Existence of Generic Trivializations (Lecture 13)

March 5, 2014

Let k be an algebraically closed field, X an algebraic curve over k, and G a smooth affine group scheme over X whose generic fiber is semisimple and simply connected. Recall that our goal is to prove (under some hypotheses on the generic fiber of G) that the map

$$\mathrm{Ran}_G(X) \to \mathrm{Bun}_G(X)$$

is a universal homology equivalence. For this, it will suffice to show that the following assertion holds for every $R \in \mathrm{Ring}_k$ and every G-bundle $\mathcal{P}$ on X_R:

($*$) The projection map $\mathrm{Sect}(\mathcal{P}) = \mathrm{Spec}\, R \times_{\mathrm{Bun}_G(X)} \mathrm{Ran}_G(X) \to \mathrm{Spec}\, R$ is a universal homology equivalence.

We begin with two observations:

(a) If $\mathcal{P}$ and $\mathcal{P}'$ are G-bundles on X_R which are isomorphic over an open set $X_R - |S|$ for some finite subset $S \subseteq X(R)$, then $\mathcal{P}$ satisfies ($*$) if and only if $\mathcal{P}'$ satisfies ($*$). This is the upshot of the previous lecture.

(b) The assertion that $\mathcal{P}$ satisfies ($*$) can be tested locally with respect to the fppf topology on $\mathrm{Spec}\, R$ (this follows from elementary formal properties of ℓ-adic cohomology).

To make use of (a), it is convenient to introduce the following:

Definition 1. Let $R \in \mathrm{Ring}_k$ and let $U \subseteq X_R$ be an open set. We will say that U is *full* if the projection map $U \to \mathrm{Spec}\, R$ is surjective (that is, U intersects each fiber of the map $X_R \to \mathrm{Spec}\, R$).

We will say that a G-bundle on $\mathcal{P}_R$ is *generically trivial* if it is trivial over some full open set $U \subseteq X_R$.

To deduce ($*$) from (a) and (b), it will suffice to prove the following:

(i) For every G-bundle $\mathcal{P}$ on X_R, there is a faithfully flat ring étale map $\mathrm{Spec}\, R' \to \mathrm{Spec}\, R$ such that $\mathcal{P} \times_{X_R} X_{R'}$ is generically trivial.

(ii) Let $\mathcal{P}$ be a generically trivial G-bundle on X_R. Then there exists a faithfully flat map $\mathrm{Spec}\, R' \to \mathrm{Spec}\, R$ and a finite subset $S \subseteq X(R')$ such that $\mathcal{P}$ is trivial when restricted to $X_{R'} - |S|$.

(iii) Every trivial G-bundle $\mathcal{P}$ satisfies ($*$).

In this lecture, we will prove (ii) and begin the proof of (i). Assertion (ii) is an immediate consequence of the following:

Proposition 2. *Let R be a finitely generated k-algebra and let $U \subseteq X_R$ be a full open subset. Then there exists a faithfully flat map $R \to R'$ and a finite subset $S \subseteq X(R')$ such that $X_{R'} - |S|$ is contained in the inverse image $U \times_{X_R} X_{R'}$.*

Proof. Choose a k-valued point y of $\operatorname{Spec} R$. Since U is full, we can choose a k-valued point (x, y) of $U \subseteq X \times_{\operatorname{Spec} k} \operatorname{Spec} R$ lying over y. Let $U(x)$ denote the intersection $(\{x\} \times_{\operatorname{Spec} k} \operatorname{Spec} R) \cap U$. Then the projection map $U(x) \to \operatorname{Spec} R$ is an open immersion whose image contains the point y. Since the desired assertion is Zariski local on $\operatorname{Spec} R$, we may assume that without loss of generality that $U(x) \to \operatorname{Spec} R$ is surjective: that is, that U contains the product $\{x\} \times \operatorname{Spec} R$.

Choose a closed subscheme $K \subseteq X_R$ whose underlying topological space is the complement of U (for example, we could endow K with the reduced structure). Let $\mathcal{I}_K \subseteq \mathcal{O}_{X_R}$ denote the ideal sheaf of K. For each integer $n \geq 0$, let $\mathcal{I}_K(nx)$ denote the tensor product of $\mathcal{I}_K$ with the pullback of the ample invertible sheaf $\mathcal{O}_X(nx)$ on X. For $n \gg 0$, the sheaf $\mathcal{I}_K(nx)$ is generated by global sections. Choose a point $x' \in X(k)$ such that (x', y) is contained in U and a global section f of $\mathcal{I}_K(nx)$ which does not vanish at (x', y). Without loss of generality (passing to a Zariski open neighborhood of y if necessary) we may assume that f does not vanish identically on any fiber of the projection map $X_R \to \operatorname{Spec} R$. Let us identify f with a section of the line bundle $\mathcal{O}_{X_R}(nx)$, so that the vanishing locus of f is a closed subscheme $D \subseteq X_R$. By construction, this closed subscheme contains K and the projection map $D \to \operatorname{Spec} R$ is finite and flat of degree n. Replacing U by $X_R - D$, we are reduced to proving Proposition 2 under the following additional assumption:

($\star$) There exists a closed subscheme $D \subseteq X_R$ such that $D \to \operatorname{Spec} R$ is finite flat of degree $n \geq 0$, and $U = X - D$.

We now proceed by induction on n. If $n = 0$, then $U = X_R$ and there is nothing to prove. Otherwise, the map $D \to \operatorname{Spec} R$ is finite flat. Replacing $\operatorname{Spec} R$ by D, we can reduce to the case where the map $D \to \operatorname{Spec} R$ admits a section s. Let $D_0 \subseteq X_R$ be the image of the section s. Then D_0 is contained in D, so we can write D as a divisorial sum $D_0 + D_1$ where $D_1 \subseteq X_R$ has degree $(n-1)$ over R. Using the inductive hypothesis, we can assume that there exists a finite set $S_0 \subseteq X(R)$ such that $X_R - |S_0| \subseteq X - D_1$. We now complete the proof by taking $S = S_0 \cup \{s\}$. □

We now turn to the proof of (i). In the case where the group scheme G is split, a strong version of this result was proven by Drinfeld and Simpson in [**?**]. Let us briefly describe the strategy of proof. Fix a closed point $u \in \operatorname{Spec} R$, and let $\mathcal{P}_u$ denote the fiber of $\mathcal{P}$ at u (so that $\mathcal{P}_u$ is a G-bundle on X). Let $\mathcal{K}_X$ denote the fraction field of X. Since $\mathcal{K}_X$ has dimension 1 (see [**?**]), any G-bundle on $\operatorname{Spec} \mathcal{K}_X$ is automatically trivial. It follows that $\mathcal{P}_u$ is trivial at the generic point of X, and therefore admits a trivialization η_u on a dense open subset $U \subseteq X$.

One would now like to show that it is possible to "deform" the trivialization η_u to obtain trivializations of $\mathcal{P}_v$ for points $v \in \operatorname{Spec} R$ which are sufficiently "near" to u. However, there is an obstacle: the trivialization η_u is defined only on a dense open subset $U \subseteq X$. The collection of trivializations is therefore a torsor for the group $G(U)$, which is an unwieldy infinite-dimensional object which is ill-suited for study by standard tools of deformation theory. On the other hand, there is no guarantee that η_u can be extended to a trivialization of $\mathcal{P}_u$ over the entire curve X (since G-bundles on X can certainly be globally nontrivial).

To circumvent this difficulty, Drinfeld and Simpson first sought after a weaker structure on $\mathcal{P}_u$: namely, a reduction of structure group from G to a Borel subgroup $B \subseteq G$. Any trivialization of $\mathcal{P}_u$ over a dense open subset $U \subseteq X$ determines a B-structure on $\mathcal{P}_u|_U$, which can then be extended to a B-structure on the entire torsor $\mathcal{P}_u$ using the valuative criterion for properness (since the quotient G/B is proper, at least in the case where G has good reduction everywhere). One might then hope to use deformation theory to show that a B-structure on $\mathcal{P}_u$ can be extended to a B-structure on $\mathcal{P}$ in a neighborhood of u (at least if the original B-structure is well-chosen). On the other hand, B-structures are easy to analyze, since B is a solvable algebraic group.

Our proof will follow the same basic strategy. However, we must be careful about the meaning of "Borel subgroup" if the group scheme G is not assumed to be split.

Notation 3. Let G_0 denote the generic fiber of G. Then G_0 is a reductive algebraic group over the fraction field K_X, which has dimension 1. It follows that G_0 is quasi-split: that is, we can choose a Borel subgroup $B_0 \subseteq G_0$ which is defined over K_X. We let B denote the scheme-theoretic closure of B_0 in G.

Warning 4. The scheme B is flat over X and is closed under multiplication and inversion in G: in particular, it can be regarded as a flat affine group scheme over X. However, there is no reason to expect that B should be smooth over X, or that the fibers of B should be connected.

Exercise 5. Let $x \in X$ be a closed point for which the fiber G_x is reductive. Show that B_x is a Borel subgroup of G_x.

Now suppose we are given a finitely generated k-algebra R and a G-bundle $\mathcal{P}$ over X_R. By definition, a *B-reduction* of $\mathcal{P}$ is a pair $(\mathcal{Q}, \alpha)$, where $\mathcal{Q}$ is a B-bundle on the relative curve X_R, and α is an isomorphism of $\mathcal{P}$ with the induced G-bundle $(\mathcal{Q} \times_X G)/B$.

Warning 6. If B is not smooth over X, we should take care in specifying what we mean by a B-bundle. In this context, we mean an X_R-scheme $\mathcal{Q}$ equpped with an action of B for which there exists an fppf surjection $U \to X_R$ and a B-equivariant isomorphism $U \times_{X_R} \mathcal{Q} \simeq U \times_X B$ (in general, there is no reason to expect that such a trivialization can be found étale-locally).

The proof of (i) can be broken into two steps:

- (i') For every G-bundle $\mathcal{P}$ on X_R, there exists an étale surjection $\operatorname{Spec} R' \to \operatorname{Spec} R$ such that $\mathcal{P} \times_{X_R} X_{R'}$ admits a B-reduction.
- (i'') Every B-bundle on X_R is generically trivial.

We will prove (i'') in this lecture, and postpone (i') until the next.

Let $\operatorname{rad}_u(B_0)$ denote the unipotent radical of B_0 and let $T_0 = B_0/\operatorname{rad}_u(B_0)$ denote the quotient torus. Then the group scheme B_0 fits into an exact sequence

$$0 \to \operatorname{rad}_u(B_0) \to B_0 \to T_0 \to 0.$$

Fix an algebraic closure $\overline{K}_X$ of K_X and let Λ denote collection of all maps from T_0 into the multiplicative group $\mathbf{G}_m$ which are defined over $\overline{K}_X$. Then Λ is a finite free abelian group with an action of $\operatorname{Gal}(\overline{K}_X/K_X)$. Since G_0 is simply connected, the lattice Λ has a canonical basis given by the simple weights of G_0 (which are in bijection with the vertices of the Dynkin diagram of G_0); this basis is stable under the action of $\operatorname{Gal}(\overline{K}_X/K_X)$. Let $\mathcal{I}$ denote the set of orbits for $\operatorname{Gal}(\overline{K}_X/K_X)$. For each $I \in \mathcal{I}$, let $\Gamma_I \subseteq \operatorname{Gal}(\overline{K}_X/K_X)$ be the (open) stabilizer of some element of I and let $K_I \subseteq \overline{K}_X$ be the fixed field of Γ_i, so that K_I is a finite separable extension of K_X of degree equal to the cardinality of I. Unwinding the definition, we see that the torus T_0 is given by

$$\operatorname{Hom}(\Lambda, \mathbf{G}_m) = \operatorname{Hom}(\bigoplus_{I \in \mathcal{I}} \mathbf{Z}[I], \mathbf{G}_m) = \prod_{I \in \mathcal{I}} \operatorname{Res}^{K_I}_{K_X} \mathbf{G}_m.$$

Here $\operatorname{Res}^{K_I}_{K_X} \mathbf{G}_m$ denotes the Weil restriction of the multiplicative group $\mathbf{G}_m$ along the map $\operatorname{Spec} K_I \to \operatorname{Spec} K_X$.

For each $I \in \mathcal{I}$, we can identify K_I with the fraction field of an algebraic curve X_I equipped with a generically étale map $\pi_I : X_I \to X$. By a direct limit argument, we can choose a dense open set $U \subseteq X$ with the following properties:

(1) Each of the maps $U_I = X_I \times_X U \to U$ is étale.

(2) The group scheme $B_U = B \times_X U$ fits into an exact sequence

$$0 \to N \to B_U \to T_U \to 0$$

where $T_U \simeq \prod_{I \in \mathcal{I}} \operatorname{Res}^{U_I}_U \mathbf{G}_m$.

(3) The group scheme N admits a finite filtration by copies of the additive group $\mathbf{G}_a$.

Let $\mathcal{Q}$ be a B-bundle on X_R. Then $\mathcal{Q}$ determines a T_U-bundle over the open set $U_R \subseteq X_R$, which we can identify with a finite collection of line bundles $\mathcal{L}_I$ over the schemes $U_{IR} = U_I \times_{\operatorname{Spec} k} \operatorname{Spec} R$. Each of these line bundles can be trivialized locally with respect to the Zariski topology. Working Zariski-locally on $\operatorname{Spec} R$, we may assume that each $\mathcal{L}_I$ is trivial over a full open set $V_I \subseteq U_{IR}$. Then $W = U_R - \bigcup_{I \in \mathcal{I}} \operatorname{Im}(U_{IR} - V_I \rightarrow U_R)$ is a full open subset of X_R. Working locally on $\operatorname{Spec} R$, we may assume that W contains a full affine open subset $W_0 \subseteq W$. By construction, the restriction $\mathcal{Q}|_{W_0}$ determines a *trivial* T_U-bundle on W_0. It follows that the structure group of $\mathcal{Q}|_{W_0}$ can be reduced to the the subgroup $N \subseteq B_U$. Using assertion (3) and our assumption that W_0 is affine, we see that every N-bundle on W_0 is trivial. It follows that $\mathcal{Q}|_{W_0}$ is trivial, so that $\mathcal{Q}$ is generically trivial as desired.