Lecture 5: Booleanization

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Recall the following definition from the previous lecture:

**Definition 1.** Let \( \mathcal{C} \) be a category. We will say that \( \mathcal{C} \) is a **coherent category** if it satisfies the following axioms:

(A1) The category \( \mathcal{C} \) admits finite limits.

(A2) Every morphism \( f : X \to Z \) in \( \mathcal{C} \) admits a factorization \( X \xrightarrow{g} Y \xrightarrow{h} Z \), where \( g \) is an effective epimorphism and \( h \) is a monomorphism.

(A3) For every object \( X \in \mathcal{C} \), the partially ordered set \( \text{Sub}(X) \) is an upper semilattice: that is, it has a least element, and every pair of subobjects \( X_0, X_1 \subseteq X \) have a least upper bound \( X_0 \lor X_1 \).

(A4) The collection of effective epimorphisms in \( \mathcal{C} \) is stable under pullback.

(A5) For every morphism \( f : X \to Y \) in \( \mathcal{C} \), the map \( f^{-1} : \text{Sub}(Y) \to \text{Sub}(X) \) is a homomorphism of upper semilattices.

**Definition 2.** Let \( \mathcal{C} \) and \( \mathcal{C}' \) be coherent categories. A **morphism of coherent categories** from \( \mathcal{C} \) to \( \mathcal{C}' \) is a functor \( F : \mathcal{C} \to \mathcal{C}' \) with the following properties:

(1) The functor \( F \) is left exact: that is, it preserves finite limits.

(2) The functor \( F \) carries effective epimorphisms to effective epimorphisms.

(3) For every object \( X \in \mathcal{C} \), the induced map \( \text{Sub}(X) \to \text{Sub}(F(X)) \) is a homomorphism of upper semilattices: that is, it preserves smallest elements and joins.

**Remark 3.** Let \( F : \mathcal{C} \to \mathcal{C}' \) satisfy condition (1) of Definition 2. Then \( F \) preserves monomorphisms. Consequently, condition (2) is equivalent to the requirement that for every morphism \( f : X \to Z \), the functor \( F \) carries the canonical factorization \( X \to \text{Im}(f) \to Z \) to the factorization \( F(X) \to \text{Im}(F(f)) \to F(Z) \). Note also that in the situation of (3), the map \( \text{Sub}(X) \to \text{Sub}(F(X)) \) is automatically a homomorphism of lower semilattices (that is, it preserves largest elements and meets).

**Example 4.** Let \( P \) and \( P' \) be distributive lattices. Then, when viewed as categories, \( P \) and \( P' \) are coherent categories. A morphism of coherent categories from \( P \) to \( P' \) is just a lattice homomorphism: that is, a map of partially ordered sets that preserves least upper bounds and greatest lower bounds for finite subsets.

**Example 5.** Let \( \mathcal{C} \) be a category containing an object \( X \). We can then form a new category \( \mathcal{C}/X \), whose objects are pairs \( (U,f) \), where \( U \in \mathcal{C} \) is an object and \( f : U \to X \) is a morphism. A morphism from \( (U,f) \) to \( (V,g) \) in \( \mathcal{C} \) is a morphism \( h : U \to V \) in \( \mathcal{C} \) such that \( f = g \circ h \). The construction \( (U,f) \mapsto U \) determines a forgetful functor \( \mathcal{C}/X \to \mathcal{C} \), and we will generally abuse notation by identifying an object of \( \mathcal{C}/X \) with its image under this forgetful functor.

**Exercise 6.** Show that if \( \mathcal{C} \) is a coherent category, then so is \( \mathcal{C}/X \). Moreover, the formation of fiber products, images, and unions of subobjects in \( \mathcal{C}/X \) can be computed in the underlying category \( \mathcal{C} \).
Beware that the forgetful functor $\mathcal{C}/X \to \mathcal{C}$ is not a morphism of coherent categories, because it does not preserve final objects. However, the forgetful functor has a right adjoint which is a morphism of coherent categories. More generally, suppose that $f : X \to Y$ is any morphism in $\mathcal{C}$. Then $f$ determines a functor $f^* : \mathcal{C}/Y \to \mathcal{C}/X$, given by the construction $U \mapsto U \times_X Y$.

**Exercise 7.** Show that if $f : X \to Y$ is a morphism in a coherent category $\mathcal{C}$, then the functor $f^* : \mathcal{C}/Y \to \mathcal{C}/X$ is a morphism of coherent categories. In fact, this is precisely the content of axioms (A4) and (A5).

**Definition 8.** Let $\mathcal{C}$ be a coherent category. A *model* of $\mathcal{C}$ is a morphism of coherent categories $M : \mathcal{C} \to \text{Set}$. We let $\text{Mod}(\mathcal{C})$ denote the full subcategory of the functor category $\text{Fun}(\mathcal{C}, \text{Set})$ spanned by the models of $\mathcal{C}$; we refer to $\text{Mod}(\mathcal{C})$ as the category of models of $\mathcal{C}$.

**Example 9.** Let $\mathcal{C} = \text{Syn}_0(T)$ be the weak syntactic category of a (typed) first-order theory $T$. Then we can identify $\text{Mod}(\mathcal{C})$ with the category of models of $T$ (with morphisms given by elementary embeddings): that is the content of the theorem from the previous lecture.

We now return to a question from the previous lecture: given a category $\mathcal{C}$, can we construct a first-order theory $T$ whose weak syntactic category is $\mathcal{C}$?

**Construction 10.** Let $\mathcal{C}$ be a small coherent category. We define a typed first-order theory $T(\mathcal{C})$ as follows:

- The types of $T(\mathcal{C})$ are the objects of $\mathcal{C}$. We use uppercase letters like $X$ and $Y$ to denote these types, and the corresponding lowercase letters $x$, $y$, etcetera to denote variables of those types.
- For every morphism $f : X \to Y$ in $\mathcal{C}$, the language of $T(\mathcal{C})$ has a single predicate $P_f$, of arity $(X,Y)$.

By definition, a structure for the language $T(\mathcal{C})$ is a rule which associates to each object $X \in \mathcal{C}$ a set $M[X]$, and to each morphism $f : X \to Y$ a relation $M[P_f] \subseteq M[X] \times M[Y]$. We now list the axioms of $T(\mathcal{C})$, along with the constraints they place on a structure $M$:

- For every $f : X \to Y$, we have an axiom $(\forall x)(\exists! y)[P_f(x,y)]$. (So that $M[P_f]$ is the graph of a function $f_M : M[X] \to M[Y]$.)
- If $i : X \to X$ is the identity morphism, we have an axiom $(\forall x)[P_i(x,x)]$. (So that $i_M : M[X] \to M[X]$ is the identity map.)
- Given a pair of composable morphisms $f : X \to Y$ and $g : Y \to Z$, we have an axiom $(\forall x,y,z)[(P_f(x,y) \land P_g(y,z)) \Rightarrow P_{gf}(x,z)]$. (So that $g_M \circ f_M = (g \circ f)_M$.)

These first axioms guarantee that a model $M$ of $T(\mathcal{C})$ can be viewed as a functor from $\mathcal{C}$ to the category of sets. We now add additional axioms to guarantee that this functor has nice properties:

- If $1$ is a final object of $\mathcal{C}$ and $e$ is a variable of type $1$, we have an axiom $(\exists! e)[e = e]$ (So that $M$ preserves final objects.)
- For every pullback square

\[
\begin{array}{ccc}
X' & \rightarrow & X \\
\downarrow & & \downarrow f \\
Y' & \rightarrow & Y
\end{array}
\]

in $\mathcal{C}$, we have an axiom $(\forall x,y,y')[P_f(x,y) \land P_g(y',y)] \Rightarrow (\exists! x')(P_{f'}(x',x) \land P_{g'}(x',y'))$ (So that $M$ preserves pullback squares.)
- If $f : X \to Y$ is an effective epimorphism in $\mathcal{C}$, we have an axiom $(\forall y)(\exists x)[P_f(x,y)]$ (so that $M$ carries effective epimorphisms to surjections of sets).
• If \( X \) is an object of \( \mathcal{C} \) and \( f : X_0 \to X \) is a monomorphism which exhibits \( X_0 \) as the smallest element of \( \text{Sub}(X) \), then we have an axiom \( \neg (\exists x_0)[x_0 = x_0] \) (so that \( M \) carries the smallest element of \( \text{Sub}(X) \) to the empty set).

• If \( X \) is an object of \( \mathcal{C} \) which is given as the join of subobjects \( f : Y \to X \) and \( g : Z \to X \), then we have an axiom \((\forall x)[(\exists y)[P_f(y,x)] \vee (\exists z)[P_g(z,x)]]\) (so that \( M \) carries joins in \( \text{Sub}(W) \) to unions of subsets of \( M[W] \)).

The theory \( T(\mathcal{C}) \) has the property that models of \( T(\mathcal{C}) \) are the same as models of \( \mathcal{C} \). Let us now make that idea more precise:

**Construction 11.** We define a functor \( \lambda : \mathcal{C} \to \text{Syn}_0(T(\mathcal{C})) \) as follows:

• For each object \( X \in \mathcal{C} \), we set \( \lambda(X) = [x = x] \), where \( x \) is some variable of the type \( X \).

• For every morphism \( f : X \to Y \) in \( \mathcal{C} \), we let \( \lambda(f) : \lambda(X) \to \lambda(Y) \) denote the morphism in \( \mathcal{C} \) defined by the formula \( P_f(x,y) \).

By construction, a model \( M \) of \( T(\mathcal{C}) \) can be viewed as a morphism of coherent categories \( M : \mathcal{C} \to \text{Set} \), so that \( M[P_f(x,y)] \) is the graph of a function \( f_M \) from \( M[X] \) to \( M[Y] \). Moreover, since we have \((g \circ f)_M = g_M \circ f_M\) for each \( M \), it follows that \( \lambda(g \circ f) = \lambda(g) \circ \lambda(f) \). Similarly, \( \lambda(\text{id}_X) = \text{id}_{\lambda(X)} \), so that \( \lambda \) is a functor from \( \mathcal{C} \) to \( \text{Syn}_0(T(\mathcal{C})) \). Moreover, this functor preserves finite limits, effective epimorphisms, and joins of subobjects (since these properties can be tested in every model of \( T(\mathcal{C}) \)). In other words, \( \lambda \) is a morphism of coherent categories.

Note that the identification
\[
\{\text{Models of } T(\mathcal{C})\} \simeq \{\text{Models of } \mathcal{C}\}
\]
is simply given by composition with the functor \( F \) of Construction 11. This composition determines a functor
\[
\text{Mod}(T(\mathcal{C})) \simeq \text{Mod}(\text{Syn}_0(T(\mathcal{C}))) \to \text{Mod}(\mathcal{C}).
\]
By construction, the composite functor is bijective on objects. Beware that it is not necessarily an equivalence of categories. Our next goal is to discuss the following:

**Theorem 12.** Let \( \mathcal{C} \) be a small coherent category. Then the functor \( \lambda : \mathcal{C} \to \text{Syn}_0(T(\mathcal{C})) \) of Construction 11 is an equivalence of categories if and only if \( \mathcal{C} \) is Boolean.

**Remark 13.** The “only if” direction is obvious, since the weak syntactic category \( \text{Syn}_0(T(\mathcal{C})) \) is Boolean.

**Remark 14.** In the situation of Theorem 12, we can think of the weak syntactic category \( \text{Syn}_0(T(\mathcal{C})) \) as a “Booleanization” of \( \text{Syn}_0(\mathcal{C}) \). If \( \mathcal{C} \to \mathcal{D} \) is any morphism of coherent categories, then \( f \) can be completed to a diagram
\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{f} & \mathcal{D} \\
\downarrow{\lambda_\mathcal{C}} & & \downarrow{\lambda_\mathcal{D}} \\
\text{Syn}_0(T(\mathcal{C})) & \xrightarrow{\text{Syn}_0(f)} & \text{Syn}_0(T(\mathcal{D})).
\end{array}
\]
which commutes up to canonical isomorphism. If \( \mathcal{D} \) is Boolean, then Theorem 12 guarantees that \( \lambda_\mathcal{D} \) is an equivalence, so that \( f \) is isomorphic to the composition \( \mathcal{C} \xrightarrow{\lambda^{-1}_\mathcal{D}} \text{Syn}_0(T(\mathcal{D})) \xrightarrow{g} \text{Syn}_0(T(\mathcal{C})) \). It is possible to show that this factorization is essentially unique (but this requires additional input).
Example 15 (The Theory of Groups). We will see later that there is a natural way to choose a coherent category $\mathcal{C}$ for which the category Mod($\mathcal{C}$) is equivalent to the category whose objects are groups and whose morphisms are group homomorphisms. In this case, $T(\mathcal{C})$ would be equivalent to the category whose objects are groups and whose morphisms are elementary embeddings of groups. Here we could replace groups by other mathematical structures of a similar flavor (abelian groups, rings, Lie algebras, etcetera).

Example 16. A topological space $X$ is said to be *spectral* if it satisfies the following conditions:

- The quasi-compact open subsets of $X$ form a basis for the topology of $X$.
- The space $X$ is quasi-compact, and the intersection $U \cap V$ is quasi-compact whenever $U, V \subseteq X$ are quasi-compact open sets.
- Every irreducible closed subset of $X$ has a unique generic point.

For example, the underlying topological space of any quasi-compact and quasi-separated scheme is spectral (and conversely, due to a theorem of Hochster).

Let $X$ be a spectral space. We say that a subset $K \subseteq X$ is *constructible* if it belongs to the Boolean algebra generated by the quasi-compact open subsets of $X$. We can then equip $X$ with a new topology, called the constructible topology, by taking the constructible subsets of $X$ as a basis. Let us denote the resulting topological space by $X^c$; one can show that it is a *Stone space* (that is, it is compact, Hausdorff, and totally disconnected).

The construction $X \mapsto X^c$ can be regarded as a special case of the Booleanization procedure of Construction 11. If $X$ is a spectral space, then the collection of quasi-compact open subsets of $X$ forms a distributive lattice $P$, which we can regard as a coherent category. Then the Booleanization $\text{Syn}_0(T(P))$ is a Boolean coherent category in which every object admits a monomorphism to the final object, and is therefore equivalent to a Boolean algebra. This turns out to be the Boolean algebra of constructible subsets of $X$, or equivalently of quasi-compact open subsets of $X^c$.

In the situation above, we can identify $X$ with the set of equivalence classes of models of $P$, and $X^c$ with the set of equivalence classes of models of $\text{Syn}_0(T(P))$. The fact that the topological spaces $X$ and $X^c$ have the same points is an illustration of the general fact that a coherent category $\mathcal{C}$ and its Booleanization $\text{Syn}_0(T(\mathcal{C}))$ have “the same” models. However, the *categories* of models need not be equivalent. In the example of a spectral space $X$, this corresponds to the observation that in general there can be closure relations between points of $X$ (that is, it is possible for a point $x \in X$ to lie in the closure of a different point $y \in X$), but not in $X^c$ (since $X^c$ is a Hausdorff space).

We now begin the proof of Theorem 12 (we will continue in the next lecture).

**Proposition 17.** Let $\mathcal{C}$ be a small Boolean coherent category. Let $X$ be an object of $\mathcal{C}$ which is given as a product $\prod_{1 \leq i \leq n} X_i$, and suppose that $\varphi(x_1, \ldots, x_n)$ is a formula in the language of $T(\mathcal{C})$ whose variables $x_i$ have type $X_i$. Then there exists a subobject $Y \subseteq X$ such that $\lambda(Y)$ and $[\varphi(x_1, \ldots, x_n)]$ coincide as subobjects of $\lambda(X_1) \times \cdots \times \lambda(X_n) \simeq \lambda(X)$.

**Proof.** We proceed by induction on the construction of the formula $\varphi$. There are five cases:

(i) Suppose $\varphi(\vec{x})$ has the form $x_i = x_j$, for some pair $i, j$ with $X_i = X_j$. In this case, we can take $Y$ to be the fiber product $X \times_{X_i \times X_j} X_i$.

(ii) Suppose that $\varphi(\vec{x})$ has the form $P_f(x_i, x_j)$, where $f : X_i \to X_j$ is a morphism in $\mathcal{C}$. In this case, we take $Y$ to be the fiber product $X \times_{X_i \times X_j} X_i$ (where $X_i$ is embedded in the product $X_i \times X_j$ as the graph of $f$).

(iii) Suppose that $\varphi(\vec{x})$ has the form $\varphi_0(\vec{x}) \lor \varphi_1(\vec{x})$. By our inductive hypothesis we can assume that there are subobjects $Y_0, Y_1 \subseteq X$ satisfying $\lambda(Y_0) = [\varphi_0(\vec{x})]$ and $\lambda(Y_1) = [\varphi_1(\vec{x})]$ (as subobjects of $\lambda(X)$). We then take $Y = Y_0 \lor Y_1$. 


Suppose that $\varphi(\vec{x})$ has the form $\neg \psi(\vec{x})$. By the inductive hypothesis, we can choose a subobject $Y' \subseteq X$ such that $\lambda(Y') = [\psi(\vec{x})]$ as subobjects of $\lambda(X)$. Our assumption that $\mathcal{C}$ is Boolean guarantees that $Y'$ has a complement $Y \in \text{Sub}(X)$. Since $F$ induces a Boolean algebra homomorphism $\text{Sub}(X) \to \text{Sub}(F(X))$, it follows that $\lambda(Y) = [\varphi(\vec{x})]$ (as subobjects of $\lambda(X)$).

Suppose that $\varphi(\vec{x})$ has the form $(\exists z)[\psi(\vec{x}, z)]$, where $z$ is a variable of type $Z$. In this case, our inductive hypothesis guarantees that there exists a subobject $Y \subseteq X \times Z$ such that $\lambda(Y) = [\psi(\vec{x}, z)]$ as subobjects of $\lambda(X \times Z) \simeq \lambda(X) \times \lambda(Z)$. Let $Y$ denote the image of the composite map $Y \hookrightarrow X \times Z \to X$. Since $F$ preserves images, it follows that $\lambda(Y)$ is the image of the map $[\psi(\vec{x}, z)] \to F(X)$, which coincides with $[\varphi(\vec{x})]$. \qed