Universal Homology Equivalences (Lecture 11)

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Throughout this lecture, we let k denote an algebraically closed field, ℓ a prime number which is invertible in k. Let X be an algebraic curve over k and G a smooth affine group scheme over X whose generic fiber is semisimple and simply connected. Our goal over the next few lectures is to prove the following result:

Theorem 1 (Nonabelian Poincare Duality)**.** *The forgetful functor* $\rho : \mathrm{Ran}_G(X) \to \mathrm{Bun}_G(X)$ *induces an isomorphism*

$$C_*(\mathrm{Ran}_G(X); \mathbf{Z}_\ell) \to C_*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell).$$

Roughly speaking, our strategy will be to prove that ρ induces an isomorphism on $\mathbf{Z}_\ell$-homology by showing that the fibers of ρ are acyclic (with respect to $\mathbf{Z}_\ell$-homology). We begin with a few general remarks.

Let $\mathcal{C}$ and $\mathcal{I}$ be categories, where $\mathcal{I}$ is small. For each object $M \in \mathcal{C}$, let $c_M : \mathcal{I} \to \mathcal{C}$ denote the constant functor with value M. If $F : \mathcal{I} \to \mathcal{C}$ is an arbitrary functor, then a *colimit* of F is an object $\varinjlim_{I \in \mathcal{I}} F(I) \in \mathcal{C}$ with the following universal property: for each object $M \in \mathcal{C}$, we have a canonical bijection

$$\mathrm{Hom}_{\mathcal{C}}(\varinjlim_{I \in \mathcal{I}} F(I), M) \simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{I},\mathcal{C})}(F, c_M).$$

If the category $\mathcal{C}$ admits small colimits, then the construction $F \mapsto \varinjlim_{I \in \mathcal{I}} F(I)$ determines a functor $\mathrm{Fun}(\mathcal{I}, \mathcal{C}) \to \mathcal{C}$. The above discussion shows that this functor can be regarded as a *left adjoint* to the diagonal embedding

$$\mathcal{C} \to \mathrm{Fun}(\mathcal{I}, \mathcal{C})$$

$$M \mapsto c_M.$$

Let us henceforth assume that the category $\mathcal{C}$ admits colimits. In this case, the above discussion can be relativized. Suppose we are given another small category $\mathcal{J}$ and a functor $\pi : \mathcal{I} \to \mathcal{J}$. Composition with π determines a functor

$$\pi^* : \mathrm{Fun}(\mathcal{J}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{I}, \mathcal{C})$$

$$(\pi^* F)(I) = F(\pi I).$$

This functor again admits a left adjoint. We will denote this left adjoint by $\pi_! : \mathrm{Fun}(\mathcal{I}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{J}, \mathcal{C})$. We refer to $\pi_!$ as the functor of *left Kan extension* along π.

Example 2. Suppose that $\mathcal{J} = \{*\}$ consists of a single object, so that $\mathrm{Fun}(\mathcal{J}, \mathcal{C}) \simeq \mathcal{C}$. Then the left Kan extension functor

$$\pi_! : \mathrm{Fun}(\mathcal{I}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{J}, \mathcal{C}) \simeq \mathcal{C}$$

is simply the colimit functor $F \mapsto \varinjlim_{I \in \mathcal{I}} F(I)$.

Remark 3. The formation of left Kan extensions is transitive in the following sense. Suppose we are given a pair of functors

$$\mathcal{I} \xrightarrow{\pi} \mathcal{J} \xrightarrow{\pi'} \mathcal{K}$$

between small categories. Then the functors

$$(\pi' \circ \pi)_!, \pi'_! \circ \pi_! : \mathrm{Fun}(\mathcal{I}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{K}, \mathcal{C})$$

are canonically isomorphic. In particular, taking $\mathcal{K} = \{*\}$, we see that

$$\varinjlim_{J \in \mathcal{J}} (\pi_! F)(J) \simeq \varinjlim_{I \in \mathcal{I}} F(I).$$

Let $\pi : \mathcal{I} \to \mathcal{J}$ be as above, and let $F : \mathcal{I} \to \mathcal{C}$ be any functor. By definition, the left Kan extension $\pi_! F$ comes equipped with a canonical map $u : F \to \pi^* \pi_! F$. Evaluating at an object $I \in \mathcal{I}$, we obtain a map $u_I : F(I) \to (\pi_! F)(\pi I)$. Consequently, any morphism $\alpha : \pi(I) \to J$ in the category $\mathcal{J}$ determines a map $F(I) \to (\pi_! F)(J)$ in the category $\mathcal{C}$. If we fix J, then these maps can be assembled to a morphism

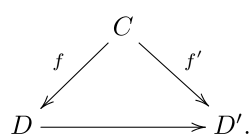
$$\varinjlim_{I \in \mathcal{I}, \alpha : \pi(I) \to J} F(I) \to (\pi_! F)(J).$$

One can show that this map is an isomorphism. More precisely, one can show that the construction

$$J \mapsto \varinjlim_{I \in \mathcal{I}, \alpha : \pi(I) \to J} F(I)$$

determines a functor from $\mathcal{J}$ to $\mathcal{C}$, and that this functor satisfies the universal property required of the left Kan extension $\pi_! F$ (this is a straightforward exercise in manipulating definitions).

Notation 4. Let $\mathcal{C}$ be any category, and let $C \in \mathcal{C}$ be an object. We let $\mathcal{C}_{C/}$ denote the category whose objects are morphisms $f : C \to D$ in $\mathcal{C}$, and whose morphisms are given by commutative diagrams



The construction $(f : C \to D) \mapsto C$ determines a forgetful functor $\mathcal{C}_{C/} \to \mathcal{C}$. We will generally abuse notation by not distinguishing between an object of $\mathcal{C}_{C/}$ and its image in $\mathcal{C}$ (in other words, we will simply refer to D as an object of $\mathcal{C}_{C/}$ if the map f is understood).

There is an evident dual construction, which produces a category $\mathcal{C}_{/C}$ whose objects are morphisms $f : D \to C$ in the original category $\mathcal{C}$.

Using Notation 4, we can rewrite our formula for the left Kan extension as

$$(\pi_! F)(J) = \varinjlim_{I \in \mathcal{I} \times_{\mathcal{J}} \mathcal{J}_{/J}} F(I).$$

All of the above remarks carry over without essential change to the setting of ∞-categories. Let us now apply it in the setting of prestacks.

Notation 5. Let $\pi : \mathcal{C} \to \mathrm{Ring}_k$ be a prestack. We define a functor $\omega_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Mod}_{\mathbf{Z}_\ell}$ by the formula

$$\omega_{\mathcal{C}}(C) = C_*(\mathrm{Spec}\, \pi(C); \mathbf{Z}_\ell).$$

Then (by definition) the ℓ-adic chain complex $C_*(\mathcal{C}; \mathbf{Z}_\ell)$ is the colimit of the functor $\omega - \mathcal{C}$.

If we are given a morphism $f : \mathcal{C} \to \mathcal{D}$ of prestacks, then we have an evident equivalence $\alpha : \omega_{\mathcal{C}} \simeq f^* \omega_{\mathcal{D}}$ of functors from $\mathcal{C}^{\mathrm{op}}$ to $\mathrm{Mod}_{\mathbf{Z}_\ell}$. This equivalence determines a natural transformation

$$\beta : f_! \omega_{\mathcal{C}} \to \omega_{\mathcal{D}}$$

of functors from $\mathcal{C}'^{\mathrm{op}}$ to $\mathrm{Mod}_{\mathbf{Z}_\ell}$. Taking colimits (and invoking Remark 3), we see that β induces a map of chain complexes

$$\begin{aligned} C_*(\mathcal{C};\mathbf{Z}_\ell) &= \varinjlim_{C \in \mathcal{C}^{\mathrm{op}}} \omega_{\mathcal{C}}(C) \\ &\simeq \varinjlim_{D \in \mathcal{D}^{\mathrm{op}}} (f_! \omega_{\mathcal{C}})(D) \\ &\xrightarrow{\beta} \varinjlim_{D \in \mathcal{D}^{\mathrm{op}}} \omega_{\mathcal{D}}(D) \\ &= C_*(\mathcal{D};\mathbf{Z}_\ell). \end{aligned}$$

This composition is just a fancy way of describing the map on chain complexes induced by f.

Definition 6. Suppose we are given a morphism of prestacks

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{D} \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi'} & \\ & \mathrm{Ring}_k\,. & \end{array}$$

We say that f is a *universal homology equivalence* if the map $\beta : f_! \omega_{\mathcal{C}} \to \omega_{\mathcal{D}}$ is an equivalence of functors from $\mathcal{D}^{\mathrm{op}}$ into $\mathrm{Mod}_{\mathbf{Z}_\ell}$. More concretely, f is a univeral homology equivalence if, for every object $D \in \mathcal{D}$, the canonical map

$$\varinjlim_{C \in \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{D/}} C_*(\operatorname{Spec} \pi(C);\mathbf{Z}_\ell) \to C_*(\operatorname{Spec} \pi'(D);\mathbf{Z}_\ell)$$

is an equivalence in $\mathrm{Mod}_{\mathbf{Z}_\ell}$.

Remark 7. Many variants of Definition 6 are possible. For example, we could take our coefficients in $\mathbf{Q}_\ell$ or $\mathbf{Z}/\ell\mathbf{Z}$, or we could work with cochain complexes instead of chain complexes. In each case, the assertion that the relevant map is a quasi-isomorphism follows from the assumption that F is a universal homological equivalence (in the sense of Definition 6). In other words, Definition 6 is the *strongest* acyclicity property of its type.

Remark 8. In the situation of Definition 6, we can identify the direct limit $\varinjlim_{C \in \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{D/}} C_*(\operatorname{Spec} \pi(C);\mathbf{Z}_\ell)$ with the complex of $\mathbf{Z}_\ell$-chains on the prestack $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{D/}$.

Remark 9. Suppose that $\mathcal{D}$ is a prestack in groupoids, and let $D \in \mathcal{D}$ be an object with $\pi'(D) = R$. Then π' induces an equivalence $\mathcal{D}_{D/} \simeq \mathrm{Ring}_R$. We may therefore identify the forgetful functor $\mathcal{D}_{D/} \to \mathcal{D}$ with a map $\operatorname{Spec} R \to \mathcal{D}$. In this case, Definition 6 requires that the canonical map

$$C_*(\mathcal{C} \times_{\mathcal{D}} \operatorname{Spec} R;\mathbf{Z}_\ell) \to C_*(\operatorname{Spec} R;\mathbf{Z}_\ell)$$

is a quasi-isomorphism, where $\mathcal{C} \times_{\mathcal{D}} \operatorname{Spec} R$ denotes the *homotopy* fiber product of $\mathcal{C}$ and $\operatorname{Spec} R$ over $\mathcal{D}$.

From the above discussion we immediately deduce the following:

Proposition 10. *Let $f : \mathcal{C} \to \mathcal{D}$ be a universal homology equivalence of prestacks. Then f induces a quasi-isomorphism $C_*(\mathcal{C};\mathbf{Z}_\ell) \to C_*(\mathcal{D};\mathbf{Z}_\ell)$.*

In the special case where $\mathcal{D}$ is a prestack in groupoids, the class of universal homology equivalences admits the following characterization:

Proposition 11. *Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of prestacks, where $\mathcal{D}$ is a prestack in groupoids. Then the following conditions are equivalent:*

(1) *The morphism f is a universal homology equivalence.*

(2) *For every finitely generated k-algebra R and every map $\operatorname{Spec} R \to \mathcal{D}$, the induced map*

$$C_*(\operatorname{Spec} R \times_{\mathcal{D}} \mathcal{C}; \mathbf{Z}_\ell) \to C_*(\operatorname{Spec} R; \mathbf{Z}_\ell)$$

is a quasi-isomorphism.

(3) *For every prestack in groupoids $\mathcal{E}$ and every map $\mathcal{E} \to \mathcal{D}$, the induced map*

$$C_*(\mathcal{E} \times_{\mathcal{D}} \mathcal{C}; \mathbf{Z}_\ell) \to C_*(\mathcal{E}; \mathbf{Z}_\ell)$$

is a quasi-isomorphism.

Proof. The equivalence of (1) and (2) was established above, and the implication (3) $\Rightarrow$ (2) is immediate. We will complete the proof by showing that (2) $\Rightarrow$ (3). Suppose that $\mathcal{E}$ is a prestack in groupoids; we wish to show that the projection map

$$\pi : \mathcal{E} \times_{\mathcal{D}} \mathcal{C} \to \mathcal{E}$$

induces an isomorphism on homology. By virtue of Proposition 10, it will suffice to show that π is a *universal* homology equivalence. Using the equivalence of (1) and (2), we are reduced to showing that for every map $\operatorname{Spec} R \to \mathcal{E}$, the projection map

$$\operatorname{Spec} R \times_{\mathcal{E}} (\mathcal{E} \times_{\mathcal{D}} \mathcal{C}) \to \operatorname{Spec} R$$

induces an isomorphism in homology, which follows from assumption (2). $\square$

Example 12. Let $\mathcal{C}$ be an arbitrary prestack. Then the projection map $f : \mathcal{C} \to \operatorname{Spec} k$ is a universal homology equivalence if and only if it induces a quasi-isomorphism $C_*(\mathcal{C}; \mathbf{Z}_\ell) \to C_*(\operatorname{Spec} k; \mathbf{Z}_\ell) \simeq \mathbf{Z}_\ell$. The "only if" direction follows from Proposition 10, and the converse follows from the Künneth formula of the previous lecture.

Let us now return to the setting where X is an algebraic curve and G is a smooth affine group scheme over X. We will actually prove the following stronger version of our main assertion:

Theorem 13. *If the generic fiber of G is semisimple and simply connected, then the forgetful functor $\operatorname{Ran}_G(X) \to \operatorname{Bun}_G(X)$ is a universal homology equivalence.*

Let us unwind what Theorem 13 is saying in more concrete terms. Note that $\operatorname{Bun}_G(X)$ is a prestack in groupoids. It will therefore suffice to show that for every map $\eta : \operatorname{Spec} R \to \operatorname{Bun}_G(X)$, the projection map

$$\operatorname{Spec} R \times_{\operatorname{Bun}_G(X)} \operatorname{Ran}_G(X) \to \operatorname{Spec} R$$

induces an isomorphism on homology with coefficients in $\mathbf{Z}_\ell$. The map η determines a G-bundle $\mathcal{P}$ on the relative curve X_R. Unwinding the definition, we can describe the category $\operatorname{Spec} R \times_{\operatorname{Bun}_G(X)} \operatorname{Ran}_G(X)$ as follows:

- The objects are quadruples $(A, S, \mu : S \to X(A), \gamma)$, where A is a finitely generated R-algebra, S is a nonempty finite set, $\mu : S \to X(A)$ is a map, and γ is a map of schemes which fits into a commutative diagram

$$\begin{array}{ccc} X_A - |\mu(S)| & \xrightarrow{\gamma} & \mathcal{P} \\ & \searrow & \downarrow \\ & & X_R \end{array}$$

- A morphism from (A, S, μ, γ) to (A', S', μ', γ') consists of an R-algebra homomorphism $\phi : A \to A'$ and a surjection of finite sets $S \to S'$ such that the diagram of sets

$$\begin{array}{ccc} S & \longrightarrow & S' \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu'} \\ X(A) & \longrightarrow & X(A') \end{array}$$

  and the diagram of schemes

$$\begin{array}{ccc} X_{A'} - |S'| & \longrightarrow & X_A - |S| \\ & {\scriptstyle \gamma'} \searrow \quad \swarrow {\scriptstyle \gamma} & \\ & \mathcal{P} & \end{array}$$

  are commutative.

Let us denote this prestack by $\mathrm{Sect}(\mathcal{P})$. Roughly speaking, its points are given by *rational* sections of the bundle $\mathcal{P} \to X_R$ (that is, sections which are defined outside of some divisor in X). We can now reformulate Theorem 13 as follows:

Theorem 14. *Suppose that the generic fiber of G is semisimple and simply connected. Let R be a finitely generated k-algebra, and let $\mathcal{P}$ be a G-bundle on X_R. Then the canonical map*

$$\mathrm{Sect}(\mathcal{P}) \to \mathrm{Spec}\, R$$

induces an isomorphism on $\mathbf{Z}_\ell$-homology.

Let us conclude this lecture by indicating a brief heuristic for why Theorem 14 should be true. Suppose that $G = \mathbf{G}_m$ is the multiplicative group, $R = k$, and $\mathcal{P}$ is the trivial G-bundle on $\mathbf{G}_m$. Roughly speaking, we can think of $\mathrm{Sect}(\mathcal{P})$ as a moduli space for rational maps from the curve X into $\mathbf{G}_m$. In other words, it is an algebro-geometric incarnation of the group of units $K_X^\times$, where K_X denotes the fraction field of X. Since K_X is a vector space of infinite dimension over k, $\mathrm{Sect}(\mathcal{P})$ behaves like an infinite-dimensional affine space with a point removed, and is therefore "contractible." Our goal over the next few lectures will be to make this argument precise (and to make it work for group schemes more general than $\mathfrak{G}_m$).

References

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[2] Gaitsgory, D. *Contracibility of the space of rational maps.*