The Local Calculation: From Germs to Bundles (Lecture 27)

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Let k be an algebraically closed field, X an algebraic curve over k, and G a smooth affine group scheme over X, and let T be a nonempty finite set. Assume that the group scheme G is split (so that $G = X \times_{\operatorname{Spec} k} G_0$, where G_0 is a split reductive algebraic group over k). In this lecture, we will prove that the restriction map

$$\operatorname{Ran}^G_{\text{germ}}(X)^T \to \operatorname{Ran}^G(X)^T$$

is a universal homology equivalence.

Fix a finitely generated k-algebra R and a map $f : \operatorname{Spec} R \to \operatorname{Ran}^G(X)^T$. Then f can be identified with a pair $(\nu, \mathcal{P}_0)$, where $\nu : T \to X(R)$ is a map of sets and $\mathcal{P}_0$ is a G-bundle on the divisor $D = |\nu(T)| \subseteq X_R$. Let $\mathcal{C}$ denote the fiber product

$$\operatorname{Ran}^G_{\text{germ}}(X)^T \times_{\operatorname{Ran}^G(X)^T} \operatorname{Spec} R.$$

Unwinding the definitions, we can identify the objects of $\mathcal{C}$ with triples $(A, \mathcal{P}, \alpha)$ where A is a finitely generated R-algebra, $\mathcal{P}$ is a G-bundle on an open subset of X_A which contains D_A (which can be extended to a G-bundle on X_A), and α is an isomorphism between $\mathcal{P}|_{D_A}$ and $\mathcal{P}_0|_{D_A}$. We wish to show that the projection map $\mathcal{C} \to \operatorname{Spec} R$ induces an isomorphism on ℓ-adic homology.

Remark 1. The argument that we give here is specific to the case of a split group scheme. In the case of a general group scheme G, our proof that the map

$$\operatorname{Spec} R \times_{\operatorname{Ran}^G(X)^T} \operatorname{Ran}^G_{\text{germ}}(X)^T \to \operatorname{Spec} R$$

would require that the map $\nu : T \to X(R)$ is in "general position" with respect to the locus where G fails to be reductive.

The assertion that the map $\mathcal{C} \to \operatorname{Spec} R$ is an an equivalence can be tested locally on $\operatorname{Spec} R$ (with respect to the étale topology). We may therefore assume without loss of generality that $\mathcal{P}_0|_D$ is trivial. Since G is smooth, this implies that the G-bundle $\mathcal{P}_0$ is trivial. We may therefore identify objects of $\mathcal{C}$ with triples $(A, \mathcal{P}, \alpha)$, where $\mathcal{P}$ is a G-bundle on some open set $U \subseteq X_A$ containing D_A, and α is a trivialization of $\mathcal{P}|_{D^\wedge_A}$.

Suppose that $(A, \mathcal{P}, \alpha)$ is an object of $\mathcal{C}$, so that $\mathcal{P}$ can be extended to a G-bundle $\overline{\mathcal{P}}$ on X_A. Let B_0 be a Borel subgroup of G_0. Using the Drinfeld-Simpson theorem, we see that after passing to an étale cover of $\operatorname{Spec} A$, we can assume that $\overline{\mathcal{P}}$ admits a reduction to a B_0-bundle $\overline{\mathcal{Q}}$. The group B_0 fits into an exact sequence

$$0 \to \operatorname{rad}_u(B_0) \to B_0 \to \mathbf{G}_m^r \to 0.$$

Consequently, $\overline{\mathcal{Q}}$ determines a finite collection of line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_r$ on the curve X_A. Passing to a Zariski covering of $\operatorname{Spec} A$, we can assume that each of the line bundles $\mathcal{L}_i$ is trivial over an affine open subset $V \subseteq X_A$ which contains D_A. In this case, we can reduce the structure group of $\overline{\mathcal{Q}}|_V$ to the unipotent radical $\operatorname{rad}_u(B_0)$. Since V is affine, it follows that $\overline{\mathcal{Q}}|_V$ is trivial, so that $\overline{\mathcal{P}}|_V$ is trivial.

Let $\mathcal{C}_0$ denote the full subcategory of $\mathcal{C}$ spanned by those triples $(A, \mathcal{P}, \alpha)$ where the G-bundle $\mathcal{P}$ is trivial. The above argument shows that the inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ induces an equivalence after sheafification for the étale topology, and therefore induces an isomorphism on $\mathbf{Z}_\ell$-homology. We are therefore reduced to proving that the map $\mathcal{C}_0 \to \operatorname{Spec} R$ induces an isomorphism on $\mathbf{Z}_\ell$-homology. This map has a canonical section $e : \operatorname{Spec} R \to \mathcal{C}_0$, given by the standard trivialization of the trivial G_0-bundle on X_R. We will complete the proof by showing that e induces an isomorphism on $\mathbf{Z}_\ell$-homology.

For any quasi-projective k-scheme Z, we let $\operatorname{Hom}_{\text{germ}}(X_A, Z)$ denote the set of germs of maps from X_A to Z around D_A: that is, the direct limit $\varinjlim \operatorname{Hom}(U, Z)$ where the limit is taken over all open subsets $U \subseteq X_A$ which contain D_A. We let $\operatorname{Hom}(D_A, Z)$ denote the set of maps from D_A into Z (in the category of formal k-schemes). We let $\operatorname{Map}_{\text{germ}}(X_R, Z)$ denote the prestack whose objects are pairs (A, f) where $A \in \operatorname{Ring}_R$ and $f \in \operatorname{Hom}_{\text{germ}}(X_A, Z)$, and we define $\operatorname{Map}(D, Z)$ similarly. Note that we can regard $\operatorname{Map}_{\text{germ}}(X_R, G_0)$ as a group object in the 2-category of prestacks over $\operatorname{Spec} R$, and we can identify $\mathcal{C}_0$ with the homotopy quotient of $\operatorname{Map}(D, G_0)$ by the action of $\operatorname{Map}_{\text{germ}}(X_R, G_0)$. Similarly, we can identify $\operatorname{Spec} R$ with the homotopy quotient of $\operatorname{Map}_{\text{germ}}(X_R, G_0)$ by the translation action of itself. It follows that e induces an identification of

$$C_*(\operatorname{Spec} R; \mathbf{Z}_\ell) \to C_*(\mathcal{C}_0; \mathbf{Z}_\ell)$$

with the geometric realization of a map of simplicial chain complexes

$$C_*(\operatorname{Map}_{\text{germ}}(X_R, G_0)^n \times_{\operatorname{Spec} R} \operatorname{Map}(D, G_0); \mathbf{Z}_\ell) \to C_*(\operatorname{Map}_{\text{germ}}(X_R, G_0)^n \times_{\operatorname{Spec} R} \operatorname{Map}_{\text{germ}}(X_R, G_0); \mathbf{Z}_\ell).$$

We will complete the proof by showing that for any prestack in groupoids $\mathcal{E}$ equipped with a map $\mathcal{E} \to \operatorname{Spec} R$, the induced map

$$\mathcal{E} \times_{\operatorname{Spec} R} \operatorname{Map}(D, G_0) \to \mathcal{E} \times_{\operatorname{Spec} R} \operatorname{Map}_{\text{germ}}(X_R, G_0)$$

induces an isomorphism on homology. Using a direct limit argument, we can reduce to the case where $\mathcal{E} = \operatorname{Spec} A$ for some finitely generated R-algebra A. Replacing R by A, we are reduced to proving the following:

Proposition 2. *The restriction map* $\operatorname{Map}_{\text{germ}}(X_R, G_0) \to \operatorname{Map}(D, G_0)$ *induces an isomorphism on* $\mathbf{Z}_\ell$*-homology.*

In fact, we will show that the restriction map is a universal homology equivalence. Choose a map $\operatorname{Spec} A \to \operatorname{Map}(D, G_0)$, corresponding to a map $u : D_A \to G_0$; we wish to prove that the projection map

$$\operatorname{Map}_{\text{germ}}(X_R, G_0) \times_{\operatorname{Map}(D, G_0)} \operatorname{Spec} A \to \operatorname{Spec} A$$

induces an isomorphism on homology. Let $V \subseteq G_0$ denote the "big cell" of the Bruhat decomposition. Then V is a dense open subset of G_0. Consequently, for any finite set of points $g_1, \ldots, g_m \in G_0(k)$, the intersection $g_1^{-1} V \cap \cdots \cap g_m^{-1} V$ contains some k-point $h \in G_0(k)$, so that $g_1, \ldots, g_m \in V h^{-1}$. In other words, every finite subset of G_0 is contained in some translate Vh^{-1} of V. Consequently, after passing to a Zariski open covering of $\operatorname{Spec} A$, we may assume that the map u factors through some translate $Vh^{-1} \subseteq G_0$. Note that the diagram of prestacks

$$\begin{array}{ccc} \operatorname{Map}_{\text{germ}}(X_R, Vh^{-1}) & \longrightarrow & \operatorname{Map}_{\text{germ}}(X_R, G_0) \\ \downarrow & & \downarrow \\ \operatorname{Map}(D, Vh^{-1}) & \longrightarrow & \operatorname{Map}(D, G_0) \end{array}$$

is a pullback square: if $f : U \to G_0$ is a map from some open subset $U \subseteq X_B$ into G_0 which carries D_B into Vh^{-1}, then (after shrinking U) we can assume that $f(U) \subseteq Vh^{-1}$. It will therefore suffice to show that the map $\operatorname{Map}_{\text{germ}}(X_R, Vh^{-1}) \to \operatorname{Map}(D, Vh^{-1})$ is a universal homology equivalence.

Note that V is isomorphic (as a k-scheme) to an open subset of an affine space $\mathbf{A}^d$, where d is the dimension of the group G. Repeating the above argument, we obtain a pullback square

$$\begin{array}{ccc} \operatorname{Map}_{\text{germ}}(X_R, Vh^{-1}) & \longrightarrow & \operatorname{Map}_{\text{germ}}(X_R, \mathbf{A}^d) \\ \downarrow & & \downarrow \\ \operatorname{Map}(D, Vh^{-1}) & \longrightarrow & \operatorname{Map}(D, \mathbf{A}^d). \end{array}$$

We are therefore reduced to proving that the map $\mathrm{Map}_{\mathrm{germ}}(X_R, \mathbf{A}^d) \to \mathrm{Map}(D, \mathbf{A}^d)$ is a universal homology equivalence. Proceeding by induction on d, we can reduce to proving that the map

$$\mathrm{Map}_{\mathrm{germ}}(X_R, \mathbf{A}^1) \to \mathrm{Map}(D, \mathbf{A}^1)$$

is a universal homology equivalence.

Fix a map $\mathrm{Spec}\, A \to \mathrm{Map}(D, \mathbf{A}^1)$, corresponding to a regular function f_0 on the affine scheme D_A. Let $\mathcal{E}$ denote the fiber product

$$\mathcal{E} = \mathrm{Map}_{\mathrm{germ}}(X_R, \mathbf{A}^1) \times_{\mathrm{Map}(D, \mathbf{A}^1)} \mathrm{Spec}\, A.$$

We wish to prove that the projection map $\pi : \mathcal{E} \to \mathrm{Spec}\, A$ induces an isomorphism on $\mathbf{Z}_\ell$-homology. This assertion is local on $\mathrm{Spec}\, A$; we may therefore assume without loss of generality that D_A is contained in an affine open subset $U \subseteq X_A$. In this case, we can lift f_0 to a regular function on U, which determines a map $s : \mathrm{Spec}\, A \to \mathcal{E}$ which is a section of π. To complete the proof, it will suffice to show that the composite map $s \circ \pi : \mathcal{E} \to \mathcal{E}$ induces an isomorphism on $\mathbf{Z}_\ell$-homology. In fact, we claim that $s \circ \pi$ is $\mathbf{A}^1$-homotopic to the identity: that is, there exists a map $h : \mathbf{A}^1 \times \mathcal{E} \to \mathcal{E}$ whose restriction to $\{0\} \times \mathcal{E}$ is the identity, and whose restriction to $\{1\} \times \mathcal{E}$ coincides with $s \circ \pi$. To prove this, we take h to be the "straight line" homotopy, which carries a regular function $\overline{f}$ on an open subset $W \subseteq X_B$ and an element $t \in B$ to the function $(1-t)\overline{f} + tf$ on $\mathbf{A}^1 \times W \cap U_B \subseteq X_B$.