ℓ-adic Sheaves (Lecture 18)

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Let k be an algebraically closed field and ℓ a prime number which is invertible in k. If G is a smooth affine group scheme over an algebraic curve X over k whose generic fiber is split reductive, then we have shown that the projection map

$$\theta : \mathrm{Ran}_G(X) \to \mathrm{Bun}_G(X)$$

induces an isomorphism on $\mathbf{Z}_\ell$-homology. Our formulation of this result uses the theory of ℓ-adic homology, but this should be regarded as inessential: the main idea is that θ behaves like a homotopy equivalence (and we can construct an analogous map in topology which *is* a homotopy equivalence). Over the next few lectures, we are going to develop a dual formulation of nonabelian Poincare duality. There does not seem to be a corresponding "unstable" picture of this result, to the effect that some map of prestack induces an isomorphism on homology. To formulate it, we will need to use sheaf theory in an essential way. Our goal in this lecture is to give a brief review of the theory of ℓ-adic sheaves.

Notation 1. Let Sch_k denote the category of quasi-projective k-schemes. Given a quasi-projective k-scheme X, we let $\mathrm{Sch}_X^{\mathrm{et}}$ denote the category whose objects are étale maps $f : U \to X$, and whose morphisms are commutative diagrams

$$\begin{array}{ccc} U & \longrightarrow & U' \\ & {\searrow}^{f} \quad {\swarrow}_{f'} & \\ & X. & \end{array}$$

Let $\Lambda = \mathbf{Z}/\ell^d\mathbf{Z}$, for some integer d, and let $\mathcal{F} \in \mathrm{Fun}((\mathrm{Sch}_X^{\mathrm{et}})^{\mathrm{op}}, \mathrm{Mod}_\Lambda)$. For each integer n, we let $\pi_n \mathcal{F}$ denote the sheafification of the presheaf of abelian groups given by $U \mapsto \mathrm{H}_n(\mathcal{F}(U))$. We will say that $\mathcal{F}$ is *locally acyclic* if $\pi_n \mathcal{F} \simeq 0$ for every integer n.

We let $\mathrm{Shv}(X; \Lambda)$ denote the full subcategory of $\mathrm{Fun}((\mathrm{Sch}_X^{\mathrm{et}})^{\mathrm{op}}, \mathrm{Mod}_\Lambda)$ spanned by those functors $\mathcal{F}$ which satisfy the following condition:

($*$) For every morphism $\alpha : \mathcal{F}' \to \mathcal{F}$ in $\mathrm{Fun}((\mathrm{Sch}_X^{\mathrm{et}})^{\mathrm{op}}, \mathrm{Mod}_\Lambda)$, if $\mathcal{F}'$ is acyclic, then α is nullhomotopic.

We will refer to $\mathrm{Shv}(X; \Lambda)$ as the *∞-category of* Mod_Λ*-valued étale sheaves on* X.

Warning 2. Let $\mathcal{F} \in \mathrm{Fun}((\mathrm{Sch}_X^{\mathrm{et}})^{\mathrm{op}}, \mathrm{Mod}_\Lambda)$. Then condition ($*$) implies that $\mathcal{F}$ satisfies the following descent condition:

($*'$) Let $U \in \mathrm{Sch}_X^{\mathrm{et}}$ and let $\{U_\alpha \to U\}$ be an étale covering of U. Then the canonical map

$$\mathcal{F}(U) \to \varprojlim_V \mathcal{F}(V)$$

is an equivalence in Mod_Λ, where V ranges over all étale U-schemes for which the map $V \to U$ factors through some U_α.

However, the converse is not true in general: condition ($*$) is generally stronger than ($*'$).

Remark 3. Let $X \in \mathrm{Sch}_k$, and let $\mathcal{A}$ denote the usual abelian category of étale sheaves of Λ-modules on X. Then the homotopy category of $\mathrm{Shv}(X;\Lambda)$ can be identified with the derived category of $\mathcal{A}$. This assertion requires that the ground field k be algebraically closed (or at least of finite cohomological dimension).

Fix a quasi-projective k-scheme X and an object $\mathcal{F} \in \mathrm{Shv}(X;\Lambda)$. The construction $U \mapsto \mathrm{H}_n(\mathcal{F}(U))$ determines a presheaf of abelian groups on the étale site of X. We will denote the sheafification of this presheaf of abelian groups by $\pi_n \mathcal{F}$. We will say that $\mathcal{F}$ is *n-connective* if $\pi_m \mathcal{F} \simeq 0$ for $n < m$, and that $\mathcal{F}$ is *n-truncated* if $\pi_m \mathcal{F} \simeq 0$ for $n > m$. We say that $\mathcal{F}$ is *discrete* if $\pi_m \mathcal{F} \simeq 0$ for $m \neq 0$. We let $\mathrm{Shv}(X;\Lambda)^{\heartsuit}$ denote the full subcategory of $\mathrm{Shv}(X;\Lambda)$ spanned by the discrete objects. This ∞-category is equivalent to an ordinary category: more precisely, the construction $\mathcal{F} \mapsto \pi_0 \mathcal{F}$ determines an equivalence

$$\mathrm{Shv}(X;\Lambda)^{\heartsuit} \to \mathcal{A}.$$

Let $f : X \to Y$ be a morphism of quasi-projective k-schemes. Then f determines a pushforward functor $f_* : \mathrm{Shv}(X;\Lambda) \to \mathrm{Shv}(Y;\Lambda)$, given by the formula $(f_* \mathcal{F})(U) = \mathcal{F}(U \times_Y X)$. This functor admits a left adjoint f^*. Given an object $\mathcal{F} \in \mathrm{Shv}(Y;\Lambda)$, we will refer to $f^* \mathcal{F}$ as the *pullback* of $\mathcal{F}$. We will sometimes denote this pullback by $\mathcal{F}|_X$, particularly if f is an immersion of schemes.

If $X = \mathrm{Spec}\, k$, then the ∞-category $\mathrm{Shv}(X;\Lambda)$ is canonically equivalent to Mod_Λ (the equivalence is given either by taking global sections, or by taking the stalk at the unique point of X). Let M be an arbitrary object of Mod_Λ, which we identify with the corresponding sheaf on $\mathrm{Spec}\, k$. If X is a quasi-projective k-scheme and $f : X \to \mathrm{Spec}\, k$ is the structure map, then we let $\underline{M}_X$ denote the pullback f^*M. We will refer to $\underline{M}_X$ as the *constant sheaf with value* M on X. We say that a sheaf $\mathcal{F}$ on X is *locally constant with value* M if there exists an étale surjection $U \to X$ such that $\mathcal{F}|_U \simeq \underline{M}_U$.

We say that a sheaf $\mathcal{F}$ on X is *constructible* if there exists a stratification of X by locally closed subschemes X_α having the following property: for each index α, there exists a perfect Λ-module M_α such that $\mathcal{F}|_{X_\alpha}$ is locally constant with value M_α. We let $\mathrm{Shv}^c(X;\Lambda)$ denote the full subcategory of $\mathrm{Shv}(X;\Lambda)$ spanned by the constructible objects.

Constructible sheaves admit the following characterization:

Proposition 4. *Let X be a quasi-projective k-scheme, and let $\mathcal{F} \in \mathrm{Shv}(X;\Lambda)$. The following conditions are equivalent:*

- (a) *The sheaf $\mathcal{F}$ is constructible.*
- (b) *The sheaf $\mathcal{F}$ is a compact object of $\mathrm{Shv}(X;\Lambda)$. That is, the construction $\mathcal{G} \mapsto \mathrm{Map}(\mathcal{F},\mathcal{G})$ commutes with filtered colimits.*

Moreover, every object of $\mathrm{Shv}(X;\Lambda)$ can be written as a filtered colimit of constructible objects of $\mathrm{Shv}(X;\Lambda)$. In other words, we have an equivalence of ∞-categories

$$\mathrm{Shv}(X;\Lambda) \simeq \mathrm{Ind}(\mathrm{Shv}^c(X;\Lambda)).$$

If $f : X \to Y$ is a map of quasi-projective k-schemes, then one can show that the pushforward functor $f_* : \mathrm{Shv}(X;\Lambda) \to \mathrm{Shv}(Y;\Lambda)$ preserves colimits (this is essentially a consequence of the fact that the étale topology is *finitary*, so that étale sheafification commutes with filtered colimits). It follows from formal nonsense that the functor f_* admits a right adjoint. In the special case where f is proper, we will denote this right adjoint by $f^! : \mathrm{Shv}(Y;\Lambda) \to \mathrm{Shv}(X;\Lambda)$.

So far, everything we have said is valid if we replace Λ by an arbitrary commutative ring. The next statements are highly nontrivial, and depend on the assumption that Λ is torsion of order invertible in k.

Proposition 5. *Let $f : X \to Y$ be a map of quasi-projective k-schemes. Then:*

- (a) *The functor f_* carries constructible sheaves to constructible sheaves.*
- (b) *If f is proper, the functor $f^!$ preserves colimits.*

(*c*) *If f is proper, the functor $f^!$ carries constructible sheaves to constructible sheaves.*

In the case where f is proper, it follows from general nonsense that assertions (a) and (b) are equivalent.

Notation 6. Let X be a proper k-scheme, and let $f : X \to \operatorname{Spec} k$ be the structure map. We define the *dualizing sheaf* of X to be the object $f^! \underline{\Lambda}_{\operatorname{Spec} k} \in \operatorname{Shv}(X; \Lambda)$. We will denote the dualizing sheaf of X by ω_X. It follows from Proposition 5 that ω_X is constructible.

Now suppose that we are given integers $d \geq d'$. Set $\Lambda = \mathbf{Z}/\ell^d \mathbf{Z}$ and $\Lambda' = \mathbf{Z}/\ell^{d'} \mathbf{Z}$, so we have a surjective ring homomorphism $\Lambda \to \Lambda'$. This ring homomorphism induces a base change functor $\operatorname{Mod}_\Lambda \to \operatorname{Mod}_{\Lambda'}$, which we will denote by $M \mapsto \Lambda' \otimes_\Lambda M$. One can show that if $\mathcal{F}$ is an object of $\operatorname{Shv}(X; \Lambda)$, then the construction

$$U \mapsto \Lambda' \otimes_\Lambda \mathcal{F}(U)$$

determines an object of $\operatorname{Shv}(X; \Lambda')$. We will denote the resulting sheaf by $\Lambda' \otimes_\Lambda \mathcal{F}$. The construction $\mathcal{F} \mapsto \Lambda' \otimes_\Lambda \mathcal{F}$ determines a functor

$$\operatorname{Shv}(X; \Lambda) \to \operatorname{Shv}(X; \Lambda').$$

Allowing d to vary, we obtain a tower of ∞-categories

$$\cdots \to \operatorname{Shv}(X; \mathbf{Z}/\ell^3 \mathbf{Z}) \to \operatorname{Shv}(X; \mathbf{Z}/\ell^2 \mathbf{Z}) \to \operatorname{Shv}(X; \mathbf{Z}/\ell \mathbf{Z})$$

One can identify the inverse limit of this tower with a full subcategory of $\operatorname{Shv}(X; \mathbf{Z}_\ell)$. An object $\mathcal{F} \in \operatorname{Shv}(X; \mathbf{Z}_\ell)$ belongs to this full subcategory if and only if it is ℓ-adically complete: in other words, if and only if the inverse limit of the tower

$$\cdots \mathcal{F} \xrightarrow{\ell} \mathcal{F} \xrightarrow{\ell} \mathcal{F}$$

vanishes.

Each of the base change functors $\mathcal{F} \mapsto \Lambda' \otimes_\Lambda \mathcal{F}$ carries constructible sheaves to constructible sheaves, so we also have a tower of "small" ∞-categories

$$\cdots \to \operatorname{Shv}^c(X; \mathbf{Z}/\ell^3 \mathbf{Z}) \to \operatorname{Shv}^c(X; \mathbf{Z}/\ell^2 \mathbf{Z}) \to \operatorname{Shv}^c(X; \mathbf{Z}/\ell \mathbf{Z}).$$

We will denote the limit of this tower by $\operatorname{Shv}^c_\ell(X)$. It can be identified with the full subcategory of $\operatorname{Shv}(X; \mathbf{Z}_\ell)$ spanned by those objects $\mathcal{F}$ which are ℓ-adically complete and have the property that

$$\mathcal{F} / \ell \, \mathcal{F} = \mathbf{Z}/\ell \mathbf{Z} \otimes_{\mathbf{Z}_\ell} \mathcal{F}$$

is constructible. We will refer to $\operatorname{Shv}^c_\ell(X)$ as the *∞-category of constructible ℓ-adic sheaves on X*.

We will need many constructions (such as direct and inverse limits) which generally take us out of the realm of constructible sheaves. It will therefore be convenient to work with the following enlargement of $\operatorname{Shv}^c_\ell(X)$:

Definition 7. Let X be a quasi-projective k-scheme. We let $\operatorname{Shv}_\ell(X)$ denote the ∞-category $\operatorname{Ind}(\operatorname{Shv}^c_\ell(X))$ of Ind-objects of $\operatorname{Shv}^c_\ell(X)$. More informally: an object $\operatorname{Shv}_\ell(X)$ is given by a filtered diagram $\{\mathcal{F}_\alpha\}$ of constructible $\mathbf{Z}_\ell$-sheaves on X, and morphisms are given by the formula

$$\operatorname{Map}(\{\mathcal{F}_\alpha\}, \{\mathcal{G}_\beta\}) = \varprojlim_\alpha \varinjlim_\beta \operatorname{Map}(\mathcal{F}_\alpha, \mathcal{G}_\beta).$$

Warning 8. The ∞-category $\operatorname{Shv}(X; \mathbf{Z}_\ell)$ admits filtered colimits, and contains $\operatorname{Shv}^c_\ell(X)$ as a full subcategory. By formal nonsense, the inclusion $\operatorname{Shv}^c_\ell(X) \hookrightarrow \operatorname{Shv}(X; \mathbf{Z}_\ell)$ admits an essentially unique extension to a functor $\operatorname{Shv}_\ell(X) \to \operatorname{Shv}(X; \mathbf{Z}_\ell)$ which preserves filtered colimits. This functor need not be fully faithful: constructible $\mathbf{Z}_\ell$-sheaves on X are generally not compact when viewed as objects of $\operatorname{Shv}(X; \mathbf{Z}_\ell)$.

Remark 9. Let $f : X \to Y$ be a map of quasi-projective k-schemes. Since the functors f^*, f_*, preserve constructible sheaves, they determine adjoint functors

$$f^* : \mathrm{Shv}^c_\ell(Y) \to \mathrm{Shv}^c_\ell(X) \qquad f_* : \mathrm{Shv}^c_\ell(Y) \to \mathrm{Shv}^c_\ell(X)$$

The pushforward functor f_* is compatible with the naive pushforward $\mathrm{Shv}(X; \mathbf{Z}_\ell) \to \mathrm{Shv}(Y; \mathbf{Z}_\ell)$, but the pullback functor f^* does not (it is necessary to first take the naive pullback, and then ℓ-adically complete). The functors f^* and f_* can be formally extended to a pair of adjoint functors

$$f^* : \mathrm{Shv}_\ell(Y) \to \mathrm{Shv}_\ell(X) \qquad f_* : \mathrm{Shv}_\ell(Y) \to \mathrm{Shv}_\ell(X)$$

which preserve filtered colimits.

If f is proper, then similar remarks apply to the functor $f^!$.

Remark 10. Let X be a quasi-projective k-scheme, and let $\mathcal{F}$ be an object of $\mathrm{Shv}_\ell(X)$. We define $\mathcal{F}[\ell^{-1}]$ to be the direct limit of the sequence

$$\mathcal{F} \xrightarrow{\ell} \mathcal{F} \xrightarrow{\ell} \mathcal{F} \xrightarrow{\ell} \cdots$$

We let $\mathrm{Shv}_{\mathbf{Q}_\ell}(X)$ to be the full subcategory of $\mathrm{Shv}_\ell(X)$ spanned by those objects $\mathcal{F}$ for which the natural map $\mathcal{F} \to \mathcal{F}[\ell^{-1}]$ is an equivalence. We will refer to $\mathrm{Shv}_{\mathbf{Q}_\ell}(X)$ as the ∞-category of $\mathbf{Q}_\ell$-sheaves on X.

Warning 11. The ∞-category $\mathrm{Shv}_{\mathbf{Q}_\ell}(X)$ is dramatically different from the naively defined ∞-category $\mathrm{Shv}(X; \mathbf{Q}_\ell)$.

References

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