ℓ-adic Cohomology (Lecture 6)

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Our goal in this course is to describe (in a convenient way) the ℓ-adic cohomology of the moduli stack of bundles on an algebraic curve. We begin in this lecture by reviewing the ℓ-adic cohomology of schemes; we will generalize to stacks (algebraic and otherwise) in the next lecture.

Notation 1. Throughout this lecture, we k be an algebraically closed field and Sch_k the category of quasi-projective k-schemes (the restriction to quasi-projective k-schemes is not important; we could just as well use some smaller or larger category in what follows).

The category $\mathrm{Sch}_k^{\mathrm{op}}$ is equipped with several Grothendieck topologies which arise naturally in algebraic geometry. In particular, we can equip Sch_k with the *étale topology*, where covering sieves are generated by mutually surjective étale maps.

Definition 2. Let $\mathcal{C}$ be an ∞-category. A $\mathcal{C}$-*valued presheaf* on Sch_k is a functor $\mathrm{Sch}_k^{\mathrm{op}} \to \mathcal{C}$ (here we abuse notation by identifying $\mathrm{Sch}_k^{\mathrm{op}}$ with the ∞-category $\mathrm{N}(\mathrm{Sch}_k^{\mathrm{op}})_\bullet$ introduced in the previous lecture).

Let $\mathcal{F} : \mathrm{Sch}_k^{\mathrm{op}} \to \mathcal{C}$ be a $\mathcal{C}$-valued presheaf on Sch_k. We will say that $\mathcal{F}$ is a $\mathcal{C}$-*valued sheaf on* Sch_k if the following condition is satisfied:

$(*)$ Let X be a quasi-projective k-scheme and suppose we are given a jointly surjective collection of étale morphisms $u_\alpha : U_\alpha \to X$. Let $\mathcal{C}$ denote the category of quasi-projective k-schemes Y equipped with a map $Y \to X$ which factors through some u_α (the factorization need not be specified). Then $\mathcal{F}$ induces an equivalence

$$\mathcal{F}(X) \to \varprojlim_{Y \in \mathcal{C}} \mathcal{F}(Y)$$

in the ∞-category $\mathcal{C}$.

Example 3. Let $\mathcal{C}$ be an ordinary category and let $\mathrm{N}(\mathcal{C})_\bullet$ be its nerve. Then giving a $\mathrm{N}(\mathcal{C})_\bullet$-valued sheaf on Sch_k is equivalent to giving a $\mathcal{C}$-valued sheaf on Sch_k, in the sense of classical category theory.

Notation 4. Let $\mathrm{Mod}_{\mathbf{Z}}$ denote the ∞-category introduced in the previous lecture: the objects of $\mathrm{Mod}_{\mathbf{Z}}$ are chain complexes of injective abelian groups, the morphisms in $\mathrm{Mod}_{\mathbf{Z}}$ are maps of chain complexes, the 2-simplices are given by chain homotopies, and so forth.

For every integer n, the construction $V_* \mapsto \mathrm{H}_n(V_*)$ determines a functor from the ∞-category $\mathrm{Mod}_{\mathbf{Z}}$ to the ordinary category Ab of abelian groups. We will say that an object $V_* \in \mathrm{Mod}_{\mathbf{Z}}$ is *discrete* if $\mathrm{H}_n(V_*) \simeq 0$ for $n \neq 0$. One can show that the construction $V_* \mapsto \mathrm{H}_0(V_*)$ induces an equivalence from the ∞-category of discrete objects of $\mathrm{Mod}_{\mathbf{Z}}$ to the ordinary category of abelian groups. The inverse of this equivalence gives a fully faithful embedding $\mathrm{Ab} \hookrightarrow \mathrm{Mod}_{\mathbf{Z}}$, whose essential image is the full subcategory of $\mathrm{Mod}_{\mathbf{Z}}$ consisting of discrete objects.

In what follows, we will often abuse terminology by identifying Ab with its essential image in $\mathrm{Mod}_{\mathbf{Z}}$. We will often refer to objects of Ab as as *discrete* $\mathbf{Z}$*-modules* or *ordinary* $\mathbf{Z}$*-modules*, to distinguish them from more general objects of $\mathrm{Mod}_{\mathbf{Z}}$.

Example 5. For any object $M \in \mathrm{Mod}_{\mathbf{Z}}$, we can consider the *constant presheaf* $c_M : \mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{Mod}_{\mathbf{Z}}$, given on objects by $c_M(X) = M$.

Notation 6. Let $\mathrm{Shv}(\mathrm{Sch}_k; \mathbf{Z})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Sch}_k^{\mathrm{op}}, \mathrm{Mod}_{\mathbf{Z}})$ spanned by those functors $\mathcal{F}$ which are $\mathrm{Mod}_{\mathbf{Z}}$-valued sheaves. The inclusion $\mathrm{Shv}(\mathrm{Sch}_k^{\mathrm{op}}; \mathbf{Z}) \hookrightarrow \mathrm{Fun}(\mathrm{Sch}_k^{\mathrm{op}}, \mathrm{Mod}_{\mathbf{Z}})$ admits a left adjoint, which assigns to each presheaf $\mathcal{F} : \mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{Mod}_{\mathbf{Z}}$ its *sheafification* $\mathcal{F}^{\dagger}$ with respect to the étale topology.

Let M be a finite abelian group, which we regard as an object of $\mathrm{Mod}_{\mathbf{Z}}$. If $c_M : \mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{Mod}_{\mathbf{Z}}$ denote the constant functor taking the value M, then we will denote the sheafification of c_M by $X \mapsto C^*(X; M)$. We will refer to $C^*(X; M)$ as the *complex of M-valued cochains on X*.

Remark 7. Let M be a finite abelian group, and let $\underline{M} : \mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{Ab}$ denote the constant sheaf associated to M (so that $\underline{M}(X)$ is the set of locally constant M-valued functions on X). In the usual abelian category of Ab-valued sheaves on Sch_k, the sheaf $\underline{M}$ admits an injective resolution

$$0 \to \underline{M} \to I^0 \to I^1 \to \cdots$$

Then the construction $X \mapsto I^*(X)$ determines a functor from $\mathrm{Sch}_k^{\mathrm{op}}$ to the ordinary category of chain complexes, hence also a functor $\mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{Mod}_{\mathbf{Z}}$. One can show that the evident maps $M \to I^*(X)$ exhibit the functor $X \mapsto I^*(X)$ as a sheafification of c_M. In other words, the complex of M-valued cochains $C^*(X; M)$ can be explicitly constructed as

$$\cdots \to 0 \to I^0(X) \to I^1(X) \to \cdots .$$

In particular, the cohomology groups of the chain complex $C^*(X; M)$ can be identified with the usual *étale cohomology groups* of X with values in M, which we will denote simply by $\mathrm{H}^*(X; M)$.

The preceding definitions make sense also in the case where the abelian group M is not finite. However, they are generally badly behaved if $M = \mathbf{Z}$ or $M = \mathbf{Q}$. Consequently, we will use the notation $C^*(X; M)$ for a slightly different chain complex in general.

Definition 8. Let ℓ be a prime number, and let $\mathbf{Z}_\ell$ denote the ring of ℓ-adic integers. For every quasi-projective k-scheme X, we let $C^*(X; \mathbf{Z}_\ell)$ denote the limit $\varprojlim C^*(X; \mathbf{Z}/\ell^d \mathbf{Z})$, where the limit is formed in the ∞-category $\mathrm{Mod}_{\mathbf{Z}}$. We will refer to $C^*(X; \mathbf{Z}_\ell)$ as the *complex of $\mathbf{Z}_\ell$-valued cochains on X*.

Warning 9. The construction $X \mapsto C^*(X; \mathbf{Z}_\ell)$ is a sheaf for the étale topology, in the sense of Definition 2. However, it is *not* the sheafification of the constant functor $c_{\mathbf{Z}_\ell}$. It can be described instead as the ℓ-adic completion of the sheafification of the constant functor $c_{\mathbf{Z}_\ell}$.

Definition 10. Let $\mathbf{Q}_\ell = \mathbf{Z}_\ell[\ell^{-1}]$ denote the field of ℓ-adic rational numbers. The inclusion $\mathbf{Z}_\ell \hookrightarrow \mathbf{Q}_\ell$ induces a base change functor $\mathrm{Mod}_{\mathbf{Z}_\ell} \to \mathrm{Mod}_{\mathbf{Q}_\ell}$, which we will denote by $M \mapsto M[\ell^{-1}]$. For every quasi-projective k-scheme X, we define

$$C^*(X; \mathbf{Q}_\ell) = C^*(X; \mathbf{Z}_\ell)[\ell^{-1}].$$

We will denote the cohomology groups of the chain complexes $C^*(X; \mathbf{Z}_\ell)$ and $C^*(X; \mathbf{Q}_\ell)$ by $\mathrm{H}^*(X; \mathbf{Z}_\ell)$ and $\mathrm{H}^*(X; \mathbf{Q}_\ell)$, respectively. We refer to either of these as the *ℓ-adic cohomology of X*.

We now discuss the cup product on ℓ-adic cohomology. Our starting point is the fact that the ∞-category $\mathrm{Mod}_{\mathbf{Z}}$ admits a *symmetric monoidal* structure: that is, it is equipped with a tensor product functor

$$\otimes_{\mathbf{Z}} : \mathrm{Mod}_{\mathbf{Z}} \times \mathrm{Mod}_{\mathbf{Z}} \to \mathrm{Mod}_{\mathbf{Z}}$$

which is commutative and associative up to *coherent* homotopy (see [2] for more details). Concretely, this can be described as the left derived functor of the usual tensor product (to compute with it, it is convenient to work with an alternative definition of $\mathrm{Mod}_{\mathbf{Z}}$ using chain complexes of free modules rather than chain complexes of injective modules).

Since $\mathrm{Mod}_{\mathbf{Z}}$ is a symmetric monoidal ∞-category, it makes sense to consider associative or commutative algebra objects of $\mathrm{Mod}_{\mathbf{Z}}$. These can be thought of as chain complexes of abelian groups which are equipped with an algebra structure which is required to be associative (respectively commutative and associative) up to coherent homotopy. In the case of associative algebras, it is always possible to *rectify* the multiplication by choosing a quasi-isomorphic chain complex which is equipped with a multiplication which is strictly associative: that is, a differential graded algebra. In the commutative case this is not always possible: in concrete terms, a commutative algebra structure on an object of $\mathrm{Mod}_{\mathbf{Z}}$ is equivalent to the data of an E_∞-algebra over $\mathbf{Z}$.

The symmetric monoidal structure on $\mathrm{Mod}_{\mathbf{Z}}$ induces a symmetric monoidal structure on the functor ∞-category $\mathrm{Fun}(\mathrm{Sch}_k^{\mathrm{op}}, \mathrm{Mod}_{\mathbf{Z}})$, given by pointwise tensor product:

$$(\mathcal{F} \otimes \mathcal{F}')(X) = \mathcal{F}(X) \otimes_{\mathbf{Z}} \mathcal{F}'(X).$$

This symmetric monoidal structure is *compatible* with the sheafification functor

$$\mathrm{Fun}(\mathrm{Sch}_k^{\mathrm{op}}, \mathrm{Mod}_{\mathbf{Z}}) \to \mathrm{Shv}(\mathrm{Sch}_k; \mathbf{Z}),$$

in the sense that the sheafification of a tensor product $\mathcal{F} \otimes \mathcal{F}'$ depends only on the sheafifications of $\mathcal{F}$ and $\mathcal{F}'$ individually. It follows formally that the sheafification functor carries commutative algebra objects of $\mathrm{Fun}(\mathrm{Sch}_k^{\mathrm{op}}, \mathrm{Mod}_{\mathbf{Z}})$ to commutative algebra objects of $\mathrm{Fun}(\mathrm{Sch}_k^{\mathrm{op}}, \mathrm{Mod}_{\mathbf{Z}})$. In particular, each of the constant functors $c_{\mathbf{Z}/\ell^d\mathbf{Z}}$ is a commutative algebra object of $\mathrm{Fun}(\mathrm{Sch}_k^{\mathrm{op}}, \mathrm{Mod}_{\mathbf{Z}})$, so that we can regard the construction $X \mapsto C^*(X; \mathbf{Z}/\ell^d\mathbf{Z})$ as a functor which takes values in commutative algebra objects of $\mathrm{Mod}_{\mathbf{Z}}$. This structure passes to the limit in d, and determines commutative algebra structures on $C^*(X; \mathbf{Z}_\ell)$ and $C^*(X; \mathbf{Q}_\ell)$. At the level of cohomology, this endows each of the cohomology groups

$$\mathrm{H}^*(X; \mathbf{Z}/\ell^d\mathbf{Z}) \qquad \mathrm{H}^*(X; \mathbf{Z}_\ell) \qquad \mathrm{H}^*(X; \mathbf{Q}_\ell)$$

with the structure of a graded-commutative ring (which agrees with the usual *cup product* of cohomology classes).

Variant 11. Let Λ be an arbitrary commutative ring. Then we can associate to Λ an ∞-category Mod_Λ, whose homotopy category is the classical derived category of Λ-modules. This ∞-category admits several equivalent descriptions:

- Let $\mathrm{Chain}(\Lambda)$ denote the category of chain complexes of Λ-modules. Then Mod_Λ can be described as the ∞-category obtained from $\mathrm{Chain}(\Lambda)$ by formally inverting all quasi-isomorphisms. In other words, Mod_Λ is universal with respect to the property that there exists a functor $\mathrm{Chain}(\Lambda) \to \mathrm{Mod}_\Lambda$), which carries each quasi-isomorphism in $\mathrm{Chain}(\Lambda)$ to an equivalence in Mod_Λ.

- We say that a chain complex $V_* \in \mathrm{Chain}(\Lambda)$ is *K-injective* if the following condition is satisfied: for every map of chain complexes $f : M_* \to V_*$ and every monomorphism $g : M_* \hookrightarrow N_*$ which is also a quasi-isomorphism, there exists a map $\overline{f} : N_* \to V_*$ such that $f = \overline{f} \circ g$. If Λ has finite injective dimension, then this is equivalent to the condition that each V_i is an injective Λ-module (in general, the injectivity of each V_i is necessary but not sufficient to guarantee the injectivity of the chain complex V_*). We can then mimic the definition of $\mathrm{Mod}_{\mathbf{Z}}$ given in the previous lecture, using K-injective chain complexes of Λ-modules rather than injective chain complexes of abelian groups. More precisely, for each $n \geq 0$, an n-simplex of Mod_Λ is given by a finite sequence of chain complexes $M(0)_*, M(1)_*, \cdots, M(n)_*$, together with Λ-module maps $f_I : M(i_-)_* \to M(i_+)_{*+m}$ for each subset $I = \{i_- < i_1 < \cdots < i_m < i_+\} \subseteq \{0, \ldots, n\}$ satisfying

$$d(f_I(x)) = (-1)^m f_I(dx) + \sum_{1 \leq j \leq m} (-1)^j (f_{I - \{i_j\}}(x) - (f_{\{i_j, \ldots, i_+\}} \circ f_{\{i_-, \ldots i_j\}})(x)).$$

- Using the commutative ring structure on Λ, we can regard Λ as a commutative algebra in the ∞-category $\mathrm{Mod}_{\mathbf{Z}}$. Then Mod_Λ can be identified with the ∞-category of Λ-*modules* in $\mathrm{Mod}_{\mathbf{Z}}$: that is, objects $M \in \mathrm{Mod}_{\mathbf{Z}}$ equipped with an action $\Lambda \otimes_{\mathbf{Z}} M \to M$ which is coherently unital and associative.

Using any of these perspectives, one can show that Mod_Λ itself inherits a symmetric monoidal structure, which we will denote by $\otimes_\Lambda : \mathrm{Mod}_\Lambda \times \mathrm{Mod}_\Lambda \to \mathrm{Mod}_\Lambda$. For $\Lambda \in \{\mathbf{Z}/\ell^d\mathbf{Z}, \mathbf{Z}_\ell, \mathbf{Q}_\ell\}$, one can show that the functor $X \mapsto C^*(X;\Lambda)$ can be regarded as a functor from $\mathrm{Sch}_k^{\mathrm{op}}$ to the ∞-category of commutative algebra objects of Mod_Λ.

Warning 12. The tensor product functor $\otimes_\Lambda$ on Mod_Λ does not agree with the usual tensor product on discrete Λ-modules. If M and N are discrete Λ-modules, then the tensor product $M \otimes_\Lambda N$ (formed in Mod_Λ) is obtained by tensoring M with some projective resolution of N, or vice versa. In particular, we have canonical isomorphisms

$$\mathrm{H}_i(M \otimes_\Lambda N) = \mathrm{Tor}_i^\Lambda(M,N).$$

In particular $M \otimes_\Lambda N$ is discrete if and only if the groups $\mathrm{Tor}_i^\Lambda(M,N) \simeq 0$ for $i > 0$ (this is automatic, for example, if M or N is flat over Λ).

Unless otherwise specified, we will always use the notation $\otimes_\Lambda$ to indicate the tensor product in the ∞-category Mod_Λ, rather than the ordinary category of discrete Λ-modules.

If X and Y are quasi-projective k-schemes, then the multiplication on $C^*(X \times_{\mathrm{Spec}\, k} Y;\Lambda)$ induces a map

$$\begin{aligned} C^*(X;\Lambda) \otimes_\Lambda C^*(Y;\Lambda) &\to C^*(X \times_{\mathrm{Spec}\, k} Y;\Lambda) \otimes_\Lambda C^*(X \times_{\mathrm{Spec}\, k} Y;\Lambda) \\ &\to C^*(X \times_{\mathrm{Spec}\, k} Y;\Lambda). \end{aligned}$$

Theorem 13 (Künneth Formula)**.** *Let k be an algebraically closed field, let ℓ be a prime number which is invertible in k, and let $\Lambda \in \{\mathbf{Z}/\ell^d\mathbf{Z}, \mathbf{Z}_\ell, \mathbf{Q}_\ell\}$. Then, for every pair of quasi-projective k-schemes X and Y, the canonical map*

$$C^*(X;\Lambda) \otimes_\Lambda C^*(Y;\Lambda) \to C^*(X \times_{\mathrm{Spec}\, k} Y;\Lambda)$$

is an equivalence in Mod_Λ.

Using the finiteness properties of étale cohomology, Theorem 13 reduces formally to the case $\Lambda = \mathbf{Z}/\ell\mathbf{Z}$, in which case it reduces to the more familiar formula

$$\mathrm{H}^*(X \times_{\mathrm{Spec}\, k} Y;\mathbf{Z}/\ell\mathbf{Z}) \simeq \mathrm{H}^*(X;\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}/\ell\mathbf{Z}} \mathrm{H}^*(Y;\mathbf{Z}/\ell\mathbf{Z}).$$

Definition 14. Let Λ be a commutative ring. We say that an object $M \in \mathrm{Mod}_\Lambda$ is *perfect* if it is dualizable with respect to the tensor product on Mod_Λ. That is, M is perfect if there exists another object $M^\vee \in \mathrm{Mod}_\Lambda$ together with maps

$$e : M^\vee \otimes_\Lambda M \to \Lambda \qquad c : \Lambda \to M \otimes_\Lambda M^\vee$$

for which the composite maps

$$M \xrightarrow{c \times \mathrm{id}} M \otimes_\Lambda M^\vee \otimes_\Lambda M \xrightarrow{\mathrm{id} \times e} M$$

$$M^\vee \xrightarrow{\mathrm{id} \times c} M^\vee \otimes_\Lambda M \otimes_\Lambda M^\vee \xrightarrow{e \times \mathrm{id}} M^\vee$$

are homotopic to the identity. In this case, the object $M^\vee \in \mathrm{Mod}_\Lambda$ is canonically determined. More precisely, the construction $M \mapsto M^\vee$ determines a contravariant functor from the full subcategory $\mathrm{Mod}_\Lambda^{\mathrm{pf}} \subseteq \mathrm{Mod}_\Lambda$ of perfect Λ-modules to itself. We will refer to $M^\vee$ as the *dual* of M, or as the Λ-*linear dual* of M if we wish to emphasize its dependence on the ring Λ.

Remark 15. Let $M \in \mathrm{Mod}_\Lambda$. One can show that M is perfect if and only if it is quasi-isomorphic to a finite complex of projective Λ-modules. If Λ is a regular Noetherian ring of finite Krull dimension (for example, if Λ is $\mathbf{Z}_\ell$, $\mathbf{Q}_\ell$, or $\mathbf{Z}/\ell\mathbf{Z}$), then a chain complex $M \in \mathrm{Mod}_\Lambda$ is perfect if and only if its total homology $\mathrm{H}_*(M)$ is finitely generated as an ordinary Λ-module.

The following is a basic finiteness theorem in the theory of étale cohomology:

Theorem 16. *Let k be an algebraically closed field, let ℓ be a prime number which is invertible in k, and let $\Lambda \in \{\mathbf{Z}_\ell, \mathbf{Q}_\ell, \mathbf{Z}/\ell^d\mathbf{Z}\}$. Then for every quasi-projective k-scheme X, the chain complex $C^*(X;\Lambda)$ is a perfect object of* Mod_Λ.

Proof. Using abstract nonsense, the proof reduces to the case $\Lambda = \mathbf{Z}/\ell\mathbf{Z}$. In this case, we must show that the étale cohomology groups $\mathrm{H}^n(X;\mathbf{Z}/\ell\mathbf{Z})$ are finite-dimensional $\mathbf{Z}/\ell\mathbf{Z}$-vector spaces for every integer n, which vanish for $n \gg 0$. For proofs, we refer the reader to [1]. □

Definition 17. Let k and Λ be as in Theorem 16. For every quasi-projective k-scheme X, we let $C_*(X;\Lambda)$ denote the Λ-linear dual of $C^*(X;\Lambda)$. We will denote the homology of $C_*(X;\Lambda)$ by $\mathrm{H}_*(X;\Lambda)$.

Remark 18. In the special case where $\Lambda \in \{\mathbf{Z}/\ell\mathbf{Z}, \mathbf{Q}_\ell\}$ is a field, the homology groups $\mathrm{H}_*(X;\Lambda)$ are simply given by the duals of the cohomology groups $\mathrm{H}^*(X;\Lambda)$ (in the category of vector spaces over Λ). More generally, we have a spectral sequence

$$\mathrm{Ext}^i_\Lambda(\mathrm{H}^*(X;\Lambda),\Lambda) \Rightarrow \mathrm{H}_{-*-i}(X;\Lambda).$$

Dualizing the Künneth formula for cohomology, we obtain a Künneth formula in homology:

Corollary 19 (Künneth Formula)**.** *Let X and Y be quasi-projective schemes over an algebraically closed field k, let ℓ be a prime number which is invertible in k, and let $\Lambda \in \{\mathbf{Z}_\ell, \mathbf{Q}_\ell, \mathbf{Z}/\ell^d\mathbf{Z}\}$. Then we have a canonical equivalence*

$$C_*(X;\Lambda) \otimes_\Lambda C_*(Y;\Lambda) \simeq C_*(X \times_{\mathrm{Spec}\, k} Y;\Lambda)$$

in the ∞-category Mod_Λ.

In the special case where Λ is a field, Corollary 19 gives an isomorphism of homology groups

$$\mathrm{H}_*(X \times_{\mathrm{Spec}\, k} Y;\Lambda) \simeq \mathrm{H}_*(X;\Lambda) \otimes_\Lambda \mathrm{H}_*(Y;\Lambda).$$

More generally, it gives a convergent spectral sequence

$$\mathrm{Tor}^\Lambda_p(\mathrm{H}_*(X;\Lambda), \mathrm{H}_*(Y;\Lambda)) \Rightarrow \mathrm{H}_{*+p}(X \times_{\mathrm{Spec}\, k} Y;\Lambda).$$

References

[1] Freitag, E. and R. Kiehl. *Etale cohomology and the Weil conjecture.* Springer-Verlag 1988.

[2] Lurie, J. *Higher Algebra.*

[3] Spaltenstein, N. *Resolutions of unbounded complexes.* Compositio Math. 65 (1988) no.2, 121-154.