Spaces of Rational Maps (Lecture 16)

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Throughout this lecture, we let k denote an algebraically closed field, ℓ a prime number which is invertible in k, and X an algebraic curve over k. Our goal is to prove the following:

Theorem 1. *Let G be a smooth affine group scheme over X whose generic fiber is semisimple and simply connected, let R be a finitely generated k-algebra and let $\mathcal{P}$ be a G-bundle on X_R. Then the projection map $\mathrm{Sect}(\mathcal{P}) \to \mathrm{Spec}\, R$ induces an isomorphism on ℓ-adic homology.*

Recall that if $\mathcal{P}$ admits a generic trivialization, then the homology of $\mathrm{Sect}(\mathcal{P})$ is the same as the homology of $\mathrm{Sect}(\mathcal{P}_{\mathrm{triv}})$, where $\mathcal{P}_{\mathrm{triv}}$ denotes the trivial G-bundle on X_R. Over the last several lectures, we proved that any G-bundle $\mathcal{P}$ admits a generic trivialization after passing to some fppf covering of $\mathrm{Spec}\, R$. It will therefore suffice to prove Theorem 1 in the special case where $\mathcal{P} = \mathcal{P}_{\mathrm{triv}}$ is a trivial G-bundle. In this case, $\mathcal{P}$ is the pullback of a G-bundle defined on the curve X itself. We may therefore reduce to the special case where $R = k$.

Let us now assume for simplicity that the group scheme G is generically split. In this case, we can choose a reductive algebraic group G' over k and a finite subset $S \subseteq X(k)$ such that $G \times_X (X - S)$ and $G' \times (X - S)$ are isomorphic (as group schemes over $(X - D)$. Let $\mathcal{P}$ and $\mathcal{P}'$ denote the trivial G and G'-bundles over X, respectively. Then we have an equivalence of prestacks

$$\mathrm{Sect}^u_{\supseteq S}(\mathcal{P}) \simeq \mathrm{Sect}^u_{\supseteq S}(\mathcal{P}').$$

Arguing as in Lecture 12, we obtain isomorphisms

$$\begin{aligned}
\mathrm{H}_*(\mathrm{Sect}(\mathcal{P}); \mathbf{Z}_\ell) &\simeq \mathrm{H}_*(\mathrm{Sect}^u(\mathcal{P}); \mathbf{Z}_\ell) \\
&\simeq \mathrm{H}_*(\mathrm{Sect}^u_{\supseteq S}(\mathcal{P}); \mathbf{Z}_\ell) \\
&\simeq \mathrm{H}_*(\mathrm{Sect}^u_{\supseteq S}(\mathcal{P}'); \mathbf{Z}_\ell) \\
&\simeq \mathrm{H}_*(\mathrm{Sect}^u(\mathcal{P}'); \mathbf{Z}_\ell) \\
&\simeq \mathrm{H}_*(\mathrm{Sect}(\mathcal{P}'); \mathbf{Z}_\ell).
\end{aligned}$$

We may therefore replace $(G, \mathcal{P})$ by $(G' \times X, \mathcal{P}')$. Changing notation, we are reduced to proving the following:

Theorem 2. *Let G be a simply connected semisimple algebraic group over k and let $\mathcal{P}$ denote the trivial G-bundle on X. Then $\mathrm{Sect}(\mathcal{P})$ is acyclic: that is, the projection map $\mathrm{Sect}(\mathcal{P}) \to \mathrm{Spec}\, k$ induces an isomorphism on ℓ-adic homology.*

In the setting of Theorem 2, we can think of $\mathrm{Sect}(\mathcal{P})$ as parametrizing maps rational from X into G. In the proof, it will be useful to consider, more generally, rational maps from X into other quasi-projective k-schemes.

Definition 3. Let Y be a quasi-projective k-scheme. We define a category $\mathrm{Map}^+_{\mathrm{rat}}(X, Y)$ as follows:

- The objects of $\mathrm{Map}^+_{\mathrm{rat}}(X, Y)$ are triples (R, S, γ), where R is a finitely generated k-algebra, S is a finite subset of $X(R)$, and $\gamma : X_R - |S| \to Y$ is a map of k-schemes.

- A morphism from (R,S,γ) to (R',S',γ') consists of a k-algebra homomorphism $\phi : R \to R'$ carrying S to a subset of S', for which the γ' is given by the composition

$$X_{R'} - |S'| \to X_R - |S| \xrightarrow{\gamma} Y.$$

The construction $(R,S,\gamma) \mapsto R$ determines a forgetful functor $\mathrm{Map}^+_{\mathrm{rat}}(X,Y) \to \mathrm{Ring}_k$, which exhibits $\mathrm{Map}^+_{\mathrm{rat}}(X,Y)$ as a prestack.

The construction $(R,S,\gamma) \mapsto (R,S)$ determines a map of prestacks $\mathrm{Map}^+_{\mathrm{rat}}(X,Y)$. We let $\mathrm{Map}_{\mathrm{rat}}(X,Y)$ and $\mathrm{Map}^u_{\mathrm{rat}}(X,Y)$ denote the fiber products

$$\mathrm{Map}^+_{\mathrm{rat}}(X,Y) \times_{\mathrm{Ran}^+(X)} \mathrm{Ran}(X) \qquad \mathrm{Map}^+_{\mathrm{rat}}(X,Y) \times_{\mathrm{Ran}^+(X)} \mathrm{Ran}^u(X).$$

Note that when $Y = G$, the prestack $\mathrm{Map}_{\mathrm{rat}}(X,Y)$ can be identified with $\mathrm{Sect}(\mathcal{P})$ where $\mathcal{P}$ is the trivial G-bundle on X.

Variant 4. Let $U \subseteq Y$ be an open subset. We let $\mathrm{Map}^+_{\mathrm{rat}}(X, U \subseteq Y)$ denote the full subcategory of $\mathrm{Map}^+_{\mathrm{rat}}(X,Y)$ spanned by those objects (R,S,γ) for which the open set $\gamma^{-1}(U) \subseteq X_R$ is full. We let $\mathrm{Map}_{\mathrm{rat}}(X, U \subseteq Y)$ and $\mathrm{Map}^u_{\mathrm{rat}}(X, U \subseteq Y)$ denote the inverse images of $\mathrm{Map}^+_{\mathrm{rat}}(X, U \subseteq Y)$ in $\mathrm{Map}_{\mathrm{rat}}(X,Y)$ and $\mathrm{Map}^u_{\mathrm{rat}}(X,Y)$, respectively.

Exercise 5. In the situation of Variant 4, the projection maps $\mathrm{Map}_{\mathrm{rat}}(X, U \subseteq Y) \to \mathrm{Map}^u_{\mathrm{rat}}(X, U \subseteq Y) \to \mathrm{Map}^+_{\mathrm{rat}}(X, U \subseteq Y)$ are a universal homology equivalences. Consequently, the prestacks $\mathrm{Map}_{\mathrm{rat}}(X, U \subseteq Y)$, $\mathrm{Map}^u_{\mathrm{rat}}(X, U \subseteq Y)$ and $\mathrm{Map}^+_{\mathrm{rat}}(X, U \subseteq Y)$ are interchangeable for purposes of computing homology.

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Proposition 6. *Let Y be a quasi-projective k-scheme and let $U \subseteq Y$ be an open set. Then the inclusion map*

$$\mathrm{Map}^+_{\mathrm{rat}}(X,U) \hookrightarrow \mathrm{Map}^+_{\mathrm{rat}}(X, U \subseteq Y)$$

is a universal homology equivalence.

Proof. Fix an object (R,S,γ) of $\mathrm{Map}^+_{\mathrm{rat}}(X, U \subseteq Y)$, and set

$$\mathcal{C} = \mathrm{Map}^+_{\mathrm{rat}}(X,U) \times_{\mathrm{Map}^+_{\mathrm{rat}}(X,U \subseteq Y)} \mathrm{Map}^+_{\mathrm{rat}}(X, U \subseteq Y)_{(R,S,\gamma)/}.$$

We wish to show that the projection map $\mathcal{C} \to \operatorname{Spec} R$ induces an isomorphism on homology. Let $K = X_R - \gamma^{-1}(U)$. Unwinding the definitions, we can identify $\mathcal{C}$ with the full subcategory of $\mathrm{Ran}^+(X) \times_{\operatorname{Spec} k} \operatorname{Spec} R$ spanned by those pairs (A, S'), where A is a finitely generated R-algebra and $S' \subseteq X(A)$ is a finite subset which contains the image of S and has the property that $|S'| \subseteq X_A$ contains the inverse image of K.

The assertion that the map $\mathcal{C} \to \operatorname{Spec} R$ induces an isomorphism on homology can be tested locally on $\operatorname{Spec} R$ (with respect to the fppf topology). We may therefore suppose that there exists a finite subset $T \subseteq X(R)$ containing S such that $K \subseteq |T|$. For each finitely generated R-algebra A, let T_A denote the image of T in $X(A)$. Let $\alpha : \mathcal{C} \hookrightarrow \mathrm{Ran}^+(X) \times_{\operatorname{Spec} k} \operatorname{Spec} R$ denote the inclusion map, and let $\beta : \mathrm{Ran}^+(X) \times_{\operatorname{Spec} k} \operatorname{Spec} R \to \mathcal{C}$ denote the morphism of prestacks given by $(A, S') \mapsto (A, S' \cup T_A)$. Then there exist natural transformations (in the 2-category of prestacks)

$$\mathrm{id} \to \alpha \circ \beta \qquad \mathrm{id} \to \beta \circ \alpha,$$

so that α and β induce (mutually inverse) isomorphisms on homology. We are therefore reduced to proving that the projection map $\mathrm{Ran}^+(X) \times_{\operatorname{Spec} k} \operatorname{Spec} R \to \operatorname{Spec} R$ induces an isomorphism on homology. This follows from the Künneth formula, since $\mathrm{Ran}^+(X)$ is acyclic. □

Let Y be a quasi-projective k-scheme and let $U \subseteq Y$ be an open subset. Then $\mathrm{Map}^u_{\mathrm{rat}}(X, U \subseteq Y)$ is a prestack in sets: it corresponds to the functor $F_{U,Y} : \mathrm{Ring}_k \to \mathcal{S}\mathrm{et}$ which assigns to each finitely generated k-algebra R the set of pairs (S, γ) where $S \subseteq X(R)$ is a nonempty finite set and $\gamma : X_R - |S| \to Y$ is a map of k-schemes such that $\gamma^{-1}(U)$ is full.

Suppose that we are given a pair of open sets $U, V \subseteq Y$. We then have a commutative diagram of inclusions of set-valued functors

$$\begin{array}{ccc} F_{U\cap V,Y} & \longrightarrow & F_{U,Y} \\ \downarrow & & \downarrow \\ F_{V,Y} & \longrightarrow & F_{U\cup V,Y}. \end{array}$$

This diagram is a pullback square, but is not quite a pushout square: given a subset $S \subseteq X(R)$ and a map $\gamma : X_R - |S| \to Y$ such that $\gamma^{-1}(U \cup V)$ is full, we cannot conclude that either $\gamma^{-1}(U)$ or $\gamma^{-1}(V)$ is full. However, the images of $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$ comprise an open covering of $\operatorname{Spec} R$, so that the inclusion map

$$F_{U,Y} \amalg_{F_{U\cap V,Y}} F_{V,Y} \hookrightarrow F_{U\cup V,Y}$$

becomes an isomorphism after sheafification with respect to the Zariski topology. It follows that the associated diagram

$$\begin{array}{ccc} C_*(\mathrm{Map}^u_{\mathrm{rat}}(X, U\cap V \subseteq Y) & \longrightarrow & \mathrm{Map}^u_{\mathrm{rat}}(X, U \subseteq Y) \\ \downarrow & & \downarrow \\ \mathrm{Map}^u_{\mathrm{rat}}(X, V \subseteq Y) & \longrightarrow & \mathrm{Map}^u_{\mathrm{rat}}(X, U\cup V \subseteq Y) \end{array}$$

is a homotopy pushout diagram of chain complexes. This proves the following:

Proposition 7. *Let Y be a quasi-projective k-scheme and let $U, V \subseteq Y$ be open sets. Suppose that the prestacks $\mathrm{Map}^u_{\mathrm{rat}}(X, U \cap V \subseteq Y)$, $\mathrm{Map}^u_{\mathrm{rat}}(X, U \subseteq Y)$, and $\mathrm{Map}^u_{\mathrm{rat}}(X, V \subseteq Y)$ are acyclic. Then $\mathrm{Map}^u_{\mathrm{rat}}(X, U \cup V \subseteq Y)$ is also acyclic.*

Now let G be a reductive algebraic group over k. Choose a Borel subgroup $B \subseteq G$ and an opposite Borel subgroup $B' \subseteq G$, so that $B \cap B' = T$ is a maximal torus of G. Let $U \subseteq B$ and $U' \subseteq B'$ be the unipotent radicals of B and B', respectively. Then the Bruhat decomposition supplies an open immersion

$$U \times T \times U' \hookrightarrow G$$

whose image is a dense open subset $V \subseteq G$. Since G is quasi-compact, we can write

$$G = \bigcup_{1 \le i \le n} g_i V$$

for some finite collection of k-valued points $g_1, \ldots, g_n \in G(k)$. We wish to show that $\mathrm{Map}^u_{\mathrm{rat}}(X, G)$ is acyclic. Applying Proposition 7 repeatedly, we see that it will suffice to show that $\mathrm{Map}^u_{\mathrm{rat}}(X, \bigcap_{i\in I} g_i V \subseteq G)$ is acyclic for each nonempty subset $I \subseteq \{1, 2, \ldots, n\}$. Note that $V_I = \bigcap_{i\in I} g_i V$ is isomorphic as a k-scheme to an open subset of V, so that we can choose an open embedding $V_I \hookrightarrow \mathbf{A}^d$ where $d = \dim(G)$. Using Proposition 6, we see that the inclusion maps

$$\mathrm{Map}^u_{\mathrm{rat}}(X, V_I \subseteq G) \hookleftarrow \mathrm{Map}^u_{\mathrm{rat}}(X, V_I) \hookrightarrow \mathrm{Map}^u_{\mathrm{rat}}(X, V_I \subseteq \mathbf{A}^d)$$

induce isomorphisms on homology. We may therefore deduce Theorem 1 immediately from the following:

Theorem 8. *Let $U \subseteq \mathbf{A}^d$ be a nonempty open subset. Then the prestack $\mathrm{Map}_{\mathrm{rat}}(X, U \subseteq \mathbf{A}^d)$ is acyclic.*

We will prove Theorem 8 in the next lecture.

References

[1] Gaitsgory, D. *Contracibility of the space of rational maps.*