Lecture 27X-Ultrafunctors

April 17, 2018

We now study functors between ultracategories.

**Definition 1.** Let \( \pi : \mathcal{E} \to \text{Stone}^{fr} \) and \( \pi' : \mathcal{E}' \to \text{Stone}^{fr} \) be ultracategory fibrations. A *morphism of ultracategory fibrations* from \( \mathcal{E} \) to \( \mathcal{E}' \) is a functor \( F : \mathcal{E} \to \mathcal{E}' \) with the following properties:

1. The diagram

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{F} & \mathcal{E}' \\
\pi \downarrow & & \downarrow \pi' \\
\text{Stone}^{fr} & & \\
\end{array}
\]

commutes. In particular, for every \( X \in \text{Stone}^{fr} \), \( F \) induces a functor \( F_X : \mathcal{E}_X \to \mathcal{E}'_X \).

2. The functor \( F \) carries locally \( \pi \)-Cartesian morphisms of \( \mathcal{E} \) to locally \( \pi' \)-Cartesian morphisms of \( \mathcal{E}' \).

We let \( \text{Mor}(\mathcal{E},\mathcal{E}') \) denote the full subcategory of \( \text{Fun}(\mathcal{E},\mathcal{E}') \times_{\text{Fun}(\mathcal{E},\text{Stone}^{op})} \{ \pi \} \) consisting of morphisms of ultracategory fibrations from \( \mathcal{E} \) to \( \mathcal{E}' \).

We will be primarily interested in the following example:

**Proposition 2.** Let \( \mathcal{C} \) and \( \mathcal{C}' \) be small pretopoi, and let \( f : \mathcal{C} \to \mathcal{C}' \) be a pretopos morphism (that is, a functor which preserves finite limits, finite coproducts, and effective epimorphisms). If \( X \) is a topological space and \( \mathcal{O}_X : \mathcal{C}' \to \text{Shv}(X) \) is an \( X \)-model of \( \mathcal{C}' \), then \( \mathcal{O}_X \circ f : \mathcal{C} \to \text{Shv}(X) \) is an \( X \)-model of \( \mathcal{C} \). It follows that the construction \( (X,\mathcal{O}_X) \to (X,\mathcal{O}_X \circ f) \) determines a functor \( F : \text{Top}_{\mathcal{C}'} \to \text{Top}_{\mathcal{C}} \). Then:

1. The functor \( F \) carries \( \text{Stone}_{\mathcal{C}'} \) into \( \text{Stone}_{\mathcal{C}} \) and \( \text{Stone}_{\mathcal{C}'}^{fr} \) into \( \text{Stone}_{\mathcal{C}}^{fr} \).

2. The induced map \( F : \text{Stone}_{\mathcal{C}'} \to \text{Stone}_{\mathcal{C}} \) is a morphism of ultracategory fibrations.

**Proof.** It is clear that \( F \) carries \( \text{Stone}_{\mathcal{C}'} \) into \( \text{Stone}_{\mathcal{C}} \). Let us identify \( \text{Stone}_{\mathcal{C}}^{op} \) and \( \text{Stone}_{\mathcal{C}'}^{op} \) with the full subcategories of \( \text{Fun}(\mathcal{C},\text{Set}) \) and \( \text{Fun}(\mathcal{C}',\text{Set}) \) spanned by those functors which preserve finite limits and effective epimorphisms. Under these identifications, the functor \( F_{|\text{Stone}_{\mathcal{C}'}} : \text{Stone}_{\mathcal{C}'} \to \text{Stone}_{\mathcal{C}} \) is given by precomposition with \( f \). It follows that \( F_{|\text{Stone}_{\mathcal{C}'}} : \text{Stone}_{\mathcal{C}'} \to \text{Stone}_{\mathcal{C}} \) commutes with coproducts (since these correspond to products in \( \text{Fun}(\mathcal{C},\text{Set}) \) and \( \text{Fun}(\mathcal{C}',\text{Set}) \)). Since \( F \) carries \( \text{Mod}(\mathcal{C}')^{op} \subseteq \text{Stone}_{\mathcal{C}'} \) into \( \text{Mod}(\mathcal{C}) \subseteq \text{Stone}_{\mathcal{C}} \), it restricts to a functor \( F : \text{Stone}_{\mathcal{C}'}^{fr} \to \text{Stone}_{\mathcal{C}}^{fr} \). By construction, this functor fits into a commutative diagram

\[
\begin{array}{ccc}
\text{Stone}_{\mathcal{C}'}^{fr} & \xrightarrow{F} & \text{Stone}_{\mathcal{C}}^{fr} \\
\pi' \downarrow & & \downarrow \pi \\
\text{Stone}_{\mathcal{C}'} & & \\
\end{array}
\]

We claim that \( F \) carries locally \( \pi \)-Cartesian morphisms in \( \text{Stone}_{\mathcal{C}'}^{fr} \) to locally \( \pi \)-Cartesian morphisms in \( \text{Stone}_{\mathcal{C}}^{fr} \). Note that a morphism \( g : (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y) \) in the category \( \text{Stone}_{\mathcal{C}'}^{fr} \) is locally \( \pi' \)-Cartesian if and only if, for every isolated point \( x \in X \), the induced map \( \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x} \) is an isomorphism in \( \text{Mod}(\mathcal{C}') \). It then follows that \( \mathcal{O}_{Y,f(x)} \circ f \to \mathcal{O}_{X,x} \circ f \) is an isomorphism of models of \( \mathcal{C} \), so that \( F(g) \) is locally \( \pi \)-Cartesian. \( \boxed{} \)
We can now give a precise statement of Makkai’s theorem:

**Theorem 3** (Strong Conceptual Completeness). Let $\mathcal{C}$ and $\mathcal{C}'$ be small pretopoi, and let $\text{Fun}^{\text{coh}}(\mathcal{C}, \mathcal{C}')$ denote the category of pretopos morphisms from $\mathcal{C}$ to $\mathcal{C}'$. Then the preceding construction induces an equivalence of categories

$$\text{Fun}^{\text{coh}}(\mathcal{C}, \mathcal{C}') \rightarrow \text{Mor}(\text{Stone}_{\mathcal{C}'}, \text{Stone}_{\mathcal{C}})^{\text{op}}.$$  

We will give the proof of Theorem 3 over the next two lectures. First, let us try to describe more concretely what it is saying. Let’s return to the case of a general pair of ultracategory fibrations

$$\pi : \mathcal{E} \rightarrow \text{Stone}^{\text{fr}} \quad \pi' : \mathcal{E}' \rightarrow \text{Stone}^{\text{fr}}.$$  

Suppose we are given a continuous map $f : \beta I \rightarrow \beta J$ in $\text{Stone}^{\text{fr}}$. Then pullback along $f$ induces functors

$$\mathcal{E}_{\beta J} \rightarrow \mathcal{E}_{\beta I} \quad \mathcal{E}'_{\beta J} \rightarrow \mathcal{E}'_{\beta I},$$

both of which we will denote by $f^*$. Condition (2) guarantees that the diagram

$$\begin{array}{ccc}
\mathcal{E}_{\beta J} & \xrightarrow{f^*} & \mathcal{E}_{\beta I} \\
\downarrow^{F_{\beta J}} & & \downarrow^{F_{\beta I}} \\
\mathcal{E}'_{\beta J} & \xrightarrow{f^*} & \mathcal{E}'_{\beta I}
\end{array}$$

Let $M = \mathcal{E}^{\text{op}}$ and $M' = \mathcal{E}'^{\text{op}}$ denote the underlying categories of $\mathcal{E}$ and $\mathcal{E}'$, respectively. Then $F$ induces a functor $F_0 : M \rightarrow M'$. For every set $I$, we have equivalences

$$\mathcal{E}^\text{op}_{\beta I} \simeq M^I \quad \mathcal{E}'^\text{op}_{\beta I} \simeq M'^I$$

which fit into a commutative diagram

$$\begin{array}{ccc}
\mathcal{E}^\text{op}_{\beta I} & \xrightarrow{F_{\beta I}} & \mathcal{E}'^\text{op}_{\beta I} \\
\downarrow \sim & & \downarrow \sim \\
M^I & \xrightarrow{F_0^I} & M'^I.
\end{array}$$

Consequently, a morphism of ultracategory fibrations $F : \mathcal{E} \rightarrow \mathcal{E}'$ is largely determined by the underlying functor $F_0 : M \rightarrow M'$. However, the functor $F_0$ is not arbitrary: applying the preceding paragraph in the case where $I = *$ is a single point and $J$ is an arbitrary set, (so that $f : \beta I \rightarrow \beta J$ determines an ultrafilter $\mathcal{U}$ on $J$), we deduce that $F_0$ “commutes with ultraproducts indexed by $\mathcal{U}$”, in the sense that we have a commutative diagram

$$\begin{array}{ccc}
M^I & \xrightarrow{F_0^I} & M'^I \\
\downarrow^{P_{\mathcal{U}}} & & \downarrow^{P'_{\mathcal{U}}} \\
M & \xrightarrow{F_0} & M'.
\end{array}$$

where $P_{\mathcal{U}} : M^I \rightarrow M$ and $P'_{\mathcal{U}} : M'^I \rightarrow M'$ are the maps given by $f^*$. This motivates the following:

**Definition 4.** Let $M$ and $M'$ be ultracategories, equipped with functors

$$P_{\mathcal{U}} : M^I \rightarrow M \quad P'_{\mathcal{U}} : M'^I \rightarrow M'$$

where $\mathcal{U}$ is an ultrafilter on a set $I$, together with isomorphisms

$$\epsilon_{I,i} : P_{\mathcal{U}} \simeq \text{ev}_i \quad \epsilon'_{I,i} : P'_{\mathcal{U}} \simeq \text{ev}_i$$
when $\mathcal{U}$ is the principal ultrafilter associated to an element $i \in I$, and “diagonal” maps
\[
\mu_{U,V} \cdot : P^{|U|,V} \to P^U \circ \{P^{|V|}_i\}_{i \in I} \quad \mu_{U,V} \cdot : P'^{|U|,V} \to P'^U \circ \{P'^{|V|}_i\}_{i \in I}.
\]

when $\mathcal{U}$ is an ultrafilter on a set $I$ and $\{V_i\}_{i \in I}$ is an $I$-indexed collection of ultrafilters on a set $J$.

An ultrafunctor from $M$ to $M'$ consists of the following data:

- A functor $F_0 : M \to M'$.
- For every set $I$ and every ultrafilter $\mathcal{U}$ on a set $I$, an isomorphism $\gamma_\mathcal{U} : P^U \circ F_0 \simeq F_0 \circ P^U$ of functors from $M'$ to $M'$.

These isomorphisms are required to satisfy the following conditions:

- If $\mathcal{U}$ is the principal ultrafilter associated to an element $i \in I$, then the diagram
  \[
  \begin{array}{ccc}
  P'^U \circ F_0 \gamma_\mathcal{U} & \xrightarrow{\gamma_{|U|,V}} & F_0 \circ P^U \\
  \downarrow_{\epsilon_{i,i}} & & \downarrow_{\epsilon_{i,i}} \\
  \text{ev}_i \circ F_0 & \simeq & F_0 \circ \text{ev}_i
  \end{array}
  \]
  commutes (in the category of functors from $M'$ to $M'$).

- If $\mathcal{U}$ is an ultrafilter on a set $I$ and $\{V_i\}_{i \in I}$ is an $I$-indexed collection of ultrafilters on a set $J$, then the diagram
  \[
  \begin{array}{ccc}
  P'^{|U|,V} \cdot \circ F_0^J & \xrightarrow{\gamma_{|U|,V} \cdot} & F_0 \circ P^{|U|,V} \\
  \downarrow_{\mu_{U,V} \cdot} & & \downarrow_{\mu_{|U|,V} \cdot} \\
  P^{|U|} \circ \{P^{|V|}_i\} \circ F_0^J & \xrightarrow{\gamma_{|U|} \cdot} & P^{|U|} \circ F_0 \circ \{P^{|V|}_i\}_{i \in I} \\
  \downarrow_{\gamma_{|U|}} & & \downarrow_{\gamma'_{|U|}} \\
  F_0 \circ P^{|U|} & \overset{\gamma_{|U|}}{\longrightarrow} & F_0 \circ P^{|U|} \\
  \end{array}
  \]
  commutes (in the category of functors from $M'$ to $M'$).

Given ultrafunctors $(F_0, \{\gamma_\mathcal{U}\})$ and $(F'_0, \{\gamma'_\mathcal{U}\})$ from $M$ to $M'$, we will say that a natural transformation $\rho : F_0 \to F'_0$ is a morphism of ultrafunctors if, for every ultrafilter $\mathcal{U}$ on a set $I$, the diagram of natural transformations
\[
\begin{array}{ccc}
  P'^{|U|,V} \cdot \circ F_0^J & \xrightarrow{\gamma_{|U|} \cdot} & F_0 \circ P^{|U|,V} \\
  \downarrow_{\gamma_{|U|}} & & \downarrow_{\gamma'_{|U|}} \\
  F_0 \circ P^{|U|} & \overset{\rho}{\longrightarrow} & F'_0 \circ P^{|U|}
  \end{array}
\]
commutes. We let $\text{Fun}_{M}^{U}(M,M')$ denote the category whose objects are ultrafunctors from $M$ to $M'$ and whose morphisms are morphisms of ultrafunctors.

**Warning 5.** Makkai introduced a notion of ultrafunctor between ultracategories which is *a priori* more restrictive than our Definition 4: that is, Makkai requires a larger collection of diagrams to commute. However, the difference turns out to be irrelevant in the primary case of interest to us (where $M$ is the category of models of a small pretopos $\mathcal{C}$), by virtue of Theorem 8 below.

**Example 6.** Let $X$ and $X'$ be compact Hausdorff spaces. We saw in the previous lecture that we can think of $X$ and $X'$ as *ultrasets*: that is, as ultracategories with only trivial morphisms. In this situation, Definition 4 simplifies considerably: an ultrafunctor from $X$ to $X'$ is simply a map of sets $F_0 : X \to X'$ with the following property: for every map of sets $f : I \to X$ and every ultrafilter $\mathcal{U}$ on $I$, we have $F_0(P^{|U|}(f)) = P^{|U|}(F_0 \circ f)$.
in $X'$. By general nonsense, it suffices to check this equality in the case where $I = X$ and $f$ is the identity. It follows that $F_0$ is an ultrafunctor if and only if it is a morphism of $\beta$-algebras: that is, the diagram

$$
\begin{array}{ccc}
\beta X & \xrightarrow{\beta(F_0)} & \beta X' \\
\downarrow & & \downarrow \\
X & \xrightarrow{F_0} & X'
\end{array}
$$

commutes, where the vertical maps are the continuous extensions of $\text{id}_X$ and $\text{id}_{X'}$, respectively. This is equivalent to the requirement that $F_0$ is continuous.

Let $\pi : \mathcal{E} \to \text{Stone}^{\text{fr}}$ and $\pi' : \mathcal{E}' \to \text{Stone}^{\text{fr}}$ be ultracategory fibrations, with underlying ultracategories $M$ and $M'$, respectively. If $F : \mathcal{E} \to \mathcal{E}'$ is a morphism of ultracategory fibrations (in the sense of Definition 1), then it is not difficult to see that the underlying functor $F_0 : M \to M'$ has the structure of an ultrafunctor (with isomorphisms $\gamma_U$ defined as in the discussion preceding Definition 4). Passage from $F$ to $F_0$ determines a functor

$$\text{Mor}(\mathcal{E}, \mathcal{E}') \to \text{Fun}^{\text{Ult}}(M, M')^{\text{op}}.$$ 

Moreover, just as an ultracategory fibration can be recovered (up to equivalence) from its underlying ultracategory, a morphism of ultracategory fibrations can be recovered (up to isomorphism) from its underlying ultrafunctor. More precisely, we can apply the analysis of Lecture 25X to obtain the following:

**Proposition 7.** Let $\pi : \mathcal{E} \to \text{Stone}^{\text{fr}}$ and $\pi' : \mathcal{E}' \to \text{Stone}^{\text{fr}}$ be ultracategory fibrations with underlying ultracategories $M = \mathcal{E}_{\ast}^{\text{op}}$ and $M' = \mathcal{E}'_{\ast}^{\text{op}}$. Then the preceding construction induces an equivalence of categories

$$\text{Mor}(\mathcal{E}, \mathcal{E}')^{\text{op}} \to \text{Fun}^{\text{Ult}}(M, M').$$

Combining this result with Theorem 3, we obtain the following reformulation of Theorem 3:

**Theorem 8 (Strong Conceptual Completeness).** Let $\mathcal{C}$ and $\mathcal{C}'$ be small pretopoi and let $\text{Fun}^{\text{coh}}(\mathcal{C}, \mathcal{C}')$ denote the category of pretopos morphisms from $\mathcal{C}$ to $\mathcal{C}'$. Then there is a canonical equivalence of categories

$$\text{Fun}^{\text{coh}}(\mathcal{C}, \mathcal{C}') \to \text{Fun}^{\text{Ult}}(\text{Mod}(\mathcal{C}'), \text{Mod}(\mathcal{C})).$$