-null spaces $\Sigma$ the inclusion of the constant maps $\Sigma \to \Sigma^{B^+}$ is an equivalence.

**Lemma** Suppose $Z \to W$ has the property that for all $A$-null spaces $D$, the map

$$D^+ \to D^Z$$

is an equivalence. Then $N_\alpha Z \to N_\alpha W$ is a weak equivalence.

pf: Immediate from the universal property.

**Lemma** If $Z \to B$ is a map having the property that for each $b \in B$ the homotopy fiber $F_b$ is $A$-periodic, then

$$F_b \to b$$

(Fibration Lemma)

then $N_\alpha Z \to N_\alpha B$ is a weak equivalence.

**proof** As in the proof of the previous fibration lemma, the assumptions imply that for each $A$-null space $D$, the map $D^{B^+} \to D^{B^+}$ is a weak equivalence. The claim follows from the lemma above.
Proposition. If $A$ is connected then $\varpi A \simeq 1^\infty \Omega^\infty A$ is $A$-periodic.

Proof. We will use the May model $\varpi A \simeq \underset{n \to \infty}{\lim} \operatorname{Fil}_n$ in which $\operatorname{Fil}_1 = A$ and for $n > 1$, $\operatorname{Fil}_n$ is defined inductively by the pushout square

\[
\begin{array}{ccc}
C_n(\mathbb{R}^\infty; A) & \longrightarrow & C_n(R^\infty; A) \\
\downarrow & & \downarrow \\
\operatorname{Fil}_{n-1} & \longrightarrow & \operatorname{Fil}_n
\end{array}
\]

in which

\[C_n(R^\infty; A) = \{ \Sigma C \subset R^\infty, \varphi : S \to A \mid |S| = n \} \]

is the configuration space of $n$ points in $R^\infty$ labeled by elements of $A$, the subspace $C_n'(R^\infty; A)$ is the subspace of $(S, \varphi)$ such that $\varphi(s) = x$ for some $s \in S$ and the left map sends $(S, \varphi)$ to $(S', \varphi')$ where $S'$ is the complement of some $s \in S$ with $\varphi(s) = x$ and $\varphi'$ is the restriction of $\varphi$. The result follows once one shows that for all $A$-null spaces $D$, the map

\[D(\operatorname{Fil}_n) \longrightarrow D(\operatorname{Fil}_{n+1}) \]

is a weak equivalence. From the pushout square it is enough to show that

\[
\begin{array}{ccc}
C_n(\mathbb{R}^\infty; A)_{+} & \longrightarrow & C_n'(R^\infty; A)_{+} \\
\downarrow & & \downarrow \\
D & \longrightarrow & D
\end{array}
\]

is a weak equivalence. Now the maps
\[ C_n(\mathbb{R}^\infty; A) \to C_n(\mathbb{R}^\infty; pt) \]
\[ C_n'(\mathbb{R}^\infty; A) \to C_n(\mathbb{R}^\infty; pt) \]

are fibrations with fibers \( A^n \) and \( T^n(A) = \{ (a_1, \ldots, a_n) | a_i = x, \text{some } i \} \).

One easily checks that \( A^n \) and \( T^nA \) are \( A \)-periodic, so the fibration lemma implies that in the diagram,

\[
\begin{array}{ccc}
C_n(\mathbb{R}^\infty; pt) & \longrightarrow & C_n(\mathbb{R}^\infty; pt) \\
\downarrow & & \downarrow \\
C_n(\mathbb{R}^\infty; A) & \longrightarrow & C_n'(\mathbb{R}^\infty; A) \\
\end{array}
\]

the vertical maps are weak equivalences.

Now one can imitate the argument in the stable case to show

**Proposition** If \( H_2(A; \mathbb{Z}_p) \neq 0 \) then \( K(\mathbb{Z}_p, d) \) is \( A \)-periodic.