In the last lecture, we introduced the notion of a pretopos. Recall that a pretopos is a coherent category \( \mathcal{C} \) with a few additional features: the category \( \mathcal{C} \) is required to admit finite coproducts (which are required to be disjoint), and every equivalence relation in \( \mathcal{C} \) is required to be effective.

Exercise 1. Let \( \mathcal{C} \) and \( \mathcal{D} \) be pretopoi, and let \( \lambda : \mathcal{C} \to \mathcal{D} \) be a functor which preserves finite limits and effective epimorphisms. Show that the following conditions are equivalent:

1. The functor \( \lambda \) is a morphism of coherent categories: that is, for every object \( X \in \mathcal{C} \), the induced map \( \text{Sub}(X) \to \text{Sub}(\lambda(X)) \) is a homomorphism of upper semilattices.
2. The functor \( \lambda \) preserves finite coproducts.

If these conditions are satisfied, we will say that \( \lambda \) is a morphism of pretopoi.

Definition 2. Let \( \mathcal{C} \) be a pretopos. A model of \( \mathcal{C} \) is a morphism of pretopoi \( M : \mathcal{C} \to \text{Set} \). We let \( \text{Mod}(\mathcal{C}) \) denote the full subcategory of \( \text{Fun}(\mathcal{C}, \text{Set}) \) spanned by the models of \( \mathcal{C} \).

Definition 3. Let \( T \) be a (typed) first-order theory. We say that \( T \) eliminates imaginaries if the weak syntactic category \( \text{Syn}_0(T) \) is a pretopos.

Every first-order theory \( T \) can be replaced by an essentially equivalent first-order theory which eliminates imaginaries. Suppose, for example, that \( X = [\varphi(x)] \) is a formula in the language of \( T \), having a single free variable \( x \) of type \( t \), and that \( R \subseteq X \times X \) is an equivalence relation defined by some formula \( \psi(x, x') \). In this case, we can enlarge the language of \( T \) by adding a new type \( s \) (to be interpreted as “\( X \) modulo the equivalence relation \( R \)”) and a new binary predicate \( P \) of type \((t, s)\) (to be interpreted as the graph of the projection map \( X \to X/R \)), and adding the axioms

\[
(\forall x,y)[P(x,y) \Rightarrow \varphi(x)]
\]

\[
(\forall x)[\varphi(x) \Rightarrow (\exists! y) P(x,y)]
\]

\[
(\forall x,x',y,y')[ (P(x,y) \land P(x',y')) \Rightarrow (y = y' \Leftrightarrow \psi(x,x'))]
\]

\[
(\forall y)(\exists x)[P(x,y)].
\]

Elaborating on this construction, one can produce a typed first-order theory \( T^{eq} \), having the same models as \( T \), for which \( T^{eq} \) eliminates imaginaries. This is a special case of a more general construction:

Theorem 4. Let \( \mathcal{C} \) be a small coherent category. Then there exists a small pretopos \( \mathcal{C}^{eq} \) and a morphism of coherent categories \( \lambda : \mathcal{C} \to \mathcal{C}^{eq} \) with the following universal property: if \( \mathcal{D} \) is any pretopos, then composition with \( \lambda \) induces an equivalence of categories \( \text{Fun}^{coh}(\mathcal{C}^{eq}, \mathcal{D}) \to \text{Fun}^{coh}(\mathcal{C}, \mathcal{D}) \). Here \( \text{Fun}^{coh}(\mathcal{C}, \mathcal{D}) \) denotes the full subcategory of \( \text{Fun}(\mathcal{C}, \mathcal{D}) \) spanned by the morphisms of coherent categories, and \( \text{Fun}^{coh}(\mathcal{C}^{eq}, \mathcal{D}) \) is defined similarly.

In particular, composition with \( \lambda \) induces an equivalence of categories \( \text{Mod}(\mathcal{C}^{eq}) \to \text{Mod}(\mathcal{C}) \).
In the situation of Theorem 4, the pretopos $\mathcal{C}^\text{eq}$ is determined (up to canonical equivalence) by the coherent category $\mathcal{C}$; we will refer to $\mathcal{C}^\text{eq}$ as the pretopos completion of $\mathcal{C}$.

**Definition 5.** Let $T$ be a (typed) first order theory. We let $\text{Syn}(T)$ denote the pretopos completion of $\text{Syn}_0(T)$. We will refer to $\text{Syn}(T)$ as the syntactic category of $T$.

One of our goals over the next several lectures is construct the pretopos completion $\mathcal{C}^\text{eq}$. To get a feeling for what we need to do, let us first suppose that we are given a triple of objects $X, Y, Z \in \mathcal{C}$. It could be that the objects $Y$ and $Z$ do not have a coproduct in $\mathcal{C}$ (they also might admit a coproduct in $\mathcal{C}$ which fails to be disjoint, in which case they will have a different coproduct in $\mathcal{C}^\text{eq}$). We would like $\mathcal{C}^\text{eq}$ to be an enlargement of the category $\mathcal{C}$ in which the coproduct $Y \amalg Z$ exists and is disjoint. Let us abuse notation by identifying objects of $\mathcal{C}$ with their image in $\mathcal{C}^\text{eq}$ (our proof of Theorem 4 will show that the functor $\lambda : \mathcal{C} \to \mathcal{C}^\text{eq}$ is fully faithful, so this abuse is mostly harmless). So, in order to make sense of the category $\mathcal{C}^\text{eq}$, we must in particular define the set of maps $\text{Hom}_{\mathcal{C}^\text{eq}}(X, Y \amalg Z)$. Note that we have some obvious candidates for elements of this set: since $Y$ and $Z$ should be subobjects of $Y \amalg Z$, we should have monomorphisms

$$\text{Hom}_{\mathcal{C}}(X, Y) \to \text{Hom}_{\mathcal{C}^\text{eq}}(X, Y \amalg Z) \to \text{Hom}_{\mathcal{C}}(X, Z).$$

However, it is not reasonable to expect to find all maps from $X$ to $Y \amalg Z$ in this way. For example, if $X$ can be written as the join of disjoint subobjects $X_0, X_1 \subseteq X$, then it is a disjoint coproduct of $X_0$ and $X_1$ in both $\mathcal{C}$ and $\mathcal{C}^\text{eq}$. Consequently, any pair of maps $f_0 : X_0 \to Y$ and $f_1 : X_1 \to Z$ in $\mathcal{C}$ can be uniquely amalgamated to obtain a map $f : X \simeq X_0 \amalg X_1 \to Y \amalg Z$ in the pretopos $\mathcal{C}^\text{eq}$. Roughly speaking, we would like to arrange that all morphisms in $\mathcal{C}^\text{eq}$ are defined by a procedure similar to this: that is, they can be built by “gluing together” morphisms that are already present in the coherent category $\mathcal{C}$. To make this precise, it will be convenient to construct $\mathcal{C}^\text{eq}$ as a subcategory of a much larger category of sheaves on the coherent category $\mathcal{C}$.

**Definition 6.** Let $\mathcal{C}$ be a category which admits finite limits. A Grothendieck topology on $\mathcal{C}$ is a specification of a collection of families of maps $\{f_i : U_i \to X\}_{i \in I}$, which we refer to as coverings, satisfying the following axioms:

1. **(T1)** If $\{f_i : U_i \to X\}$ is a covering and $g : Y \to X$ is any morphism in $\mathcal{C}$, then the collection of projection maps $\{U_i \times_X Y \to Y\}_{i \in I}$ is a covering.

2. **(T2)** Let $\{f_i : U_i \to X\}_{i \in I}$ is a covering and let $\{g_j : V_j \to X\}$ be a collection of maps. If, for each $i$, the projection maps $\{V_j \times_X U_i \to U_i\}$ form a covering, then $\{g_j : V_j \to X\}$ is a covering.

3. **(T3)** If $\{f_i : U_i \to X\}$ is a collection of morphisms such that some $f_i$ admits a section (that is, a morphism $s : X \to U_i$ satisfying $f_i \circ s = \text{id}_X$), then $\{f_i : U_i \to X\}$ is a covering.

**Remark 7.** One can also consider Grothendieck topologies on categories which do not admit finite limits. In this case, requirement (T1) must be rephrased. In this course, we will not need to consider Grothendieck topologies on such categories.

**Exercise 8.** Let $\mathcal{C}$ be a category which admits fiber products which is equipped with a Grothendieck topology, and suppose that $\{U_i \to X\}$ is a covering. Show that any larger collection of morphisms (with the same target $X$) is also a covering.

**Definition 9.** A Grothendieck site is a category $\mathcal{C}$ together with a Grothendieck topology on $\mathcal{C}$.

**Example 10.** Let $X$ be a topological space and let $\mathcal{U}$ be the collection of all open subsets of $X$, regarded as a partially ordered set with respect to inclusions. Then, when regarded as a category, the poset $\mathcal{U}$ carries a Grothendieck topology, where a collection of maps $\{U_i \to U\}_{i \in I}$ is a covering if $\bigcup_{i \in I} U_i = U$.

The original motivation for Definition 6 came from algebraic geometry:
Definition 13. Let $\mathcal{C}$ be the category of schemes $U$ equipped with an étale map $U \to X$. Then $\mathcal{C}$ can be equipped with a Grothendieck topology, where a collection of maps $\{U_i \to U\}$ is a covering if the induced map $\prod_i U_i \to U$ is surjective. When endowed with this topology, $\mathcal{C}$ is referred to as the small étale site of the scheme $X$.

For the moment, we will be primarily interested in Grothendieck sites which arise from coherent categories:

Proposition 12. Let $\mathcal{C}$ be a coherent category. Then $\mathcal{C}$ admits a Grothendieck topology which can be described as follows: a collection of morphisms $\{f_i : U_i \to X\}_{i \in I}$ is a covering if there exists some finite subset $I_0 \subseteq I$ such that $\bigvee_{i \in I_0} \text{Im}(f_i) = X$ (in the lattice $\text{Sub}(X)$).

Proof. Since the formation of images and joins of subobjects is compatible with pullback, it is clear that this notion of covering satisfies axiom $(T1)$.

Note that a collection of maps $\{f_i : U_i \to X\}_{i \in I}$ is a covering if, for some finite subset $I_0 \subseteq I$, there is no proper subobject $X' \subseteq X$ such that $f_i$ factors through $X'$, for each $i \in I_0$. From this description, it is clear that $(T3)$ is satisfied.

To verify $(T2)$, suppose that we are given a covering $\{U_i \to X\}_{i \in I}$ and a collection of maps $\{V_j \to X\}_{j \in J}$ with the property that, for each $i \in I$, the collection $\{V_j \times_X U_i \to U_i\}_{j \in J}$ is a covering. We wish to show that $\{V_j \to X\}_{j \in J}$ is a covering. Without loss of generality, we may assume that $I$ is finite and $J$ are finite. Suppose that there exists a subobject $X' \subseteq X$ such that each of the maps $V_j \to X$ factors through $X'$. Then each $V_j \to_X U_i \to U_i$ factors through $X' \times_X U_i$. It follows that $X' \times_X U_i = U_i$ (as subobjects of $U_i$). In other words, the maps $U_i \to X$ factor through $X'$. Using our assumption that $\{U_i \to X\}$ is a covering, we conclude that $X' = X$. \qed

Definition 13. Let $\mathcal{C}$ be a category. A presheaf (of sets) on $\mathcal{C}$ is a functor $\mathcal{F} : \mathcal{C}^{\text{op}} \to \text{Set}$.

Suppose that $\mathcal{C}$ admits fiber products and is equipped with a Grothendieck topology. We will say that a presheaf $\mathcal{F}$ is a sheaf if, for every covering $\{U_i \to X\}_{i \in I}$, the diagram of sets

$$
\mathcal{F}(X) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i) \longrightarrow \prod_{i,j \in I} \mathcal{F}(U_i \times_X U_j)
$$

is an equalizer: that is, we can $\mathcal{F}(X)$ with the set of tuples $\{s_i \in \mathcal{F}(U_i)\}$ having the property that, for every pair $i, j \in I$, the elements $s_i$ and $s_j$ have the same image in $\mathcal{F}(U_i \times_X U_j)$. We let $\text{Shv}(\mathcal{C})$ denote the full subcategory of $\text{Fun}(\mathcal{C}^{\text{op}}, \text{Set})$ spanned by the sheaves on $\mathcal{C}$.

Example 14. Let $X$ be a topological space and let $\mathcal{U}$ be the partially ordered set of open subsets of $X$. A sheaf on $X$ is defined to be a sheaf on the category $\mathcal{U}$, where we equip $\mathcal{U}$ with the Grothendieck topology of Example 10. We will denote the category of sheaves on $X$ by $\text{Shv}(X)$.

Example 15 (Representable Sheaves). Let $\mathcal{C}$ be any category. For each object $Y \in \mathcal{C}$, let $h_Y$ denote the functor represented by $Y$, given by the formula $h_Y(X) = \text{Hom}_{\mathcal{C}}(X,Y)$. We say that a Grothendieck topology on $\mathcal{C}$ is subcanonical if $h_Y$ is a sheaf, for each $Y \in \mathcal{C}$. If this condition is satisfied, then the construction $Y \mapsto h_Y$ determines a fully faithful embedding $\mathcal{C} \hookrightarrow \text{Shv}(\mathcal{C})$.

Proposition 16. Let $\mathcal{C}$ be a coherent category. Then the Grothendieck topology of Proposition 12 is subcanonical. In particular, we can view $\mathcal{C}$ as a full subcategory of $\text{Shv}(\mathcal{C})$.

Proof. Fix an object $Y \in \mathcal{C}$; we wish to show that the functor $h_Y$ is a sheaf. To prove this, suppose we are given a covering $\{U_i \to X\}$. We wish to show that the diagram of sets

$$
\text{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \prod_{i \in I} \text{Hom}_{\mathcal{C}}(U_i,Y) \longrightarrow \prod_{i,j \in I} \text{Hom}_{\mathcal{C}}(U_i \times_X U_j,Y)
$$

is an equalizer. In other words, we wish to show every collection of morphisms $f_i : U_i \to Y$ which is compatible, in the sense that $f_i$ and $f_j$ define the same morphism from $U_i \times_X U_j$ into $Y$, define an essentially unique map $f : X \to Y$. Without loss of generality, we can assume that $I$ is finite (exercise: justify this). For
each $i \in I$, let $X_i$ denote the image of the map $U_i \to X$. Then we have an effective epimorphism $U_i \to X_i$, and the map coequalizes the projection maps

$$\pi, \pi' : U_i \times_X U_i \cong U_i \times_X U_i \to U_i.$$  

It follows that each $f_i$ factors uniquely as a composition $U_i \to X_i \xrightarrow{\tau_i} Y$. Moreover, for $i \neq j$, the maps $\tau_i$ and $\tau_j$ have the same restriction to $X_i \wedge X_j$ (since this can be checked after composition with the effective epimorphism

$$U_i \times_X U_j \to X_i \times_X X_j = X_i \wedge X_j.$$  

It will therefore suffice to show that we can (uniquely) amalgamate the maps $\tau_i$ to a single map $f : X \to Y$.

We now proceed as in the previous lecture. Each $\tau_i$ has a graph $\Gamma(\tau_i)$, which we can view as a subobject of $X \times Y$. Set $Z = \bigwedge \Gamma(\tau_i)$. We will complete the proof by showing that $Z$ is the graph of a morphism from $X$ to $Y$. To prove this, it will suffice to show that the composition

$$Z \hookrightarrow X \times Y \to X$$

is a monomorphism (it will then necessarily be an isomorphism, since its image contains $\bigvee X_i$ which is equal to $X$ by virtue of our assumption that $\{U_i \to X\}$ is covering). In other words, it will suffice to show that the diagonal map $Z \hookrightarrow Z \times_X Z$ is an isomorphism. Note that, as subobjects of $X \times Y \times Y$, $Z$ can be identified with the join of the graphs of the maps $(\tau_i, \tau_i) : X_i \to Y \times Y$, while $Z \times_X Z$ can be identified with the join of the graphs of the morphisms

$$(\tau_i|_{X_i \wedge X_j}, \tau_j|_{X_i \wedge X_j}) : X_i \wedge X_j \to Y \times Y.$$  

We complete the proof by observing that we have inclusions

$$\Gamma(\tau_i|_{X_i \wedge X_j}, \tau_j|_{X_i \wedge X_j}) \subseteq \Gamma(\tau_i, \tau_i),$$

since $\tau_i$ and $\tau_j$ have the same restriction to $X_i \wedge X_j$.  

We can now describe our approach to the proof of Theorem 4. Given a coherent category $\mathcal{C}$, we can view it as a subcategory of the sheaf category $\text{Shv}(\mathcal{C})$. We will realize the pretopos completion $\mathcal{C}^{eq}$ as a slightly larger subcategory of $\text{Shv}(\mathcal{C})$. In order to show that this works, we will need to study the structural features of the category $\text{Shv}(\mathcal{C})$, and other categories like it.

**Definition 17.** Let $\mathcal{X}$ be a category. We say that $\mathcal{X}$ is a topos if there is an equivalence of categories $\mathcal{X} \cong \text{Shv}(\mathcal{C})$, where $\mathcal{C}$ is a small Grothendieck site which admits finite limits.

**Example 18.** For every topological space $X$, the category $\text{Shv}(X)$ is a topos.

**Example 19.** Let $T$ be a first-order theory and let $\text{Syn}_0(T)$ be the weak syntactic category of $T$, equipped with the Grothendieck topology of Proposition 12. Then $\text{Shv}(\text{Syn}_0(T))$ is a topos. We will refer to $\text{Shv}(\text{Syn}_0(T))$ as the classifying topos of $T$.

A useful heuristic is that a topos is a kind of generalized topological space; the classifying topos of a first-order theory $T$ can be viewed as a kind of “space of all models of $T$.”

**Warning 20.** In the situation of Definition 17, the category $\mathcal{C}$ is not uniquely determined by $\mathcal{X}$. In general, many different Grothendieck sites can give rise to the same topos. For example, we will see that if $\mathcal{C}$ is a small coherent category and $\mathcal{C}^{eq}$ is its pretopos completion, then the topoi $\text{Shv}(\mathcal{C})$ and $\text{Shv}(\mathcal{C}^{eq})$ are equivalent.

**Remark 21.** In the situation of Definition 17, the requirement that $\mathcal{C}$ admits finite limits is not necessary: we impose it only because we have not defined the notion of Grothendieck topology in full generality. However, it is a harmless assumption (if a category has the form $\text{Shv}(\mathcal{C})$ for an arbitrary small Grothendieck site $\mathcal{C}$, then it also has the form $\text{Shv}(\mathcal{C}')$ where $\mathcal{C}'$ is a small Grothendieck site which admits finite limits).