First Steps (Lecture 12)

March 1, 2014

Throughout this lecture, we let k denote an algebraically closed field and ℓ a prime number which is invertible in k. Let X be an algebraic curve over k, G a smooth affine group scheme over X. Our goal over the next several lectures is to prove that, if the generic fiber of G is semisimple and simply connected, then the canonical map

$$\mathrm{Ran}_G(X) \to \mathrm{Bun}_G(X)$$

is a universal homology equivalence.

Since $\mathrm{Bun}_G(X)$ is a prestack in groupoids, this reduces to a local assertion: we wish to show that for every map $\phi : \operatorname{Spec} R \to \mathrm{Bun}_G(X)$, the projection

$$\operatorname{Spec} R \times_{\mathrm{Bun}_G(X)} \mathrm{Ran}_G(X) \to \operatorname{Spec} R$$

induces an isomorphism on homology. If the map ϕ corresponds to a G-bundle $\mathcal{P}$ over X_R, we will denote the fiber product $\operatorname{Spec} R \times_{\mathrm{Bun}_G(X)} \mathrm{Ran}_G(X)$ by $\mathrm{Sect}(\mathcal{P})$. Roughly speaking, we can think of $\mathrm{Sect}(\mathcal{P})$ which parametrizes trivializations of $\mathcal{P}$ that are defined on a dense open subset of X. However, this description is slightly misleading: we can identify objects of $\mathrm{Sect}(\mathcal{P})$ with quadruples (A, S, μ, γ) where $A \in \mathrm{Ring}_k$, S is a nonempty finite set, $\mu : S \to X(A)$ is a map, and γ is a trivialization of $\mathcal{P}$ over $X_A - |\mu(S)|$. Here the finite set S is an essential datum: if we replace S by a larger set S' equipped with a map $\mu' : S' \to X(A)$, then we can of course restrict γ to a trivialization γ' of $\mathcal{P}$ over the smaller open set $X_A - |\mu'(S')|$, but there is no relationship the objects (A, S, μ, γ) is not related to (A, S', μ', γ') in the category $\mathrm{Sect}(\mathcal{P})$. To remedy this, it will be convenient to introduce another version of the Ran space $\mathrm{Ran}(X)$.

Definition 1. Let G be smooth affine group scheme over X. We define a category $\mathrm{Ran}^+_G(X)$ as follows:

- The objects of $\mathrm{Ran}^+_G(X)$ are quadruples $(R, \mathcal{P}, S, \gamma)$ where R is a finitely generated k-algebra, $\mathcal{P}$ is a G-bundle on the relative curve $X_R = \operatorname{Spec} R \times_{\operatorname{Spec} k} X$, S is a finite subset of $X(R)$, and γ is a trivialization of $\mathcal{P}$ on the open set $X_R - |S|$ determined by S.
- A morphism from $(R, \mathcal{P}, S, \gamma)$ to $(R', \mathcal{P}', S', \gamma')$ in $\mathrm{Ran}^+_G(X)$ consists of a k-algebra homomorphism $\phi : R \to R'$ which carries $S \subseteq X(R)$ into $S' \subseteq X(R)$, and an isomorphism of G-bundles $X_R \times_{X_R} \mathcal{P} \simeq \mathcal{P}'$ which carries γ to γ'.

The construction $(R, \mathcal{P}, S, \gamma) \mapsto R$ determines an op-fibration $\mathrm{Ran}^+_G(X) \to \mathrm{Ring}_k$, so that we can regard $\mathrm{Ran}^+_G(X)$ as a prestack.

Notation 2. In the special case where the group G is trivial, we will denote $\mathrm{Ran}^+_G(X)$ simply by $\mathrm{Ran}^+(X)$. We will identify objects of $\mathrm{Ran}^+(X)$ with pairs (R, S) where R is a finitely generated k-algebra and S is a finite subset of $X(R)$. We can regard $\mathrm{Ran}^+(X)$ as a (non-full) subcategory of the category $\mathrm{Ran}^u(X)$ introduced in Lecture 10. They differ in two respects:

- The category $\mathrm{Ran}^+(X)$ has more objects, because we allow pairs (R, S) where the set S is empty.
- The category $\mathrm{Ran}^+(X)$ has more morphisms. For every finitely generated k-algebra R, the fiber $\mathrm{Ran}^+(X) \times_{\mathrm{Ring}_k} \{R\}$ is the category of all finite subsets of $X(R)$, with morphisms given by inclusions (in the category $\mathrm{Ran}^u(X) \times_{\mathrm{Ring}_k} \{R\}$, there are no non-identity morphisms).

Notation 3. For any smooth affine group scheme G over X, there is an evident forgetful functor $\mathrm{Ran}^+_G(X) \to \mathrm{Ran}^+(X)$, given on objects by $(R, \mathcal{P}, S, \gamma) \mapsto (R, S)$. We define $\mathrm{Ran}^u_G(X)$ to be the fiber product

$$\mathrm{Ran}^u(X) \times_{\mathrm{Ran}^+(X)} \mathrm{Ran}^+_G(X).$$

In other words, we define $\mathrm{Ran}^u_G(X)$ to be the subcategory of $\mathrm{Ran}^+_G(X)$ where we consider only those objects $(R, \mathcal{P}, S, \gamma)$ where S is nonempty and only those morphisms $(R, \mathcal{P}, S, \gamma) \to (R', \mathcal{P}', S', \gamma')$ where the map $S \to S'$ is surjective.

Note that the fiber product $\mathrm{Ran}(X) \times_{\mathrm{Ran}^+(X)} \mathrm{Ran}^+_G(X)$ can be identified with the prestack $\mathrm{Ran}_G(X)$ introduced in a previous lecture. We therefore have a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Ran}_G(X) & \xrightarrow{\phi} & \mathrm{Ran}^u_G(X) & \xrightarrow{\psi} & \mathrm{Ran}^+_G(X) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Ran}(X) & \longrightarrow & \mathrm{Ran}^u(X) & \longrightarrow & \mathrm{Ran}^+(X).
\end{array}$$

Proposition 4. *The maps ϕ and ψ are universal homology equivalences.*

Remark 5. When G is empty, Proposition 4 asserts that the maps

$$\mathrm{Ran}(X) \to \mathrm{Ran}^u(X) \to \mathrm{Ran}^+(X)$$

are universal homology equivalences: in particular, they induce isomorphisms on homology. Note that the prestack $\mathrm{Ran}^+(X)$ has an initial object, given by the pair $(k, \emptyset)$. It follows that the cochain complex $C_*(\mathrm{Ran}^+(X); \mathbf{Z}_\ell) = \varinjlim_{(R,S) \in \mathrm{Ran}^+(X)^{\mathrm{op}}} C_*(\mathrm{Spec}\, R; \mathbf{Z}_\ell)$ is equivalent to $C_*(\mathrm{Spec}\, k; \mathbf{Z}_\ell)$. Consequently, Proposition 4 implies the acyclicity of $\mathrm{Ran}(X)$, which we proved in Lecture 10. Our actual logic will proceed in reverse: we will use the acyclicity of $\mathrm{Ran}^u(X)$ to deduce Proposition 4.

Proof of Proposition 4. The proof that ϕ is a universal homology equivalence is similar to our proof that the forgetful functor $\mathrm{Ran}(X) \to \mathrm{Ran}^u(X)$ induces an isomorphism on homology, and will be left to the reader. We will show that ψ is a universal homology equivalence. Fix an object of $D \in \mathrm{Ran}^+_G(X)$, given by a quadruple $(R, \mathcal{P}, S, \gamma)$ where R is a finitely generated k-algebra, S is a subset of $X(R)$, $\mathcal{P}$ is a G-bundle on X_R, and γ is a trivialization of $\mathcal{P}$ restricted to $X_R - |S|$. Let $\mathcal{C}$ denote the fiber product $\mathrm{Ran}^u_G(X) \times_{\mathrm{Ran}^+_G(X)} \mathrm{Ran}^+_G(X)_{D/}$. We wish to show that the canonical map $\theta : C_*(\mathcal{C}; \mathbf{Z}_\ell) \to C_*(\mathrm{Spec}\, R; \mathbf{Z}_\ell)$ is a quasi-isomorphism.

Unwinding the definitions, we see that objects of $\mathcal{C}$ can be identified with pairs (A, T), where $A \in \mathrm{Ring}_R$ is a finitely generated R-algebra and $T \subseteq X(A)$ is a nonempty finite subset which contains the image of S. More precisely, we can identify $\mathcal{C}$ with the full subcategory of the product $\mathrm{Ran}^u(X) \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, R$ spanned by those pairs (A, T) where T contains the image of S. Note that the inclusion $\mathcal{C} \hookrightarrow \mathrm{Ran}^u(X) \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, R$ admits a left inverse f, given on objects by $(A, T) \mapsto (A, T \cup \mathrm{Im}(S))$. We have a commutative diagram of chain complexes

$$\begin{array}{ccccc}
C_*(\mathcal{C}; \mathbf{Z}_\ell) & \xrightarrow{f} & C_*(\mathrm{Spec}\, R \times_{\mathrm{Spec}\, k} \mathrm{Ran}^u(X); \mathbf{Z}_\ell) & \longrightarrow & C_*(\mathcal{C}; \mathbf{Z}_\ell) \\
\downarrow \theta & & \downarrow \theta' & & \downarrow \theta \\
C_*(\mathrm{Spec}\, R; \mathbf{Z}_\ell) & \longrightarrow & C_*(\mathrm{Spec}\, R; \mathbf{Z}_\ell) & \longrightarrow & C_*(\mathrm{Spec}\, R; \mathbf{Z}_\ell),
\end{array}$$

where the upper horizontal composition is the identity map. By a diagram chase, we are reduced to proving that the map θ' is a quasi-isomorphism. This follows from the Künneth formula, together with the acyclicity of the chain complex $C_*(\mathrm{Ran}^u(X); \mathbf{Z}_\ell)$. $\square$

Notation 6. Let R be a finitely generated k-algebra and let $\mathcal{P}$ be a G-bundle on X_R. We define prestacks

$$\mathrm{Sect}(\mathcal{P}) = \operatorname{Spec} R \times_{\mathrm{Bun}_G(X)} \mathrm{Ran}_G(X)$$

$$\mathrm{Sect}^u(\mathcal{P}) = \operatorname{Spec} R \times_{\mathrm{Bun}_G(X)} \mathrm{Ran}_G^u(X)$$

$$\mathrm{Sect}^+(\mathcal{P}) = \operatorname{Spec} R \times_{\mathrm{Bun}_G(X)} \mathrm{Ran}_G^+(X).$$

We have evident projection maps

$$\mathrm{Sect}(\mathcal{P}) \to \mathrm{Sect}^u(\mathcal{P}) \to \mathrm{Sect}^+(\mathcal{P}).$$

It can be deduced from Proposition 4 (or by repeating the proof of Proposition 4) that these maps are also universal homology equivalences.

Let us now explain why the preceding observations are helpful.

Notation 7. Let R be a finitely generated k-algebra and let $\mathcal{P}$ be a G-bundle on X_R. We can identify $\mathrm{Sect}^+(\mathcal{P})$ with a category whose objects are triples (A, S, γ) where $A \in \mathrm{Ring}_R$ is a finitely generated R-algebra, $S \subseteq X(A)$ is a finite subset, and γ is a trivialization of $\mathcal{P}$ over the open set $X_A - |S| \subseteq X_A$.

If we are given a subset $S_0 \subseteq X(R)$, then we let $\mathrm{Sect}^+_{\supseteq S_0}(\mathcal{P})$ denote the subcategory of $\mathcal{P}$ consisting of those triples (A, S, γ) where $S \subseteq X(A)$ contains the image of S_0.

The inclusion map $\iota : \mathrm{Sect}^+_{\supseteq S_0}(\mathcal{P}) \hookrightarrow \mathrm{Sect}^+(\mathcal{P})$ admits a right adjoint, given on objects by

$$(A, S, \gamma) \mapsto (A, S \cup \mathrm{Im}(S_0), \gamma|_{X_A - |S \cup \mathrm{Im}(S_0)|}).$$

It follows (as explained in a previous lecture) that ι induces an isomorphism in homology.

A consequence of the above discussion is that the question of whether or not the projection map $\mathrm{Sect}(\mathcal{P}) \to \operatorname{Spec} R$ induces an isomorphism on homology depends only on the generic behavior of the G-bundle $\mathcal{P}$. More precisely, suppose that we are given a pair of G-bundles $\mathcal{P}$ and $\mathcal{P}'$ over X_R, and that they are isomorphic away from a divisor $|S_0|$ for some finite subset $S_0 \subseteq X_R$. Then the prestacks $\mathrm{Sect}^+_{\supseteq S_0}(\mathcal{P})$ and $\mathrm{Sect}^+_{\supseteq S_0}(\mathcal{P}')$ are equivalent. We therefore have canonical isomorphisms

$$\begin{aligned}
\mathrm{H}_*(\mathrm{Sect}(\mathcal{P}); \mathbf{Z}_\ell) &\simeq \mathrm{H}_*(\mathrm{Sect}^+(\mathcal{P}); \mathbf{Z}_\ell) \\
&\simeq \mathrm{H}_*(\mathrm{Sect}^+_{\supseteq S_0}(\mathcal{P}); \mathbf{Z}_\ell) \\
&\simeq \mathrm{H}_*(\mathrm{Sect}^+_{\supseteq S_0}(\mathcal{P}'); \mathbf{Z}_\ell) \\
&\simeq \mathrm{H}_*(\mathrm{Sect}^+(\mathcal{P}'); \mathbf{Z}_\ell) \\
&\simeq \mathrm{H}_*(\mathrm{Sect}(\mathcal{P}'); \mathbf{Z}_\ell).
\end{aligned}$$

These isomorphisms fit into a commutative diagram

$$\begin{array}{ccc}
\mathrm{H}_*(\mathrm{Sect}(\mathcal{P}); \mathbf{Z}_\ell) & \longrightarrow & \mathrm{H}_*(\mathrm{Sect}(\mathcal{P}'); \mathbf{Z}_\ell) \\
& \searrow \quad \swarrow & \\
& \operatorname{Spec} R. &
\end{array}$$

Consequently, the left vertical map is an isomorphism if and only if the right vertical map is an isomorphism. Over the next several lectures, we will use this idea to reduce to the case where the G-bundle $\mathcal{P}$ is *trivial*.