LECTURE XX: KOSZUL DUALITY, PART II

In the previous lecture, we discussed a covariant version of Koszul duality: given an augmented associative algebra $A$ in a monoidal $\infty$-category $C$, we can (under mild hypotheses) form the bar construction $\text{Bar}(A) = 1 \otimes_A 1$, and endow it with the structure of an associative coalgebra object of $C$. When $C$ is equipped with some notion of duality $D : C^{\text{op}} \to C$ which is lax monoidal, we can then take the dual $D \text{Bar}(A)$ and regard it as an associative algebra object of $C$. We will be particularly interested in the special case where $C = \text{SSeq} = \text{Fun}(\text{Set}^\text{fin}_\text{uni}, \text{Sp})$ is the $\infty$-category of symmetric sequences of spectra.

**Construction 1** (Spanier-Whitehead Duality for Symmetric Sequences). For every spectrum $E$, we let $D(E)$ denote the Spanier-Whitehead dual $\text{Map}(E,S)$. The construction $E \mapsto D(E)$ induces a functor from the $\infty$-category $\text{Sp}$ of spectra to $\text{Sp}^{\text{op}}$. Consequently, if $\mathcal{O} : \text{Set}^\text{fin}_\text{uni} \to \text{Sp}$ is a symmetric sequence of spectra, then the composite functor

$$\text{Set}^\text{fin}_\text{uni} \cong (\text{Set}^\text{fin}_\text{uni})^{\text{op}} \xrightarrow{\mathcal{O}} \text{Sp}^{\text{op}} \xrightarrow{D} \text{Sp}$$

is also a symmetric sequence of spectra, which we will denote by $D(\mathcal{O})$ and refer to as the *Spanier-Whitehead dual of* $\mathcal{O}$. Concretely, it is given by the formula

$$D(\mathcal{O})(T) = D(\mathcal{O}(T)).$$

**Variant 2.** Let $\text{SSeq}_{T(n)}$ denote the $\infty$-category $\text{Fun}(\text{Set}^\text{fin}_\text{uni}, \text{Sp}_{T(n)})$ of symmetric sequences of $T(n)$-local spectra. There is a $T(n)$-local version of Spanier-Whitehead duality, which we will denote by $D_{T(n)} : \text{Sp}_{T(n)} \to \text{Sp}_{T(n)}^{\text{op}}$, given by

$$D_{T(n)}(E) = \text{Map}(E, L_{T(n)} S).$$

Composition with $D_{T(n)}$ induces an operation of $T(n)$-local Spanier-Whitehead duality on the $\infty$-category $\text{SSeq}_{T(n)}$, given by the formula

$$D_{T(n)}(\mathcal{O})(T) = D_{T(n)}(\mathcal{O}(T)).$$

Let us regard the $\infty$-category $\text{SSeq}$ as endowed with the monoidal structure given by the *composition product* of symmetric sequences. In this case, the Spanier-Whitehead duality functor $D : \text{SSeq}^{\text{op}} \to \text{SSeq}$ is lax monoidal: in particular, for every pair of symmetric sequences symmetric sequences $\mathcal{O}$ and $\mathcal{O}'$, there is a canonical map

$$D(\mathcal{O}) \circ D(\mathcal{O}') \to D(\mathcal{O} \circ \mathcal{O}').$$
Remark 3 (Preliminary Explanation). Let us suppose that the symmetric sequences \( \mathcal{O} \) and \( \mathcal{O}' \) are reduced. In this case, the composition products in question are given concretely by the formulae

\[
(D(\mathcal{O}) \circ D(\mathcal{O}'))(T) = \bigoplus_{E \in \text{Equiv}(T)} D(\mathcal{O}(T/E)) \wedge \bigwedge_{T' \in T/E} D(\mathcal{O}'(T'))
\]

\[
(D(\mathcal{O} \circ \mathcal{O}'))(T) = \bigoplus_{E \in \text{Equiv}(T)} D(\mathcal{O}(T/E)) \wedge \bigwedge_{T' \in T/E} D(\mathcal{O}'(T')).
\]

There is a canonical map from the first spectrum to the second, given by the lax symmetric monoidal structure on the usual Spanier-Whitehead duality functor \( D : \text{Sp}^{\text{op}} \to \text{Sp} \).

Let us now give another explanation for the lax monoidal structure on the Spanier-Whitehead duality functor \( D : \text{SSeq}^{\text{op}} \to \text{SSeq} \). Recall that every symmetric sequence \( \mathcal{O} \in \text{SSeq} \) determines a functor \( F_{\mathcal{O}} : \text{Sp} \to \text{Sp} \), given by the formula

\[
F_{\mathcal{O}}(X) = \lim_{T \in \text{Set}_{\text{fin}}} \mathcal{O}(T) \wedge X^{\wedge T} \simeq \bigoplus_{n \geq 0} (\mathcal{O}(n) \wedge X^{\wedge n})_{h\Sigma_n}.
\]

The construction \( \mathcal{O} \mapsto F_{\mathcal{O}} \) determines a (monoidal) functor \( \text{SSeq} \to \text{Fun}(\text{Sp}, \text{Sp}) \). This functor has a right adjoint, given by the construction

\[
(F \in \text{Fun}(\text{Sp}, \text{Sp})) \mapsto \{\partial^n(F)\}_{n \geq 0};
\]

here \( \partial^n(F) \) denotes the \( n \)th coderivative of \( F \) in the sense of Lecture 8 (in that case, we were working with \( T(n) \)-local spectra, which guarantees that the construction \( \mathcal{O} \mapsto F_{\mathcal{O}} \) is fully faithful; however, the right adjoint exists in general).

Let us say that a functor \( \text{Sp} \to \text{Sp} \) is coanalytic if it has the form \( F_{\mathcal{O}} \), for some symmetric sequence \( \mathcal{O} \). In this case, we can form a new functor from \( \text{Sp} \) to \( \text{Sp} \), given by the composition \( D \circ F_{\mathcal{O}} \circ D \). This functor is usually not coanalytic. However, we can approximate it by a coanalytic functor, by applying the coderivative construction \( \partial^\bullet \).

**Proposition 4.** Let \( \mathcal{O} \) be a symmetric sequence of spectra. Then the Spanier-Whitehead dual symmetric sequence \( D(\mathcal{O}) \) is given by \( \{\partial^n(D \circ F_{\mathcal{O}} \circ D)\}_{n \geq 0} \).

**Proof.** Unwinding the definitions, we have the formula

\[
(D \circ F_{\mathcal{O}} \circ D)(X) = D(\lim_{T \in \text{Set}_{\text{fin}}} (\mathcal{O}(T) \wedge D(X)^{\wedge T})) = \lim_{T \in \text{Set}_{\text{fin}}} D(\mathcal{O}(T) \wedge D(X)^{\wedge T}).
\]
For each \( n \geq 0 \), the coderivative functor \( \partial^n \) commutes with inverse limits, so we have
\[
\partial^n(D \circ F \circ D) = \lim_{\text{Set}_{\text{fin}}^n} \partial^n(X \mapsto D(O(T) \wedge D(X)^{\wedge T}))
\]
\[
\simeq \lim_{\text{Set}_{\text{fin}}^n} \partial^n(X \mapsto D(O(T)) \wedge X^{\wedge T})
\]
where the second equality follows because the canonical map
\[
D(O(T)) \wedge X^{\wedge T} \to D(O(T) \wedge D(X)^{\wedge T})
\]
is an equivalence when \( X \) is a finite spectrum. It now suffices to observe that the functor \( D(O(T)) \wedge X^{\wedge T} \) is cohomogeneous of degree \( |T| \), so that the coderivative \( \partial^n(X \mapsto D(O(T)) \wedge X^{\wedge T}) \) vanishes for \( |T| \neq n \). For \( |T| = n \), the coderivative is given by the smash product \( D(O(T)) \wedge (\Sigma_n)_+ \). Passing to the inverse limit over \( T \), we obtain the desired equivalence
\[
\partial^n(D \circ F \circ D) \simeq D(O(n)).
\]
\( \square \)

The \( T(n) \)-local version of Proposition 3 follows by the same argument, and is perhaps a bit easier to formulate.

**Notation 5.** Let \( F : \text{Sp}_{T(n)} \to \text{Sp}_{T(n)} \) be a coanalytic functor, so that \( F \) has an essentially unique expression \( F(X) = L_{T(n)}F_{O}(X) \) for some \( T(n) \)-local symmetric sequence \( O \in \text{SSeq}_{T(n)}^{T(n)} \). We let \( F^D \) denote the coanalytic functor \( L_{T(n)} \circ F_{D_{T(n)}} \) given by the symmetric sequence which is \( T(n) \)-locally Spanier-Whitehead dual to \( O \). The proof of Proposition ?? shows that \( F^D \) is universal among coanalytic functors equipped with a natural transformation
\[
F^D \to D_{T(n)} \circ F \circ D_{T(n)}
\]
(or, equivalently, with a natural transformation \( F \circ D_{T(n)} \to D_{T(n)} \circ F^D \)).

**Remark 6.** Let \( F, G : \text{Sp}_{T(n)} \to \text{Sp}_{T(n)} \) be coanalytic functors. Then the composition \( F^D \circ G^D \) is also a coanalytic functor, equipped with a canonical map
\[
F^D \circ G^D \to (D_{T(n)} \circ F) \circ D_{T(n)} = (D_{T(n)} \circ G) \circ D_{T(n)} \]
Invoking the universal property of \( (F \circ G)^D \), we obtain a comparison map
\[
(F \circ G)^D \to F^D \circ G^D.
\]
It is not difficult to see that this map is given by the analogue of Remark 3 carried out in the \( T(n) \)-local setting (at least in the case where the functors \( F \) and \( G \) are reduced).

Let us describe a third way of understanding the lax compatibility of Spanier-Whitehead duality with composition.
Definition 7. Let $S$ denote the ∞-category of spaces. We define a functor

$$\mu : \text{Sp}^{op} \times \text{Sp}^{op} \to S$$

by the formula

$$\mu(X, Y) = \text{Map}_{\text{Sp}}(X \wedge Y, S) \simeq \text{Map}_{\text{Sp}}(X, D(Y)) \simeq \text{Map}_{\text{Sp}}(Y, D(X)).$$

The map $\mu$ classifies a right fibration of ∞-categories

$$\mathcal{E} \to \text{Sp} \times \text{Sp}.$$ 

More informally, $\mathcal{E}$ is the ∞-category whose objects are triples $(X, Y, b)$, where $X$ and $Y$ are spectra and $b : X \wedge Y \to S$ is a map of spectra.

Suppose we are given symmetric sequences $O, O' \in \text{SSeq}$. We define a pairing of $O$ with $O'$ to be an endomorphism of the right fibration $\mathcal{E} \to \text{Sp} \times \text{Sp}$ which restricts to the endomorphism of the base $\text{Sp} \times \text{Sp}$ given by

$$(X, Y) \mapsto (F_O(X), F_{O'}(Y)).$$

More informally: a pairing of $O$ with $O'$ is a rule which associates to each “bilinear form” $b : X \wedge Y \to S$ a new “bilinear form” $F_O(X) \wedge F_{O'}(Y) \to S$. We $\mathcal{P}$ denote the ∞-category whose objects are triples $(O, O', \phi)$, where $O, O' \in \text{SSeq}$ and $\phi$ is a pairing of $O$ with $O'$. This is a monoidal ∞-category which acts compatibly on $\mathcal{E}$ and $\text{Sp} \times \text{Sp}$.

Remark 8. Let $O$ and $O'$ be symmetric sequences. To specify a pairing of $O$ with $O'$, we must associate to every map of spectra $b : X \to D(Y)$ another map of spectra $F_O(X) \to D(F_{O'}(Y))$. To supply such a map functorially in $X$ and $Y$, it suffices to specify it in the universal case where $X = D(Y)$ and $b$ is the identity map. In this case, we are giving a map

$$(F_O \circ D)(Y) \to (D \circ F_{O'})(Y)$$

which depends functorially on $Y$, or equivalently (by duality) a natural transformation of functors

$$F_{O'} \to D \circ F_O \circ D.$$

The calculation of Proposition 4 shows that this is the same data as a map of symmetric sequences $O' \to D(O)$. Or, by symmetry, the same data as a map of symmetric sequences $O \to D(O')$.

The forgetful functor $\mathcal{P} \to \text{SSeq} \times \text{SSeq}$ is an example of a pairing of monoidal ∞-categories. Applying some general nonsense, one can deduce the following:

- The construction $O \mapsto D(O)$ determines a lax monoidal functor $\text{SSeq}^{op} \to \text{SSeq}$ (concretely, this lax monoidal structure induces the comparison maps $D(O) \circ D(O') \to D(O \circ O')$ of Remark 3). In particular, it carries (augmented) coalgebra objects of $\text{SSeq}$ to (augmented) algebra objects of $\text{SSeq}$.
- The induced map $\text{Alg}(\mathcal{P}) \to \text{Alg}(\text{SSeq} \times \text{SSeq})$ is a right fibration.
We can identify objects of \( \text{Alg}(\mathcal{P}) \) with triples \((\mathcal{O}, \mathcal{O}', f)\) where \( \mathcal{O} \) and \( \mathcal{O}' \) are augmented algebra objects of \( \text{Alg}(\text{SSeq}) \) (that is, augmented operads) and \( f \) is a map of augmented operads \( \mathcal{O}' \to \text{D}(\text{Bar}(\mathcal{O})) \) (or, by symmetry, a map of augmented operads \( \mathcal{O} \to \text{D}(\text{Bar}(\mathcal{O}')) \)).

**Notation 9.** Let \( \mathcal{O} \) be an augmented operad. We let \( \text{KD}(\mathcal{O}) = \text{D}(\text{Bar}(\mathcal{O})) \). We will refer to \( \text{KD}(\mathcal{O}) \) as the Koszul dual of \( \mathcal{O} \).

The construction \( \mathcal{O} \mapsto \text{KD}(\mathcal{O}) \) induces a functor

\[
\text{KD} : \text{Alg}^{\text{nu}}(\text{SSeq})^{\text{op}} \to \text{Alg}^{\text{nu}}(\text{SSeq}),
\]

which we will refer to as Koszul duality for operads. By construction, it is self-adjoint: that is, we have canonical homotopy equivalences

\[
\text{Map}_{\text{Alg}^{\text{nu}}(\text{SSeq})^{\text{op}}}(\mathcal{O}, \text{KD}(\mathcal{O}')) \simeq \text{Map}_{\text{Alg}^{\text{nu}}(\text{SSeq})^{\text{op}}}(\mathcal{O}', \text{KD}(\mathcal{O})).
\]

We now explain a less symmetric way to think about Koszul duality. In what follows, let us regard the \( \infty \)-category \( \text{Sp} \) of spectra as equipped with a left action of \( \text{SSeq} \), via the construction

\[
\text{SSeq} \times \text{Sp} \to \text{Sp},
\]

\[
(\mathcal{O}, X) \mapsto F_\mathcal{O}(X) = \lim_{T \in \text{Set}^\text{fin}} \mathcal{O}(T) \wedge X^\wedge T.
\]

**Warning 10.** The construction \( (\mathcal{O}, X) \mapsto F_\mathcal{O}(X) \) determines a left action of \( \text{SSeq} \) on \( \text{Sp} \) with respect to the convention where the composition product on reduced symmetric sequences is given by the formula

\[
(\mathcal{O} \circ \mathcal{O}')(T) = \bigoplus_{E \in \text{Equiv}(T)} (\mathcal{O}(T/E) \wedge \bigwedge_{T' \in T/E} \mathcal{O}'(T')).
\]

Beware that this is actually the opposite of the multiplication induced by the identification \( \text{SSeq} \simeq \text{LFun}^\text{op}(\text{SSeq}, \text{SSeq}) \) of the previous lecture.

Let \( \mathcal{O} \) be an operad, and let \( \mathcal{A} = \text{LMod}_\mathcal{O}(\text{Sp}) \) denote the \( \infty \)-category of spectra equipped with an \( \mathcal{O} \)-algebra structure. There is a forgetful functor \( G : \mathcal{A} \to \text{Sp} \) with a left adjoint \( F : \text{Sp} \to \mathcal{A} \) (given by functor \( F_\mathcal{O} \)). An augmentation \( \epsilon : \mathcal{O} \to \mathcal{O}_{\text{triv}} \) induces another functor \( G' : \text{Sp} \to \mathcal{A} \), which admits a left adjoint \( F' : \mathcal{A} \to \text{Sp} \). Composing \( F' \) with Spanier-Whitehead duality, we obtain a functor

\[
\mathcal{A}^{\text{op}} \xrightarrow{F'} \text{Sp}^{\text{op}} \xrightarrow{\text{D}} \text{Sp}.
\]

**Proposition 11.** Let \( \mathcal{O} \) be an augmented operad, and let \( \mathcal{A} = \text{LMod}_\mathcal{O}(\text{Sp}) \) be as above. Then the Koszul dual \( \text{KD}(\mathcal{O}) \) can be identified with the endomorphism algebra of the object \((\text{D} \circ F') \in \text{Fun}(\mathcal{A}^{\text{op}}, \text{Sp}) \) (which we regard as equipped with a left action of \( \text{SSeq} \)).
Proof. For every symmetric sequence \( O' \), let \( F_{O'} \) denote the associated endofunctor of \( \text{Sp} \). By definition, an endomorphism object \( \text{End}(D \circ F') \in \text{Seq} \), if it exists, is a symmetric sequence with the universal property

\[
\text{Map}_{\text{Seq}}(O', \text{End}(D \circ F')) = \text{Map}_{\text{Fun}(A^{op}, \text{Sp})}(F_{O'} \circ D \circ F', D \circ F')
\]

\[
= \text{Map}_{\text{Fun}(\text{Sp}^{op}, \text{Sp})}(F_{O'} \circ D, D \circ F' \circ G')
\]

\[
= \text{Map}_{\text{Fun}(\text{Sp}, \text{Sp})}(F_{O'}, D \circ F' \circ G' \circ D)
\]

\[
= \text{Map}_{\text{Fun}(\text{Sp}, \text{Sp})}(F_{O'}, D \circ \text{Bar}(G \circ F) \circ D)
\]

\[
= \text{Map}_{\text{Fun}(\text{Sp}, \text{Sp})}(F_{O'}, D \circ F_{\text{Bar}(O)} \circ D)
\]

\[
= \text{Map}_{\text{Seq}}(O', D(\text{Bar}(O))).
\]

Clearly, this universal property is enjoyed by the Koszul dual \( KD(O) = D(\text{Bar}(O)) \). (Technically this argument only establishes an equivalence \( \text{End}(D \circ F') \simeq KD(O) \) as objects of the \( \infty \)-category \( \text{Seq} \), but a more careful analysis gives an identification as algebra objects of \( \text{Seq} \).) \( \square \)

Let us now return to the case of interest. Let \( S_{\infty}^{v_n} \) denote the \( \infty \)-category of \( v_n \)-periodic spaces, so that we can identify \( S_{\infty}^{v_n} \) with the \( \infty \)-category of left modules over the coanalytic monad \( \Phi \circ \Theta : \text{Sp}_{T(n)} \to \text{Sp}_{T(n)} \). In the previous lecture, we described an augmentation on the monad \( \Phi \circ \Theta \) for which the associated map

\[
S_{\infty}^{v_n} \simeq \text{LMod}_{\Phi \circ \Theta}(\text{Sp}_{T(n)}) \to \text{Sp}_{T(n)}
\]

is given by \( \Sigma_{T(n)}^{\infty} \). Composing with the \( T(n) \)-local Spanier-Whitehead duality functor \( D_{T(n)} \), we obtain the map

\[
D_{T(n)} \circ \Sigma_{T(n)}^{\infty} : (S_{\infty}^{v_n})^{op} \to \text{Sp}_{T(n)}.
\]

\[
X \mapsto \text{Map}(\Sigma_{T(n)}^{\infty} X, L_{T(n)} S).
\]

We can identify \( \Phi \circ \Theta \) with an augmented operad in the \( \infty \)-category \( \text{Sp}_{T(n)} \) of \( T(n) \)-local spectra. This operad has a \( T(n) \)-local Koszul dual, given by

\[
KD_{T(n)}(\Phi \circ \Theta) = \text{Bar}(\Phi \circ \Theta)^{D}.
\]

Using the \( T(n) \)-local analogue of Proposition \([\mathbb{L}]\) we can identify \( KD_{T(n)}(\Phi \circ \Theta) \) with the endomorphism algebra of the functor \( D_{T(n)} \circ \Sigma_{T(n)}^{\infty} \).

Note that for any object \( X \in S_{\infty}^{v_n} \), the \( T(n) \)-local Spanier-Whitehead dual

\[
(D_{T(n)} \circ \Sigma_{T(n)}^{\infty})(X) = \text{Map}(\Sigma_{T(n)}^{\infty} X, L_{T(n)} S)
\]

can be regarded as a nonunital \( E_{\infty} \)-ring spectrum (with the \( E_{\infty} \) structure induced from the \( E_{\infty} \)-structure on \( L_{T(n)} S \)). In other words, the nonunital commutative operad \( O_{\text{Comm}}^{nu} \) (in its \( T(n) \)-local incarnation) acts on the functor \( D_{T(n)} \circ \Sigma_{T(n)}^{\infty} \). This action is then encoded by a canonical map

\[
O_{\text{Comm}}^{nu} \to \text{End}(D_{T(n)} \circ \Sigma_{T(n)}^{\infty}) = KD_{T(n)}(\Phi \circ \Theta),
\]
or equivalently by the adjoint map $\Phi \circ \Theta \to KD_{T(n)}(O_{Comm}^{nu})$. We can now precisely formulate the end of the story:

**Theorem 12.** The maps

\[
O_{Comm}^{nu} \to KD_{T(n)}(\Phi \circ \Theta)
\]

\[
\Phi \circ \Theta \to KD_{T(n)}(O_{Comm}^{nu})
\]

are homotopy equivalences of operads in $T(n)$-local spectra. In particular, $\Phi \circ \Theta$ is the Lie operad in $T(n)$-local spectra.

We postpone the proof to the next lecture.