The Main Calculation (Lecture 17)

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Throughout this lecture, we let k denote an algebraically closed field, ℓ a prime number which is invertible in k, and X an algebraic curve over k. Our goal is to prove the following:

Theorem 1. *Let $U \subseteq \mathbf{A}^d$ be a nonempty open subset. Then the prestack $\mathrm{Map}_{\mathrm{rat}}(X, U \subseteq \mathbf{A}^d)$ is acyclic (in other words, the map $\mathrm{Map}_{\mathrm{rat}}(X, U \subseteq \mathbf{A}^d) \to \operatorname{Spec} k$ induces an isomorphism on ℓ-adic homology.*

Before embarking on the proof of Theorem 1, let us give a rough idea of what is involved. By definition, $\mathrm{Map}_{\mathrm{rat}}(X, U \subseteq \mathbf{A}^n)$ is a full subcategory of the prestack $\mathrm{Map}_{\mathrm{rat}}(X, \mathbf{A}^n)$ of rational maps from X into $\mathbf{A}^n$. This latter prestack is the nth power of $\mathrm{Map}_{\mathrm{rat}}(X, \mathbf{A}^1)$, which can be roughly described as "rational functions on X". As such, it behaves like an infinite-dimensional affine space over $\operatorname{Spec} k$. Consequently, $\mathrm{Map}_{\mathrm{rat}}(X, U \subseteq \mathbf{A}^n)$ behaves like an open subset of an infinite-dimensional affine space, which is complementary to the subspace $\mathrm{Map}_{\mathrm{rat}}(X, \mathbf{A}^n - U)$ consisting of rational maps from X into the closed subset $\mathbf{A}^n - U \subseteq \mathbf{A}^n$. The idea is that because $\mathbf{A}^n - U$ has dimension smaller than n, the space of rational maps $\mathrm{Map}_{\mathrm{rat}}(X, \mathbf{A}^n - U)$ behaves as if it has infinite codimension in $\mathrm{Map}_{\mathrm{rat}}(X, \mathbf{A}^n)$, so that its removal does not change the homotopy type of $\mathrm{Map}_{\mathrm{rat}}(X, \mathbf{A}^n)$.

Let $\mathrm{Fin}^{\mathrm{s}}$ denote the category whose objects are nonempty finite sets and whose morphisms are surjections. The construction $(R, S, \mu, \gamma) \mapsto S$ determines a fibration of categories $\psi : \mathrm{Map}_{\mathrm{rat}}(X, U \subseteq \mathbf{A}^n) \to \mathrm{Fin}^{\mathrm{s}}$. For each nonempty finite set S, let $\mathrm{Map}_{\mathrm{rat}}(X, U \subseteq \mathbf{A}^n)_S$ denote the fiber of ψ over S. Then $\mathrm{Map}_{\mathrm{rat}}(X, U \subseteq \mathbf{A}^n)_S$ is a prestack, whose objects can be identified with triples (R, μ, γ), where R is a finitely generated k-algebra, $\mu : S \to X(R)$ is a map of sets, and $\gamma : X_R - |S| \to \mathbf{A}^n$ is a map of schemes having the property that $\gamma^{-1} U$ intersects each fiber of the projection $X_R \to \operatorname{Spec} R$. The construction $(R, S, \gamma) \mapsto (R, S)$ determines a map of prestacks $\mathrm{Map}_{\mathrm{rat}}(X, U \subseteq Y)_S \to X^S$. We will prove:

Proposition 2. *For every nonempty finite set S, the map $\mathrm{Map}_{\mathrm{rat}}(X, U \subseteq \mathbf{A}^n)_S \to X^S$ induces an isomorphism on homology.*

Assuming Proposition 2, we can deduce Theorem 1 from the calculation

$$\begin{aligned}
C_*(\mathrm{Map}_{\mathrm{rat}}(X, U \subseteq \mathbf{A}^n); \mathbf{Z}_\ell) &\simeq \varinjlim_S C_*(\mathrm{Map}_{\mathrm{rat}}(X, U \subseteq \mathbf{A}^n)_S; \mathbf{Z}_\ell) \\
&\simeq \varinjlim_S C_*(X^S; \mathbf{Z}_\ell) \\
&\simeq C_*(\mathrm{Ran}(X); \mathbf{Z}_\ell) \\
&\simeq C_*(\operatorname{Spec} k; \mathbf{Z}_\ell).
\end{aligned}$$

We now turn to the proof of Proposition 2. If Note that if $(R, \mu : S \to X(R))$ is an object of $\mathrm{Ran}(X)$, then we can identify the inverse image of (R, μ) in $\mathrm{Map}_{\mathrm{rat}}(X, \mathbf{A}^n)_S$ with the set of n-tuples

$$\gamma_1, \ldots, \gamma_n \in \Gamma(X_R - |\mu(S)|, \mathcal{O}_{X_R}|_{X_R - |\mu(S)|}) = \varinjlim_{m \geq 0} \Gamma(X_R; \mathcal{O}_{X_R}(m|\mu(S)|))$$

Here we identify $|\mu(S)|$ with the divisor in X_R given by the sum of the degree 1 divisors corresponding to the points $\{\mu(s)\}_{s \in S}$. We can therefore write $\mathrm{Map}_{\mathrm{rat}}(X, \mathbf{A}^n)_S$ as a direct limit $\varinjlim Z_m$, where an R-valued point of Z_m consists of a map $\mu : S \to X(R)$ together with an n-tuple of elements of $\Gamma(X_R; \mathcal{O}_{X_R}(m|\mu(S)|))$. More informally, Z_m is the prestack parametrizing maps $\mu : S \to X$ together with rational maps from X into $\mathbf{A}^n$ "having poles of order $\leq m$ along the divisor $|\mu(S)|$".

Note that if $m|S| > 2g-2$, where g is the genus of X, then the Riemann-Roch theorem implies that

$$\mathrm{H}^1(X; \mathcal{O}_X(m|\mu(S)|)) \simeq 0$$

for any map $\mu : S \to X(k)$. In this case, we see that Z_m can be identified with the total space of a vector bundle of rank $n(1-g+m|S|)$ over X^S; in particular, it is a smooth k-scheme of dimension $|S|+n(1-g+m|S|)$ and the projection map

$$Z_m \to X^S$$

induces an isomorphism on homology.

We let Z_m^0 denote the intersection $Z_m \cap \mathrm{Map}_{\mathrm{rat}}(X, U \subseteq \mathbf{A}^m)$. Then Z_m^0 can be identified with an open subscheme of Z_m, whose complement is the collection of rational maps (having poles of order at most m along the image of S) which factor through the closed subset $\mathbf{A}^d - U \subseteq \mathbf{A}^d$. We wish to show that the composite map

$$\varinjlim_m \mathrm{H}_*(Z_m^0; \mathbf{Z}_\ell) \xrightarrow{\alpha} \varinjlim_m \mathrm{H}_*(Z_m; \mathbf{Z}_\ell) \xrightarrow{\beta} \mathrm{H}_*(X^S; \mathbf{Z}_\ell)$$

is an isomorphism. We saw above that β is an isomorphism. Consequently, it will suffice to show that for each integer $*$, the map

$$\alpha_m : \mathrm{H}_*(Z_m^0; \mathbf{Z}_\ell) \to \mathrm{H}_*(Z_m; \mathbf{Z}_\ell)$$

is an isomorphism for $m \gg 0$.

Since Z_m and Z_m^0 are smooth k-schemes of dimension $d_m = |S| + n(1-g+m|S|)$, we have Poincare duality isomorphisms

$$\mathrm{H}_*(Z_m^0; \mathbf{Z}_\ell) \simeq \mathrm{H}_c^{2d_m-*}(Z_m^0; \mathbf{Z}_\ell) \qquad \mathrm{H}_*(Z_m; \mathbf{Z}_\ell) \simeq \mathrm{H}_c^{2d_m-*}(Z_m; \mathbf{Z}_\ell).$$

Let $Y_m = Z_m - Z_m^0$, so that α_m fits into a long exact sequence

$$\mathrm{H}_c^{2d_m-*-1}(Y_m; \mathbf{Z}_\ell) \to \mathrm{H}_c^{2d_m-*}(Z_m^0; \mathbf{Z}_\ell) \xrightarrow{\alpha_m} \mathrm{H}_c^{2d_m-*}(Z_m; \mathbf{Z}_\ell) \to \mathrm{H}_c^{2d_m-*}(Y_m; \mathbf{Z}_\ell).$$

It will therefore suffice to show that the groups $\mathrm{H}_c^{2d_m-*}(Y_m; \mathbf{Z}_\ell)$ and $\mathrm{H}_c^{2d_m-*-1}(Y_m; \mathbf{Z}_\ell)$ vanish for $m \gg 0$. Since these cohomology groups are concentrated in degrees $\leq 2\dim(Y_m)$, we are reduced to proving the following:

Proposition 3. *Fix an integer $*$. Then $2\dim(Y_m) < 2d_m - * - 1$ for $m \gg 0$.*

Proof. Using Noether normalization, we can choose a linear projection map $\pi : \mathbf{A}^n \to \mathbf{A}^{n-1}$ whose restriction to $\mathbf{A}^n - U$ is finite. Then composition with π induces a map $Y_m \to \mathrm{Map}_{\mathrm{rat}}(X, \mathbf{A}^{n-1})$ with finite fibers, whose image is contained in the locus $Y'_m \subseteq \mathrm{Map}_{\mathrm{rat}}(X, \mathbf{A}^{n-1})$ parametrizing maps which have poles of order at most m along the image of S. Arguing as above, we see that for $m \gg 0$, Y'_m is a smooth S-scheme of dimension $|S| + (n-1)(1-g+m|S|)$. We therefore have

$$2(d_m - \dim(Y_m)) \geq 2(d_m - \dim(Y'_m)) = 2(1-g+m|S|),$$

which grows arbitrarily large as $m \to \infty$. □

References

[1] Gaitsgory, D. *Contracibility of the space of rational maps.*