Fix a prime number $p$ and an integer $n > 0$. We have seen that there is a monadic adjunction

$$\text{Sp}_{T(n)} \xleftarrow{\Phi} \mathcal{S}^*_n,$$

and that the associated monad $\Phi \Theta : \text{Sp}_{T(n)} \to \text{Sp}_{T(n)}$ is coanalytic, and can therefore be identified with an operad in $T(n)$-local spectra. In the last lecture, we proved that the Koszul dual $KD_{T(n)}(\Phi \Theta)$ can be identified with the nonunital commutative operad $\text{Sym}^*_{\text{red}}$. Our goal in this lecture is to prove the following closely related result:

**Theorem 1.** The Koszul biduality map

$$\Phi \Theta \to KD_{T(n)}(KD_{T(n)}(\Phi \Theta)) \simeq KD_{T(n)}(\text{Sym}^*_{\text{red}})$$

is an equivalence of operads in $T(n)$-local spectra. In other words, $\Phi \Theta$ is equivalent to the $T(n)$-local Lie operad (so that $\mathcal{S}^*_n$ is equivalent to the $\infty$-category of Lie algebras in $T(n)$-local spectra).

Theorem 1 is mostly a consequence of the following formal result:

**Proposition 2.** Let $\mathcal{O} = \{\mathcal{O}(k)\}_{k \geq 0}$ be an augmented operad in $T(n)$-local spectra. Assume that:

1. Each $\mathcal{O}(k)$ is dualizable as a $T(n)$-local spectrum.
2. The $T(n)$-local spectrum $\mathcal{O}(0)$ vanishes.
3. The augmentation on $\mathcal{O}$ induces an equivalence $\mathcal{O}(1) \to L_{T(n)} S$.

Then the biduality map $\mathcal{O} \to KD_{T(n)}(KD_{T(n)}(\mathcal{O}))$ is an equivalence.

**Remark 3.** In the statement of Proposition 2, there is nothing special about the setting of $T(n)$-local spectra: we could replace $\text{Sp}_{T(n)}$ by any presentable symmetric monoidal stable $\infty$-category (such as spectra, or chain complexes over a field).

The proof of Proposition 2 will require some preliminaries.

**Lemma 4.** Let $\mathcal{O}$ be an augmented operad in $T(n)$-local spectra satisfying the hypotheses of Proposition 2. Then the cooperad $\mathcal{O}' = \text{Bar}(\mathcal{O})$ satisfies the analogous conditions:

1. Each $\mathcal{O}'(k)$ is dualizable as a $T(n)$-local spectrum.
2. The $T(n)$-local spectrum $\mathcal{O}'(0)$ vanishes.
3. The augmentation of $\mathcal{O}'$ induces an equivalence $L_{T(n)} S \to \mathcal{O}'(1)$. 


Proof. Let us say that a symmetric sequence $\mathcal{E}$ is concentrated in degrees $\geq m$ if $\mathcal{E}(k) \neq 0$ for $k < m$. It follows immediately from the definitions that if $\mathcal{E}$ and $\mathcal{E}'$ are concentrated in degrees $\geq m$ and $\geq m'$, respectively, then the composition product $\mathcal{E} \circ \mathcal{E}'$ is concentrated in degrees $\geq mm'$. For any right $\mathcal{O}$-module $\mathcal{E}$, let $\mathcal{E} \circ_\mathcal{O} \mathcal{O}_{\text{triv}}$ denote the relative composition product of $\mathcal{E}$ with $\mathcal{O}_{\text{triv}}$ over $\mathcal{O}$, given by the geometric realization of a simplicial object given in simplicial degree $k$ by the iterated composition product
\[
\mathcal{E} \circ_\mathcal{O} \mathcal{O} \circ \cdots \circ_\mathcal{O} \mathcal{O} \circ \mathcal{O}_{\text{triv}}
\]
(in which the operad $\mathcal{O}$ appears $k$ times). Suppose that $\mathcal{E}$ is concentrated in degrees $\geq m$. Since $\mathcal{O}$ and $\mathcal{O}_{\text{triv}}$ are concentrated in degrees $\geq 1$, it follows that each of these iterated composition products is concentrated in degrees $\geq m$, so that the geometric realization $\mathcal{E} \circ_\mathcal{O} \mathcal{O}_{\text{triv}}$ is concentrated in degrees $\geq m$. Moreover, it follows from assumption $(c)$ that the simplicial spectrum
\[
[k] \mapsto (\mathcal{E} \circ_\mathcal{O} \cdots \circ_\mathcal{O} \circ_\mathcal{O} \circ \mathcal{O}_{\text{triv}})(m)
\]
is constant with value $\mathcal{E}(m)$, so that the canonical map $\mathcal{E} \to \mathcal{E} \circ_\mathcal{O} \mathcal{O}_{\text{triv}}$ is an equivalence in degree $m$. Taking $\mathcal{E} = \mathcal{O}_{\text{triv}}$, we immediately deduce $(b')$ and $(c')$.

To deduce $(a')$, it will suffice to prove the following:

$(\ast)$ Let $\mathcal{E}$ be a right $\mathcal{O}$-module and let $m \geq 0$ be an integer. Then the following conditions are equivalent:

$(i)$ The $T(n)$-local spectra $\mathcal{E}(k)$ are dualizable for $k \leq m$.

$(ii)$ The $T(n)$-local spectra $(\mathcal{E} \circ_\mathcal{O} \mathcal{O}_{\text{triv}})$ are dualizable for $k \leq m$.

Let us regard $m$ as fixed. We will show that $(\ast)$ holds in the case where $\mathcal{E}$ is concentrated in degrees $\geq m'$, using descending induction on $m'$. If $m' > m$, then $\mathcal{E}(k)$ and $(\mathcal{E} \circ_\mathcal{O} \mathcal{O}_{\text{triv}})(k) \neq 0$ for $k \leq m$, so there is nothing to prove. To carry out the inductive step, let $\mathcal{E}'$ be the symmetric sequence which agrees with $\mathcal{E}$ in degree $m'$, and vanishes in all other degrees. Note that we have $\mathcal{E}'(m') = \mathcal{E}(m') = (\mathcal{E} \circ_\mathcal{O} \mathcal{O}_{\text{triv}})(m')$, so that we may assume that $\mathcal{E}'$ is dualizable in each degree (as this follows from either $(i)$ or $(ii)$). The inclusion $\mathcal{E}' \to \mathcal{E}$ induces a map of right $\mathcal{O}$-modules $\mathcal{E}' \circ_\mathcal{O} \mathcal{O} \to \mathcal{E}$, whose cofiber is some right $\mathcal{O}$-module $\mathcal{E}''$. Using the equivalence $\mathcal{E}' \circ_\mathcal{O} \mathcal{O} \circ_\mathcal{O} \mathcal{O}_{\text{triv}} \simeq \mathcal{E}'$, we obtain a cofiber sequence of symmetric sequences
\[
\mathcal{E}' \to \mathcal{E} \circ_\mathcal{O} \mathcal{O}_{\text{triv}} \to \mathcal{E}'' \circ_\mathcal{O} \mathcal{O}_{\text{triv}}.
\]

If condition $(i)$ is satisfied, then $\mathcal{E}''$ is dualizable in degrees $\leq m$ (since this is true for both $\mathcal{E}' \circ_\mathcal{O} \mathcal{O}$ and $\mathcal{E}$). Applying our inductive hypothesis, we conclude that $\mathcal{E}'' \circ_\mathcal{O} \mathcal{O}_{\text{triv}}$ is dualizable in degrees $\leq m$. The cofiber sequence then shows that $\mathcal{E} \circ_\mathcal{O} \mathcal{O}_{\text{triv}}$ is dualizable in degrees $\leq m$, proving $(ii)$.

Conversely, if $(ii)$ is satisfied, then the cofiber sequence shows that $\mathcal{E}'' \circ_\mathcal{O} \mathcal{O}_{\text{triv}}$ is dualizable in degrees $\leq m$. Applying our inductive hypothesis, we conclude
that $E''$ is dualizable in degrees $\leq m$. Using the fiber sequence

$$E' \circ O \rightarrow E \rightarrow E''$$

we conclude that $E$ is dualizable in degrees $\leq m$. $\square$

**Corollary 5.** Let $O$ be an augmented operad satisfying the hypotheses of Proposition \[2\]. Then the Koszul dual $KD_{T(n)}(O)$ also satisfies the hypotheses of Proposition \[2\].

In what follows, we let $A$ denote the full subcategory of $\text{Alg}^{\text{aug}}(\text{SSeq}_{T(n)})$ spanned by those augmented operads satisfying the hypotheses of Proposition \[2\]. We wish to show that for each $O$ in $A$, the biduality map

$$u : O \rightarrow KD_{T(n)}(KD_{T(n)}(O))$$

is an equivalence. It follows from Corollary \[5\] that $u$ is a morphism in $A$. It will therefore suffice to show that for each object $O' \in A$, composition with $u$ induces a homotopy equivalence

$$\text{Map}_A(O', O) \rightarrow \text{Map}_A(O', KD_{T(n)}(KD_{T(n)}(O))) \simeq \text{Map}_A(KD_{T(n)}(O), KD_{T(n)}(O'))$$

In other words, we are reduced to showing that the Koszul duality functor is fully faithful, when regarded as a contravariant functor from $A$ to itself. Note that this functor factors as a composition

$$A \xrightarrow{\text{Bar}} A' \xrightarrow{D_{T(n)}} A^{\text{op}},$$

where $A'$ is the full subcategory of $\text{coAlg}^{\text{aug}}(\text{SSeq}_{T(n)})$ spanned by those augmented cooperads satisfying conditions (a'), (b'), and (c') of Lemma \[4\]. It follows immediately from the definitions that the Spanier-Whitehead duality functor $D_{T(n)} : A' \rightarrow A^{\text{op}}$ is an equivalence of $\infty$-categories. Consequently, Proposition \[2\] is a consequence of the following general result about the comparison between operads and cooperads:

**Proposition 6.** Let $\overline{A} \subseteq \text{Alg}^{\text{aug}}(\text{SSeq}_{T(n)})$ denote the full subcategory of $\text{Alg}^{\text{aug}}(\text{SSeq}_{T(n)})$ spanned by those symmetric sequences satisfying conditions (b) and (c) of Proposition \[4\], and define $\overline{A'} \subseteq \text{coAlg}^{\text{aug}}(\text{SSeq}_{T(n)})$ similarly. Then the bar construction $O \mapsto \text{Bar}(O)$ induces an equivalence of $\infty$-categories $\overline{A} \rightarrow \overline{A'}$.

Let us postpone the proof of Proposition \[6\] for the moment, and return to the proof of Theorem \[1\]. By virtue of Proposition \[2\] it will suffice to prove the following:

**Proposition 7.** Let $O$ be the symmetric sequence of coderivatives of $\Phi \circ \Theta$. Then:

(a) Each $O(k)$ is dualizable as a $T(n)$-local spectrum.

(b) The $T(n)$-local spectrum $O(0)$ vanishes.

(c) The unit map $L_{T(n)}S \rightarrow O(1)$ is an equivalence.
Proof. Assertion (b) is obvious (since \((\Phi \circ \Theta)(0) \simeq \Phi(\ast) \simeq 0\)). To prove (a), it will suffice to show that the bar construction \(\text{Bar}(\mathcal{O})\) is dualizable in each degree. This can be identified with the symmetric sequence of coderivatives of \(\Sigma_{T(n)}^\infty \circ \Omega_{T(n)}^\infty\), which we showed (in the previous lecture) to agree with the \(T(n)\)-local sphere in every positive degree.

It remains to prove (c). Let \(\text{Sp}(S_{*}^{\infty})\) denote the stabilization of the \(\infty\)-category \(S_{*}^{\infty}\); and let \(\Sigma : S_{*}^{\infty} \rightarrow \text{Sp}(S_{*}^{\infty})\) denote the left adjoint to the 0th space functor. Then \(\Sigma\) is universal among colimit-preserving functors from \(S_{*}^{\infty}\) to a presentable stable \(\infty\)-category. In particular, the functor \(\Sigma_{T(n)}^\infty : S_{*}^{\infty} \rightarrow \text{Sp}(T(n))\) admits an essentially unique factorization

\[
S_{*}^{\infty} \xrightarrow{\Sigma^\infty} \text{Sp}(S_{*}^{\infty}) \xrightarrow{F} \text{Sp}(T(n)).
\]

Since \(S_{*}^{\infty}\) can be identified with the \(\infty\)-category of left modules over the (reduced) monad \(\Phi \circ \Theta\) on \(\text{Sp}(T(n))\), it follows that the stabilization \(\text{Sp}(S_{*}^{\infty})\) can be identified with the \(\infty\)-category of left modules over the first derivative \(\partial_1(\Phi \circ \Theta) : \text{Sp}(\text{Sp}(T(n))) \rightarrow \text{Sp}(\text{Sp}(T(n)))\). In other words, we obtain an equivalence of \(\infty\)-categories \(\text{Sp}(S_{*}^{\infty}) \simeq \text{LMod}_{O(1)}(\text{Sp}(T(n)))\), where we regard \(O(1)\) as an associative ring spectrum (using the operad structure on \(O\)). Under this equivalence, the functor \(F : \text{Sp}(S_{*}^{\infty}) \rightarrow \text{Sp}(T(n))\) is given by extension of scalars along the augmentation \(\epsilon : O(1) \rightarrow L_{T(n)}(S)\) determined by the augmentation of \(O\). We may therefore reformulate (c) as follows:

(c') The functor \(F : \text{Sp}(S_{*}^{\infty}) \rightarrow \text{Sp}(T(n))\) is an equivalence of \(\infty\)-categories. In other words, the functor \(\Sigma_{T(n)}^\infty : S_{*}^{\infty} \rightarrow \text{Sp}(T(n))\) exhibits \(\text{Sp}(T(n))\) as the stabilization of \(S_{*}^{\infty}\).

Let us prove (c'). If \(\mathcal{C}\) is a pointed \(\infty\)-category which admits finite limits, we let \(\text{Sp}(\mathcal{C})\) denote the \(\infty\)-category of spectrum objects of \(\mathcal{C}\), given by the inverse limit

\[
\cdots \rightarrow \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C}.
\]

The usual \(\infty\)-category of spectra is obtained by applying this construction in the case where \(\mathcal{C} = S_{*}\) is the \(\infty\)-category of pointed spaces. However, it can just as well be obtained by applying the same construction to the \(\infty\)-category \(S_{*}^{(d)}\) of \(d\)-connected pointed spaces, for any \(d \geq 0\). Let \(A\) be a finite \((p)\)-local suspension space of type \((n + 1)\), and let \(d\) be the smallest integer such that \(H_d(A; F_p) \neq 0\). Recall that \(L_n^d S_{*}^{(d)}\) denotes the full subcategory of \(S_{*}^{(d)}\) spanned by those \(d\)-connected pointed spaces which are \(P_A\)-local and \(p\)-local. Then the identification \(\text{Sp} \simeq \text{Sp}(S_{*}^{(d)})\) restricts to an identification \(\mathcal{X} \simeq \text{Sp}(L_n^d S_{*}^{(d)})\), where \(\mathcal{X} \subseteq \text{Sp}\) is the full subcategory spanned by those spectra \(X\) such that each of the spaces \((\Omega^{\infty+k} X)(d)\) is \(P_{\mathcal{X}}\)-local and \(p\)-local. This is equivalent to the requirement that \(X\) is \(p\)-local and the mapping spectrum \(X^A\) is contractible; that is, that \(X\) is an \(L_n^d\)-local spectrum.
Let us identify $S^v_{\omega}$ with the full subcategory of $L^k_{\omega} S^v_{*}$ spanned by those objects whose $v_m$-local homotopy groups vanish for $0 \leq m < n$. Then the equivalence $\text{Sp}(L^k_{\omega} S^v_{*}) \simeq X$ restricts to an equivalence $\text{Sp}(S^v_{*}) \simeq \mathcal{Y}$, where $\mathcal{Y}$ is the full subcategory of $\text{Sp}$ spanned by those spectra $X$ which are $L^k_{\omega}$-local and have the property that each of the spaces $(\Omega^{\infty+k} X)(d)$ has trivial $v_m$-periodic homotopy for $0 \leq m < n$. This is equivalent to the requirement that $X$ is rationally trivial and that the Bousfield-Kuhn functors $\Phi$ and $\Omega$ carry the functor $F$ to the $T(n)$-localization functor $L_{T(n)}: \text{Sp}(S^v_{*}) \to \text{Sp}(S^v_{*})$, which we will denote by $\text{Sp}(S^v_{*})$. Arguing as in the previous lecture, we see that this identification carries the functor $F: \text{Sp}(S^v_{*}) \to \text{Sp}(S^v_{*})$ to the $T(n)$-localization functor $L_{T(n)}: \text{Sp}(S^v_{*}) \to \text{Sp}(S^v_{*})$, which is an equivalence as desired. \hfill \Box

We conclude this lecture by sketching a proof of Proposition 6. The bar construction

$$\text{Bar}: \text{Alg}_{\text{aug}}(\text{SSeq}_{T(n)}) \to \text{coAlg}_{\text{aug}}(\text{SSeq}_{T(n)})$$

has a right adjoint, given by the cobar construction $\text{Cobar}: \text{coAlg}_{\text{aug}}(\text{SSeq}_{T(n)}) \to \text{Alg}_{\text{aug}}(\text{SSeq}_{T(n)})$. (that is, applying the bar construction in the opposite category $\text{SSeq}_{T(n)}^\text{op}$; beware however that the composition product does not preserve geometric realizations in $\text{SSeq}_{T(n)}^\text{op}$). To show that the bar construction is fully faithful, it will suffice to show that if $\mathcal{O}$ satisfies conditions (b) and (c) of Proposition 2 then the unit map

$$u_\mathcal{O}: \mathcal{O} \to \text{Cobar}(\text{Bar}(\mathcal{O}))$$

is an equivalence (one can use essentially the same argument to show that the cobar construction is fully faithful on objects of $\overline{\mathcal{A}}';$ we leave the details to the reader).

As in the proof of Lemma 4 it will be convenient to prove a more general result concerning right modules over the operad $\mathcal{O}$. Note that if $\mathcal{E}$ is a right module over $\mathcal{O}$, then the relative composition product $\mathcal{E} \circ \mathcal{O} \mathcal{O}_{\text{triv}}$ can be regarded as a right comodule over $\text{Bar}(\mathcal{O})$, with structure map given by

$$\mathcal{E} \circ \mathcal{O} \mathcal{O}_{\text{triv}} \simeq \mathcal{E} \circ \mathcal{O} \mathcal{O} \circ \mathcal{O}_{\text{triv}}$$

$$\to \mathcal{E} \circ \mathcal{O}_{\text{triv}} \circ \mathcal{O}_{\text{triv}}$$

$$\simeq (\mathcal{E} \circ \mathcal{O}_{\text{triv}}) \circ \text{Bar}(\mathcal{O})$$

Let us denote $\mathcal{E} \circ \mathcal{O} \mathcal{O}_{\text{triv}}$ by $F(\mathcal{E})$, so that $F$ determines a functor $\text{RMod}_{\mathcal{O}}(\text{SSeq}_{T(n)}) \to \text{coRMod}_{\text{Bar}(\mathcal{O})}(\text{SSeq}_{T(n)})$. This functor has a right adjoint $G: \text{coRMod}_{\text{Bar}(\mathcal{O})}(\text{SSeq}_{T(n)}) \to \text{RMod}_{\mathcal{O}}(\text{SSeq}_{T(n)})$ (given by applying a similar construction in the opposite category) which we will denote by $G(F) = \mathcal{F} \circ \text{Bar}(\mathcal{O}) \mathcal{O}_{\text{triv}}$.

For every right $\mathcal{O}$-module $\mathcal{E}$, we have a map of symmetric sequences

$$u_\mathcal{E}: \mathcal{E} \to (G \circ F)(\mathcal{E}) = (\mathcal{E} \circ \mathcal{O} \mathcal{O}_{\text{triv}}) \circ \text{Bar}(\mathcal{O}) \mathcal{O}_{\text{triv}}.$$
In the special case where $\mathcal{E} = \mathcal{O}$ (regarded as a right module over itself), the map $u_\mathcal{E}$ underlies the map of operads $\mathcal{O} \to \text{Cobar}(\text{Bar}(\mathcal{O}))$ given by the adjointness of the bar and cobar constructions. It will therefore suffice to prove the following:

\((\ast)\) For any right $\mathcal{O}$-module $\mathcal{E}$, the unit map $u_\mathcal{E} : \mathcal{E} \to (G \circ F)(\mathcal{E})$ is an equivalence.

It will suffice to prove the following assertion for each $m \geq 0$:

\((\ast_m)\) For any right $\mathcal{O}'$-module $\mathcal{E}$, the unit map $u_\mathcal{E} : \mathcal{E} \to (G \circ F)(\mathcal{E})$ is an equivalence in degrees $\leq m$.

We will show that $(\ast_m)$ holds for every right $\mathcal{O}$-module $\mathcal{E}$ which is concentrated in degrees $\geq m'$. This is clear for $m' > m$ (since then both sides vanish in degrees $\leq m$). We handle the remaining cases by descending induction on $m'$. Assume that $\mathcal{E}$ is concentrated in degrees $\geq m'$. Then $\mathcal{E}$ fits into a fiber sequence of right $\mathcal{O}$-modules

$$\mathcal{E}' \rightarrow \mathcal{E} \rightarrow \mathcal{E}'',$$

where $\mathcal{E}'$ is concentrated in degrees $> m'$, and the action of $\mathcal{O}$ on $\mathcal{E}''$ is trivial (here we invoke our assumption that $\mathcal{O}$ satisfies $(b)$ and $(c)$ of Proposition 2).

Using the exactness of the functors $F$ and $G$, we obtain a commutative diagram of fiber sequences

$$\begin{array}{ccc}
\mathcal{E}' & \xrightarrow{u_{\mathcal{E}'}} & \mathcal{E} & \xrightarrow{u_{\mathcal{E}}} & \mathcal{E}'' \\
(G \circ F)(\mathcal{E}') & \rightarrow & (G \circ F)(\mathcal{E}) & \rightarrow & (G \circ F)(\mathcal{E}'').
\end{array}$$

Our inductive hypothesis guarantees that $u_{\mathcal{E}'}$ is an equivalence in degrees $\leq m$. Consequently, to prove $(\ast_m)$, it will suffice to show that the map $u_{\mathcal{E}''}$ is an equivalence. We are therefore reduced to proving assertion $(\ast)$ in the special case where the action of $\mathcal{O}$ on $\mathcal{E}$ is trivial: that is, it factors through the augmentation $\mathcal{O} \to \mathcal{O}_{\text{triv}}$. In this case, we compute

$$(G \circ F)(\mathcal{E}) \simeq G(\mathcal{E} \circ \mathcal{O}_{\text{triv}}) \simeq G(\mathcal{E} \circ \mathcal{O}_{\text{triv}} \circ \mathcal{O}_{\text{triv}}) \simeq G(\mathcal{E} \circ \text{Bar}(\mathcal{O})) \simeq \mathcal{E},$$

where the final equivalence follows from the observation that $\mathcal{E}' = \mathcal{E} \circ \text{Bar}(\mathcal{O})$ is cofree as a right comodule over $\text{Bar}(\mathcal{O})$ (so that the cobar construction on $\mathcal{E}'$ splits).