Let $\mathcal{C}$ be an essentially small pretopos, which we regard as fixed throughout this lecture. In Lecture 15X, we constructed a fully faithful embedding

$$\text{Shv}(\mathcal{C}) \rightarrow \text{Shv}(\text{Pro}(\mathcal{C})) \simeq \text{Shv}(\text{Pro}^{\text{op}}(\mathcal{C})) \simeq \text{Shv}(\text{Stone}_{\mathcal{C}}) \subseteq \text{Fun}(\text{Stone}_{\mathcal{C}}^{\text{op}}, \text{Set}).$$

The essential image of this embedding consists of those functors $F: \text{Stone}_{\mathcal{C}}^{\text{op}} \rightarrow \text{Set}$ which commute with filtered colimits and are sheaves with respect to the Grothendieck topology of Lecture 18X. Any such functor must satisfy the following condition:

(a) The functor $F: \text{Stone}_{\mathcal{C}}^{\text{op}} \rightarrow \text{Set}$ preserves finite products: that is, it carries finite coproducts in $\text{Stone}_{\mathcal{C}}$ to finite products in the category of sets.

In Lecture 17X, we proved that if $F$ is a functor satisfying (a), then it commutes with filtered colimits if and only if it satisfies the following additional conditions:

(b) For every object $(X, O_X) \in \text{Stone}_{\mathcal{C}}$ and every point $x \in X$, the canonical map

$$\lim_{x \in U} F(U, O_X |_U) \rightarrow F(\{x\}, O_{X,x})$$

is bijective; here the colimit is taken over all clopen neighborhoods $U \subseteq X$ of the point $x$.

(c) The composite functor

$$\text{Mod}(\mathcal{C}) \hookrightarrow \text{Stone}_{\mathcal{C}}^{\text{op}} \xrightarrow{F} \text{Set}$$

commutes with filtered colimits.

Our goal in this section is to characterize those functors $F: \text{Stone}_{\mathcal{C}}^{\text{op}} \rightarrow \text{Set}$ which are sheaves. It is easy to see that if $F$ is a sheaf, then it must satisfy condition (a) above. We will prove the following partial converse:

**Theorem 1.** Let $F: \text{Stone}_{\mathcal{C}}^{\text{op}} \rightarrow \text{Set}$ be a functor satisfying conditions (a) and (b). Then $F$ is a sheaf if and only if it satisfies the following further condition:

(d) For every elementary morphism $f: M \rightarrow N$ in $\text{Mod}(\mathcal{C})$, we have an equalizer diagram

$$F(M) \rightarrow F(N) \rightrightarrows \prod F(P)$$

where the product is taken over all commutative diagrams

$$M \xrightarrow{f} N \rightrightarrows P$$

in $\text{Mod}(\mathcal{C})$. 

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Here we identify $\text{Mod}(\mathcal{C})$ with the full subcategory of $\text{Stone}^{\text{op}}_e$ spanned by those pairs $(X, \mathcal{O}_X)$, where $X$ is a singleton.

**Warning 2.** In the formulation of condition $(d)$, the product $\prod \mathcal{F}(P)$ is an ill-defined object, because it is indexed by a proper class. However, the equalizer of the diagram $\mathcal{F}(N) \rightrightarrows \prod \mathcal{F}(P)$ is still well-defined as a subset of $\mathcal{F}(N)$.

**Remark 3.** Condition $(d)$ is equivalent to the requirement that the restriction $\mathcal{F}|_{\text{Mod}(\mathcal{C})}$ is a sheaf on the category $\text{Mod}(\mathcal{C})^{\text{op}}$, where we consider a collection of morphisms $\{M \to N_i\}_{i \in I}$ in $\text{Mod}(\mathcal{C})$ to be a covering in $\text{Mod}(\mathcal{C})^{\text{op}}$ if at least one of the maps $M \to N_i$ is elementary (it follows from the amalgamation property of the previous lecture that this notion of covering defines a Grothendieck topology on $\text{Mod}(\mathcal{C})^{\text{op}}$).

**Corollary 4.** The topos $\text{Shv}(\mathcal{C})$ can be identified with the full subcategory of $\text{Fun}(\text{Stone}^{\text{op}}_e, \text{Set})$ spanned by those functors $\mathcal{F}$ which satisfy conditions $(a)$, $(b)$, $(c)$ and $(d)$ above.

Before giving the proof of Theorem 1, it will be convenient to revisit a construction from Lecture 17X, which gives a convenient reformulation of conditions $(a)$ and $(b)$.

**Notation 5.** Let $(X, \mathcal{O}_X)$ be an object of $\text{Stone}_e$, and let $\mathcal{F} : \text{Stone}^{\text{op}}_e \to \text{Set}$ be a functor. Let $\mathcal{U}_0(X)$ denote the collection of all clopen subsets of $X$. We define a functor $\mathcal{F}(\mathcal{O}_X) : \mathcal{U}_0(X)^{\text{op}} \to \text{Set}$ by the formula

$$\mathcal{F}(\mathcal{O}_X)(U) = \mathcal{F}(\mathcal{U}(U), \mathcal{O}_X|_U).$$

Note that the functor $\mathcal{F}$ satisfies condition $(a)$ above if and only if, for every object $(X, \mathcal{O}_X) \in \text{Stone}_e$, the functor $\mathcal{F}(\mathcal{O}_X) : \mathcal{U}_0(X)^{\text{op}} \to \text{Set}$ carries disjoint unions in $\mathcal{U}_0(X)$ to products in $\text{Set}$. In this case, $\mathcal{F}(\mathcal{O}_X)$ extends uniquely to a sheaf of sets on $X$, which we will also denote by $\mathcal{F}(\mathcal{O}_X)$.

By construction, the stalk of $\mathcal{F}(\mathcal{O}_X)$ at a point $x \in X$ is given by the direct limit $\lim_{x \in U} \mathcal{F}(\mathcal{U}(U), \mathcal{O}_X|_U)$. We therefore have a canonical map $\mathcal{F}(\mathcal{O}_X)|_x \to \mathcal{F}(\mathcal{O}_{X,x})$ (here we abuse notation by identifying the model $\mathcal{O}_{X,x}$ with the object $(\{x\}, \mathcal{O}_{X,x}) \in \text{Stone}_e$). Condition $(b)$ can then be restated as follows:

$(b')$ For each $(X, \mathcal{O}_X) \in \text{Stone}_e$ and each point $x \in X$, the map $\mathcal{F}(\mathcal{O}_X)|_x \to \mathcal{F}(\mathcal{O}_{X,x})$ is a bijection.

**Remark 6.** Let $(Y, \mathcal{O}_Y)$ be an object of $\text{Stone}_e$ and suppose we are given a map of Stone spaces $f : X \to Y$, so that $f^* \mathcal{O}_Y$ is an $X$-model of $\mathcal{C}$. If $\mathcal{F} : \text{Stone}^{\text{op}}_e \to \text{Set}$ is a functor satisfying condition $(a)$, then Notation 5 determines set-valued sheaves

$$\mathcal{F}(\mathcal{O}_Y) \in \text{Shv}(Y) \quad \mathcal{F}(f^* \mathcal{O}_Y) \in \text{Shv}(X)$$

together with a comparison map $f^* \mathcal{F}(\mathcal{O}_Y) \to \mathcal{F}(f^* \mathcal{O}_Y)$ in $\text{Shv}(X)$. If $\mathcal{F}$ satisfies condition $(b)$, then this comparison map is an isomorphism of sheaves on $X$ (this can be checked on stalks, where it follows from $(b')$).

**Remark 7.** Let $\mathcal{F} : \text{Stone}^{\text{op}}_e \to \text{Set}$ be a functor satisfying $(a)$ and $(b)$. Then, for each $(X, \mathcal{O}_X) \in \text{Stone}_e$, the canonical map

$$\mathcal{F}(X, \mathcal{O}_X) \to \prod_{x \in X} \mathcal{F}(\{x\}, \mathcal{O}_{X,x})$$

is injective. This follows from the fact that a section $s$ of the sheaf $\mathcal{F}(\mathcal{O}_X) \in \text{Shv}(X)$ is determined by its stalks $\{s_x\}_{x \in X}$.

We can now prove the “easy” direction of Theorem 1. Let

$$\mathcal{F} : \text{Stone}^{\text{op}}_e \to \text{Set}$$
be a sheaf (so that it satisfies condition (a)), and suppose that $\mathcal{F}$ also satisfies condition (b). We wish to show that it satisfies condition (d). Suppose we are given an elementary morphism $M \to N$ in $\text{Mod}(\mathcal{E})$. Then the induced map

$$(*) \to (\cdot, M)$$

is a covering in $\text{Stone}_\mathcal{E}$ (see Lecture 18X). It follows that the canonical map $\mathcal{F}(M) \to \mathcal{F}(N)$ is injective, and that its image consists of those elements $s \in \mathcal{F}(N)$ which satisfy the following condition:

$$(*) \text{ For every object } (X, \mathcal{O}_X) \in \text{Stone}_\mathcal{E} \text{ equipped with a pair of maps } (X, \mathcal{O}_X) \to (\cdot, N) \text{ which are coequalized by } (\cdot, N) \to (\cdot, M), \text{ the element } s \text{ belongs to the equalizer } \text{Eq}(\mathcal{F}(N) \to \mathcal{F}(X, \mathcal{O}_X)).$$

To verify (d), we must show that it suffices to check the criterion of (a) in the case where $X$ is a single point. This follows from the injectivity of the map $\mathcal{F}(X, \mathcal{O}_X) \to \prod_{x \in X} \mathcal{F}(\{x\}, \mathcal{O}_{X,x})$ (Remark 7).

We now tackle the hard direction. Assume that $\mathcal{F} : \text{Stone}^{\text{op}}_\mathcal{E} \to \text{Set}$ satisfies conditions (a), (b), and (d); we wish to show that $\mathcal{F}$ is a sheaf. Choose a covering $\{(X_i, \mathcal{O}_{X_i}) \to (X, \mathcal{O}_X)\}_{i \in I}$ in the category $\text{Stone}_\mathcal{E}$. For every pair $i, j \in I$, we can identify $(X_i, \mathcal{O}_{X_i}), (X_j, \mathcal{O}_{X_j})$, and $(X, \mathcal{O}_X)$ with weakly projective pro-objects $\Gamma(X_i; \mathcal{O}_{X_i}), \Gamma(X_j; \mathcal{O}_{X_j})$, and $\Gamma(X; \mathcal{O}_X)$. We can then form the fiber product

$$\Gamma(X_i; \mathcal{O}_{X_i}) \times_{\Gamma(X; \mathcal{O}_X)} \Gamma(X_j; \mathcal{O}_{X_j})$$

in $\text{Pro}(\mathcal{E})$. This fiber product might not be weakly projective. However, if we can choose a covering by a weakly projective pro-object, which we can then write in the form $\Gamma(X_{ij}, \mathcal{O}_{X_{ij}})$ for some $(X_{ij}, \mathcal{O}_{X_{ij}}) \in \text{Stone}_\mathcal{E}$). In order to show that $\mathcal{F}$ is a sheaf, we must verify that the diagram

$$\mathcal{F}(X, \mathcal{O}_X) \to \prod_i \mathcal{F}(X_i, \mathcal{O}_{X_i}) \to \prod_{i,j} \mathcal{F}(X_{ij}, \mathcal{O}_{X_{ij}})$$

is an equalizer diagram in the category of sets. Since every covering admits a finite subcover, it suffices to check this in the case where the set $I$ is finite. In this case, we can form the coproducts

$$(Y, \mathcal{O}_Y) = \coprod_{i \in I} (X_i, \mathcal{O}_{X_i}) \quad (Z, \mathcal{O}_Z) = \coprod_{i,j \in I} (X_{ij}, \mathcal{O}_{X_{ij}}).$$

Using condition (a), we are reduced to showing that the diagram

$$\mathcal{F}(X, \mathcal{O}_X) \to \mathcal{F}(Y, \mathcal{O}_Y) \Rightarrow \mathcal{F}(Z, \mathcal{O}_Z)$$

is an equalizer diagram of sets.

Let $\mathcal{G}$ denote the direct image of $\mathcal{F}(\mathcal{O}_Y) \in \text{Shv}(Y)$ along the projection map $Y \to X$, and let $\mathcal{H}$ denote the direct image of $\mathcal{F}(\mathcal{O}_Z)$ along the projection $Z \to X$. We then have a commutative diagram

$$\mathcal{F}(\mathcal{O}_X) \to \mathcal{G} \Rightarrow \mathcal{H}$$

in $\text{Shv}(X)$, and we wish to show that it becomes an equalizer diagram in $\text{Set}$ after taking global sections. To prove this, it will suffice to show that the above diagram is an equalizer in $\text{Shv}(X)$. This can be checked stalkwise: that is, we are reduced to showing that the map

$$\mathcal{F}(\mathcal{O}_X)_x \to \mathcal{G}_x \Rightarrow \mathcal{H}_x$$

is an equalizer diagram of sets, for each point $x \in X$. Let $Y_x \subseteq Y$ and $Z_x \subseteq Z$ denote the inverse images of $x$. Using (b) (in the form of Remark 7), we are reduced to showing that the diagram

$$\mathcal{F}({\{x\}}, \mathcal{O}_{X,x}) \Rightarrow \mathcal{F}(Y_x, \mathcal{O}_Y |_{Y_x}) \Rightarrow \mathcal{F}(Z_x, \mathcal{O}_Z |_{Z_x})$$

is an equalizer diagram of sets.
Since the map \((Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)\) is a covering, Lecture 18X shows that we can choose a point \(y \in Y_z\) for which the map of stalks \(\mathcal{O}_{X,z} \to \mathcal{O}_{Y,y}\) is elementary. Using condition \((d)\), we deduce that the composite map

\[ \mathcal{F}(\{x\}, \mathcal{O}_{X,x}) \xrightarrow{\psi} \mathcal{F}(Y_z, \mathcal{O}_Y | Y_z) \to \mathcal{F}(\{y\}, \mathcal{O}_{Y,y}) \]

is injective, so that \(\psi\) is injective. We will complete the proof by showing that every element \(s\) of the equalizer

\[ \text{Eq}(\mathcal{F}(Y_z, \mathcal{O}_Y | Y_z) \Rightarrow \mathcal{F}(Z_x, \mathcal{O}_Z | Z_x)) \]

belongs the image of \(\psi\). Let \(s_y \in \mathcal{F}(\{y\}, \mathcal{O}_{Y,y})\) denote the stalk of \(s\) at the point \(y\). We first claim that \(s_y\) belongs to the image of the composite map

\[ \mathcal{F}(\{x\}, \mathcal{O}_{X,x}) \xrightarrow{\psi} \mathcal{F}(Y_z, \mathcal{O}_Y | Y_z) \to \mathcal{F}(\{y\}, \mathcal{O}_{Y,y}). \]

By virtue of \((d)\), it will suffice to prove the following:

\((s')\) Given a model \(P \in \text{Mod}(\mathfrak{C})\) and a commutative diagram of models

\[ \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y} \Rightarrow P, \]

the stalk \(s_y\) belongs to the equalizer \(\text{Eq}(\mathcal{F}(\mathcal{O}_{Y,y}) \Rightarrow \mathcal{F}(P)). \)

Let us identify \(P\) with an object of \(\text{Pro}(\mathfrak{C})\), and form a pullback diagram

\[
\begin{array}{ccc}
P & \xrightarrow{} & \Gamma(Z; \mathcal{O}_Z) \\
\downarrow & & \downarrow \\
\bar{P} & \xrightarrow{} & \Gamma(Y; \mathcal{O}_Y) \times_{\Gamma(X; \mathcal{O}_X)} \Gamma(Y; \mathcal{O}_Y).
\end{array}
\]

in \(\text{Pro}(\mathfrak{C})\). Here the right vertical map is an effective epimorphism, so the left vertical map is an effective epimorphism as well. The object \(\bar{P}\) might not be weakly projective. However, we can choose an effective epimorphism \(Q \to \bar{P}\), where \(Q\) is weakly projective. We can then write \(Q = \Gamma(W, \mathcal{O}_W)\), for some object \((W, \mathcal{O}_W)\) in \(\text{Stone}_\mathfrak{C}\). By construction, the map \((W, \mathcal{O}_W) \to (\ast, P)\) is a covering in \(\text{Stone}_\mathfrak{C}\). Using Lecture 18X, we see that there exists a point \(w \in W\) for which the map of models \(P \to \mathcal{O}_{W,w}\) is elementary. Using condition \((d)\), we see that the map \(\mathcal{F}(P) \to \mathcal{F}(\mathcal{O}_{W,w})\) is injective. Consequently, to verify \((s')\), we are free to replace \(P\) by \(\mathcal{O}_{W,w}\). Let \(z \in Z\) denote the image of \(w\) under the map \((W, \mathcal{O}_W) \to (Z, \mathcal{O}_Z)\) in \(\text{Stone}_\mathfrak{C}\), so that the diagram of \((s')\) refines to a diagram

\[ \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y} \Rightarrow \mathcal{O}_{Z,z} \to P. \]

We are therefore reduced to showing that \(s_y\) belongs to the equalizer \(\text{Eq}(\mathcal{F}(\mathcal{O}_{Y,y}) \Rightarrow \mathcal{F}(\mathcal{O}_{Z,z}))\), which follows from our assumption that \(s \in \text{Eq}(\mathcal{F}(Y_z, \mathcal{O}_Y | Y_z) \Rightarrow \mathcal{F}(Z_x, \mathcal{O}_Z | Z_x)). \)

The above argument shows that we can write \(s_y\) as the image of an element \(\bar{s} \in \mathcal{F}(\{x\}, \mathcal{O}_{X,x})\). We will complete the proof by showing that \(\psi(\bar{s}) = s\). For this, it will suffice to show that \(\psi(\bar{s})\) and \(s\) have the same image in the stalk \(\mathcal{F}(\{y'\}, \mathcal{O}_{Y,y'})\) for each point \(y'\) in the fiber \(Y_x\). Using the amalgamation property of the previous lecture, we see that there exists a commutative diagram of models \(\sigma : \)

\[
\begin{array}{ccc}
\mathcal{O}_{X,x} & \xrightarrow{} & \mathcal{O}_{Y,y} \\
\downarrow & & \downarrow \\
\mathcal{O}_{Y,y'} & \xrightarrow{} & N.
\end{array}
\]
where the bottom horizontal map is elementary. As above, we can form a pullback diagram

\[
\begin{array}{ccc}
\tilde{N} & \rightarrow & \Gamma(Z;\mathcal{O}_Z) \\
\downarrow & & \downarrow \\
N & \rightarrow & \Gamma(Y;\mathcal{O}_Y) \times_{\Gamma(X;\mathcal{O}_X)} \Gamma(Y;\mathcal{O}_Y)
\end{array}
\]

in \text{Pro}(\mathcal{C}) and choose an effective epimorphism \(\Gamma(V;\mathcal{O}_V) \rightarrow \tilde{N}\) for some \((V,\mathcal{O}_V) \in \text{Stone}_\mathcal{C}\). The map \((V,\mathcal{O}_V) \rightarrow (\ast, N)\) is a covering, so there exists some point \(v \in V\) for which the map of models \(N \rightarrow \mathcal{O}_{V,v}\) is elementary. Our assumption that \(s\) belongs to the equalizer \(\text{Eq}(\mathcal{F}(Y_x;\mathcal{O}_Y|_{Y_x}) \Rightarrow \mathcal{F}(Z_x;\mathcal{O}_Z|_{Z_x}))\) then implies that the stalks \(s_y = \psi(s)_y\) and \(s_{y'}\) have the same image in \(\mathcal{F}(\{v\};\mathcal{O}_{V,v})\). It follows that \(\psi(s)_y\) and \(s_{y'}\) have the same image in \(\mathcal{F}(\{v\};\mathcal{O}_{V,v})\). Since the composite map \(\mathcal{O}_{Y,y'} \rightarrow N \rightarrow \mathcal{O}_{V,v}\) is elementary, assumption \((d)\) guarantees the injectivity of \(\mathcal{F}(\{y\};\mathcal{O}_{Y,y'}) \rightarrow \mathcal{F}(\{v\};\mathcal{O}_{V,v})\), so that we must have \(\psi(s)_y = s_{y'}\), as desired.