

Tamagawa Numbers in the Function Field Case (Lecture 2)

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In the previous lecture, we defined the *Tamagawa measure* associated to a connected semisimple algebraic group G over the field \mathbf{Q} and formulated *Weil's conjecture*: if G is simply connected, then the Tamagawa measure of $G(\mathbf{Q}) \backslash G(\mathbf{A})$ is equal to 1. In this section, we will review the definition of Tamagawa measure for algebraic groups G which are defined over function fields. We will then state a function field analogue of Weil's conjecture, and explain how to reformulate it as a counting problem (by applying the logic of the previous lecture in reverse).

Notation 1. Let \mathbf{F}_q denote a finite field with q elements, and let X be an algebraic curve over \mathbf{F}_q (which we assume to be smooth, proper, and geometrically connected). We let K_X denote the function field of the curve X (that is, the residue field of the generic point of X).

We will write $x \in X$ to mean that x is a *closed* point of the curve X . For each point $x \in X$, we let $\kappa(x)$ denote the residue field of X at the point x . Then $\kappa(x)$ is a finite extension of the finite field \mathbf{F}_q . We will denote the degree of this extension by $\deg(x)$ and refer to it as the *degree* of x . We let \mathcal{O}_x denote the completion of the local ring of X at the point x : this is a complete discrete valuation ring with residue field $\kappa(x)$, noncanonically isomorphic to a power series ring $\kappa(x)[[t]]$. We let K_x denote the fraction field of \mathcal{O}_x .

For every finite set S of closed point of X , let \mathbf{A}^S denote the product $\prod_{x \in S} K_x \times \prod_{x \notin S} \widehat{\mathcal{O}}_x$. We let \mathbf{A} denote the direct limit

$$\varinjlim_{S \subseteq X} \mathbf{A}^S.$$

We will refer to \mathbf{A} as the *ring of adèles* of K_X . It is a locally compact commutative ring, equipped with a ring homomorphism $K_X \rightarrow \mathbf{A}$ which is an isomorphism of K_X onto a discrete subset of \mathbf{A} . We let $\mathbf{A}_0 = \prod_{x \in X} \mathcal{O}_x$ denote the ring of *integral adèles*, so that \mathbf{A}_0 is a compact open subring of \mathbf{A} .

Let G_0 be an affine algebraic group of dimension d defined over the field K_X . It will often be convenient to assume that we are given an *integral model* of G_0 : that is, that G_0 is given as the generic fiber of a smooth affine group scheme G over the curve X .

Remark 2. If G_0 is a simply connected semisimple algebraic group over K_X , then it is always possible to find a smooth affine group scheme with generic fiber G_0 . This follows from the work of Bruhat and Tits (see [1]).

Remark 3. In what follows, it will sometimes be convenient to assume that the group scheme $G \rightarrow X$ has connected fibers. This can always be arranged by passing to an open subgroup of G .

For every commutative ring R equipped with a map $\text{Spec } R \rightarrow X$, we let $G(R)$ denote the group of R -points of G . Then:

- For each closed point $x \in X$, $G(K_x)$ is a locally compact group, which contains $G(\mathcal{O}_x)$ as a compact open subgroup.
- We can identify $G(\mathbf{A})$ with the restricted product $\prod_{x \in X}^{\text{res}} G(K_x)$: that is, with the subgroup of the product $\prod_{x \in X} G(K_x)$ consisting of those elements $\{g_x\}_{x \in X}$ such that $g_x \in G(\mathcal{O}_x)$ for all but finitely many values of x .

- The group $G(\mathbf{A})$ is locally compact, containing $G(K_X)$ as a discrete subgroup and

$$G(\mathbf{A}_0) \simeq \prod_{x \in X} G(\mathcal{O}_x)$$

as a compact open subgroup.

Let us now review the construction of Tamagawa measure on the locally compact group $G(\mathbf{A})$. Let $\Omega_{G/X}$ denote the relative cotangent bundle of the smooth morphism $\pi : G \rightarrow X$. Then $\Omega_{G/X}$ is a vector bundle on G of rank $d = \dim(G_0)$. We let $\Omega_{G/X}^d$ denote the top exterior power of $\Omega_{G/X}$, so that $\Omega_{G/X}^d$ is a line bundle on G . Let \mathcal{L} denote the pullback of $\Omega_{G/X}^d$ along the identity section $e : X \rightarrow G$. Equivalently, we can identify \mathcal{L} with the subbundle of $\pi_* \Omega_{G/X}^d$ consisting of left-invariant sections. Let \mathcal{L}_0 denote the generic fiber of \mathcal{L} , so that \mathcal{L}_0 is a 1-dimensional vector space over the function field K_X . Fix a nonzero element $\omega \in \mathcal{L}_0$, which we can identify with a left-invariant differential form of top degree on the algebraic group G_0 .

For every point $x \in X$, ω determines a Haar measure $\mu_{x,\omega}$ on the locally compact topological group $G(K_x)$. Concretely, this measure can be defined as follows. Let t denote a uniformizing parameter for \mathcal{O}_x (so that $\mathcal{O}_x \simeq \kappa(x)[[t]]$), and let $G_{\mathcal{O}_x}$ denote the fiber product $\text{Spec } \mathcal{O}_x \times_X G$. Choose a local coordinates y_1, \dots, y_d for the group $G_{\mathcal{O}_x}$ near the identity: that is, coordinates which induce a map $u : G_{\mathcal{O}_x} \rightarrow \mathbb{A}_{\mathcal{O}_x}^d$ which is étale at the origin of $G_{\mathcal{O}_x}$. Let $v_x(\omega)$ denote the order of vanishing of ω at the point x . Then, in a neighborhood of the origin in $G_{\mathcal{O}_x}$, we can write $\omega = t^{v_x(\omega)} \lambda dy_1 \wedge \dots \wedge dy_d$, where λ is an invertible regular function. Let \mathfrak{m}_x denote the maximal ideal of \mathcal{O}_x , and let $G(\mathfrak{m}_x)$ denote the kernel of the reduction map $G(\mathcal{O}_x) \rightarrow G(\kappa(x))$. Since y_1, \dots, y_d are local coordinates near the origin, the map u induces a bijection $G(\mathfrak{m}_x) \rightarrow \mathfrak{m}_x^d$. The measure defined by the differential form $dy_1 \wedge \dots \wedge dy_d$ on $G(\mathfrak{m}_x)$ is obtained by pulling back the “standard” measure on K_x^d along the map u , where this standard measure is normalized so that \mathcal{O}_x^d has measure 1. It follows that the measure of $G(\mathfrak{m}_x)$ (with respect to the differential form $dy_1 \wedge \dots \wedge dy_d$) is given by $\frac{1}{|\kappa(x)|^d}$. We then define

$$\mu_{x,\omega}(G(\mathfrak{m}_x)) = q^{-\deg(x)v_x(\omega)} \frac{1}{|\kappa(x)|^d}.$$

The smoothness of G implies that the map $G(\mathcal{O}_x) \rightarrow G(\kappa(x))$ is surjective, so that we have

$$\mu_{x,\omega}(G(\mathcal{O}_x)) = q^{-\deg(x)v_x(\omega)} \frac{|G(\kappa(x))|}{|\kappa(x)|^d}.$$

Remark 4. Since $G(\mathcal{O}_x)$ is a compact open subgroup of $G(K_x)$, there is a unique left-invariant measure μ on $G(\mathcal{O}_x)$ satisfying

$$\mu(G(\mathcal{O}_x)) = q^{-\deg(x)v_x(\omega)} \frac{|G(\kappa(x))|}{|\kappa(x)|^d}.$$

The reader can therefore take this expression as the *definition* of the measure $\mu_{x,\omega}$. However, the analytic perspective is useful for showing that this measure is independent of the choice of integral model chosen. We refer the reader to [4] for more details.

The key fact is the following:

Proposition 5. *Suppose that G_0 is connected and semisimple, and let ω be a nonzero element of \mathcal{L}_0 . Then the measures $\mu_{x,\omega}$ on the groups $G(K_x)$ determine a well-defined product measure on $G(\mathbf{A}) = \prod_{x \in X}^{\text{res}} G(K_x)$. Moreover, this product measure is independent of ω .*

Proof. To check that the product measure is well-defined, it suffices to show that it is well-defined when evaluated on a compact open subgroup of $G(\mathbf{A})$, such as $G(\mathbf{A}_0)$. This is equivalent to the absolute convergence of the infinite product

$$\prod_{x \in X} \mu_{x,\omega}(G(\mathcal{O}_x)) = \prod_{x \in X} q^{-\deg(x)v_x(\omega)} \frac{|G(\kappa(x))|}{|\kappa(x)|^d},$$

which we will discuss later.

The fact that the product measure is independent of the choice of ω follows from the fact that the infinite sum

$$\sum_{x \in X} \deg(x) v_x(\omega) = \deg(\mathcal{L})$$

is independent of ω . □

Definition 6. Let G_0 be a connected semisimple algebraic group over K_X . Let d denote the dimension of G_0 , and let g denote the genus of the curve X . The *Tamagawa measure* on $G(\mathbf{A})$ is the Haar measure given informally by the product

$$\mu_{\text{Tam}} = q^{d(1-g)} \prod_{x \in X} \mu_{x, \omega}$$

Remark 7. Equivalently, we can define Tamagawa measure μ_{Tam} to be the unique Haar measure on $G(\mathbf{A})$ which is normalized by the requirement

$$\mu_{\text{Tam}}(G(\mathbf{A}_0)) = q^{d(1-g) - \deg(\mathcal{L})} \prod_{x \in X} \frac{|G(\kappa(x))|}{|\kappa(x)|^d}.$$

Remark 8. To ensure that the Tamagawa measure μ_{Tam} is well-defined, it is important that the quotients $\frac{|G(\kappa(x))|}{|\kappa(x)|^d}$ converge swiftly to 1, so that the infinite product $\prod_{x \in X} \frac{|G(\kappa(x))|}{|\kappa(x)|^d}$ is absolutely convergent. This can fail dramatically if G_0 is not connected. However, it is satisfied for some algebraic groups which are not semisimple: for example, the additive group \mathbf{G}_a .

Remark 9. If the group G_0 is semisimple, then any left-invariant differential form ω of top degree on G_0 is also right-invariant. It follows that the group $G(\mathbf{A})$ is unimodular. In particular, the measure μ_{Tam} on $G(\mathbf{A})$ descends to a measure on the quotient $G(K_X) \backslash G(\mathbf{A})$, which is invariant under the action of $G(\mathbf{A})$ by right translation. We will denote this measure also by μ_{Tam} , and refer to it as *Tamagawa measure*. It is characterized by the following requirement: for every positive measurable function f on $G(\mathbf{A})$, we have

$$\int_{x \in G(\mathbf{A})} f(x) d\mu_{\text{Tam}} = \int_{y \in G(K_X) \backslash G(\mathbf{A})} \left(\sum_{\pi(x)=y} f(x) \right) d\mu_{\text{Tam}}, \quad (1)$$

where $\pi : G(\mathbf{A}) \rightarrow G(K_X) \backslash G(\mathbf{A})$ denotes the projection map.

An important special case occurs when f is the characteristic function of a coset γH for some compact open subgroup $H \subseteq G(\mathbf{A})$. In this case, each element of $\pi(\gamma H)$ has exactly $o(\gamma)$ preimages in γH , where $o(\gamma)$ denotes the order of the finite group $G(K_X) \cap \gamma H \gamma^{-1}$ (this group is finite, since it is the intersection of a discrete subgroup of $G(\mathbf{A})$ with a compact subgroup of $G(\mathbf{A})$). Applying formula 1, we deduce that $\mu_{\text{Tam}}(\pi(\gamma H)) = \frac{\mu_{\text{Tam}}(H)}{o(\gamma)}$.

Example 10. Let $G = \mathbf{G}_a$ be the additive group. Then the dimension d of G is equal to 1, and the line bundle \mathcal{L} of left-invariant top forms is isomorphic to the structure sheaf \mathcal{O}_X of X . Moreover, we have an equality $|G(\kappa(x))| = |\kappa(x)|$ for each $x \in X$. Consequently, the Tamagawa measure μ_{Tam} is characterized by the formula $\mu_{\text{Tam}}(G(\mathbf{A}_0)) = q^{1-g}$. Note that we have an exact sequence of locally compact groups

$$0 \rightarrow H^0(X; \mathcal{O}_X) \rightarrow G(\mathbf{A}_0) \rightarrow G(K_X) \backslash G(\mathbf{A}) \rightarrow H^1(X; \mathcal{O}_X) \rightarrow 0,$$

so that the Tamagawa measure of the quotient $G(K_X) \backslash G(\mathbf{A})$ is given by

$$\frac{|H^1(X; \mathcal{O}_X)|}{|H^0(X; \mathcal{O}_X)|} \mu_{\text{Tam}}(G(\mathbf{A}_0)) = \frac{q^g}{q} q^{1-g} = 1.$$

Remark 11. One might ask the motivation for the auxiliary factor $q^{d(1-g)}$ appearing in the definition of the Tamagawa measure. Remark 10 provides one answer: the auxiliary factor is exactly what we need in order to guarantee that Weil's conjecture holds for the additive group \mathbf{G}_a .

Another answer is that the auxiliary factor is necessary to obtain invariance under *Weil restriction*. Suppose that $f : X \rightarrow Y$ is a separable map of algebraic curves over \mathbf{F}_q . Let K_Y be the fraction field of Y (so that K_X is a finite separable extension of K_Y), let \mathbf{A}_Y denote the ring of adèles of K_Y , and let H_0 denote the algebraic group over K_Y obtained from G_0 by Weil restriction along the field extension $K_Y \hookrightarrow K_X$. Then we have a canonical isomorphism of locally compact groups $G_0(\mathbf{A}) \simeq H_0(\mathbf{A}_Y)$. This isomorphism is compatible with the Tamagawa measures on each side, but only if we include the auxiliary factor $q^{d(1-g)}$ indicated in Definition 6.

Our goal in this paper is to address the following version of Weil's conjecture:

Conjecture 12 (Weil). Suppose that G_0 is semisimple and simply connected. Then

$$\mu_{\text{Tam}}(G(K_X)\backslash G(\mathbf{A})) = 1.$$

Let us now reformulate Conjecture 12 in more elementary terms. Note that the quotient $G(K_X)\backslash G(\mathbf{A})$ carries a right action of the compact group $G(\mathbf{A}_0)$. We may therefore write $G(K_X)\backslash G(\mathbf{A})$ as a union of orbits, indexed by the collection of double cosets

$$G(K_X)\backslash G(\mathbf{A})/G(\mathbf{A}_0).$$

Applying Remark 9, we calculate

$$\begin{aligned} \mu_{\text{Tam}}(G(K_X)\backslash G(\mathbf{A})) &= \sum_{\gamma} \frac{\mu_{\text{Tam}}(G(\mathbf{A}_0))}{|G(\mathbf{A}_0) \cap \gamma^{-1}G(K_X)\gamma|} \\ &= q^{d(1-g)-\deg(\mathcal{L})} \left(\prod_{x \in X} \frac{|G(\kappa(x))|}{|\kappa(x)|^d} \right) \sum_{\gamma} \frac{1}{|G(\mathbf{A}_0) \cap \gamma^{-1}G(K_X)\gamma|}. \end{aligned}$$

We may therefore reformulate Weil's conjecture as follows:

Conjecture 13 (Weil). Suppose that G_0 is semisimple and simply connected. Then we have an equality

$$\prod_{x \in X} \frac{|\kappa(x)|^d}{|G(\kappa(x))|} = q^{d(1-g)-\deg(\mathcal{L})} \sum_{\gamma} \frac{1}{|G(\mathbf{A}_0) \cap \gamma^{-1}G(K_X)\gamma|},$$

where the sum on the right hand side is taken over a set of representatives for the double quotient

$$G(K_X)\backslash G(\mathbf{A})/G(\mathbf{A}_0).$$

Remark 14. In the statement of Conjecture 13, the product on the left hand side and the sum on the right hand side are generally both infinite. The convergence of the left hand side is equivalent to the well-definedness of Tamagawa measure μ_{Tam} , and the convergence of the right hand side is equivalent to the statement that $\mu_{\text{Tam}}(G(K_X)\backslash G(\mathbf{A}))$ is finite.

We now give an algebro-geometric interpretation of the sum appearing on the right hand side of Conjecture 13. In what follows, we will assume that the reader is familiar with the theory of principal G -bundles.

Construction 15 (Regluing). Let γ be an element of the group $G(\mathbf{A})$. We can think of γ as given by a collection of elements $\gamma_x \in G(K_x)$, having the property that there exists a finite set S such that $\gamma_x \in G(\mathcal{O}_x)$ whenever $x \notin S$.

We define a G -bundle \mathcal{P}_{γ} on X as follows:

- (a) The bundle \mathcal{P}_γ is equipped with a trivialization ϕ on the open set $U = X - S$.
- (b) The bundle \mathcal{P}_γ is equipped with a trivialization ψ_x over the scheme $\text{Spec } \mathcal{O}_x$ of each point $x \in S$.
- (c) For each $x \in S$, the trivializations of $\mathcal{P}_\gamma|_{\text{Spec } K_x}$ determined by ϕ and ψ_x differ by multiplication by the element $\gamma_x \in G(K_x)$.

Note that the G -bundle \mathcal{P}_γ is canonically independent of the choice of S , so long as S contains all points x such that $\gamma_x \notin G(\mathcal{O}_x)$.

Remark 16. Let $\gamma, \gamma' \in G(\mathbf{A})$. The G -bundles \mathcal{P}_γ and $\mathcal{P}_{\gamma'}$ come equipped with trivializations at the generic point of X . Consequently, giving an isomorphism between the restrictions $\mathcal{P}_\gamma|_{\text{Spec } K_X}$ and $\mathcal{P}_{\gamma'}|_{\text{Spec } K_X}$ is equivalent to giving an element $\beta \in G(K_X)$. Unwinding the definitions, we see that this isomorphism admits an (automatically unique) extension to an isomorphism of \mathcal{P}_γ with $\mathcal{P}_{\gamma'}$ if and only if $\gamma'^{-1}\beta\gamma$ belongs to $G(\mathbf{A}_0)$. This has two consequences:

- (a) The G -bundles \mathcal{P}_γ and $\mathcal{P}_{\gamma'}$ are isomorphic if and only if γ and γ' determine the same element of $G(K_X)\backslash G(\mathbf{A})/G(\mathbf{A}_0)$.
- (b) The automorphism group of the G -torsor \mathcal{P}_γ is the intersection $G(\mathbf{A}_0) \cap \gamma^{-1}G(K_X)\gamma$.

Remark 17. Let \mathcal{P} be a G -bundle on X . Then \mathcal{P} can be obtained from Construction 15 if and only if the following two conditions are satisfied:

- (i) There exists an open set $U \subseteq X$ such that $\mathcal{P}|_U$ is trivial.
- (ii) For each point $x \in X - U$, the restriction of \mathcal{P} to $\text{Spec } \mathcal{O}_x$ is trivial.

By a direct limit argument, condition (i) is equivalent to the requirement that $\mathcal{P}|_{\text{Spec } K_X}$ be trivial: that is, that \mathcal{P} is classified by a trivial element of $H^1(\text{Spec } K_X; G_0)$. If G_0 is semisimple and simply connected, then $H^1(\text{Spec } K_X; G_0)$ vanishes (see [2]).

If the map $G \rightarrow X$ is smooth and has connected fibers, then condition (ii) is automatic (the restriction $\mathcal{P}|_{\text{Spec } \kappa(x)}$ can be trivialized by Lang's theorem (see [3]), and any trivialization of $\mathcal{P}|_{\text{Spec } \kappa(x)}$ can be extended to a trivialization of $\mathcal{P}|_{\text{Spec } \mathcal{O}_x}$ by virtue of the assumption that G is smooth.

Suppose now that G has connected fibers. Combining Remarks 16 and 17, we obtain the formula

$$\mu_{\text{Tam}}(G(K_X)\backslash G(\mathbf{A})) \simeq q^{d(1-g)-\deg(\mathcal{L})} \left(\prod_{x \in X} \frac{|G(\kappa(x))|}{|\kappa(x)|^d} \right) \sum_{\mathcal{P}} \frac{1}{|\text{Aut}(\mathcal{P})|}.$$

Here the sum is taken over all isomorphism classes of generically trivial G -bundles on X . We may therefore reformulate Conjecture 12 as follows:

Conjecture 18 (Weil). Let $G \rightarrow X$ be a smooth affine group scheme with connected fibers whose generic fiber is semisimple and simply connected. Then

$$\prod_{x \in X} \frac{|\kappa(x)|^d}{|G(\kappa(x))|} = q^{d(1-g)-\deg(\mathcal{L})} \sum_{\mathcal{P}} \frac{1}{|\text{Aut}(\mathcal{P})|}.$$

The assertion of Conjecture 18 can be regarded as a function field version of the Siegel mass formula. More precisely, we have the following table of analogies:

Classical Mass Formula	Conjecture 18
Number field \mathbf{Q}	Function field K_X
Quadratic space $(V_{\mathbf{Q}}, q_{\mathbf{Q}})$ over \mathbf{Q}	Algebraic Group G_0
Even lattice (V, q)	Integral model G
Even lattice (V', q') of the same genus	Principal G -bundle \mathcal{P}
$\sum_{q'} \frac{1}{ \mathcal{O}_{q'}(\mathbf{Z}) }$	$\sum_{\mathcal{P}} \frac{1}{ \text{Aut}(\mathcal{P}) }$

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