Lecture 19: Reconstruction of Localic Morphisms

March 9, 2018

Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of topoi. In the previous lecture, we saw that there is an object $\Omega_{\mathcal{X}/\mathcal{Y}}$ of $\mathcal{Y}$ with the following universal property: for each object $Y \in \mathcal{Y}$, we have a canonical bijection

$$\text{Hom}_Y(Y, \Omega_{\mathcal{X}/\mathcal{Y}}) \simeq \text{Sub}(f^*Y).$$

Our goal in this lecture is to explain that, if the morphism $f$ is localic, then we can recover $\mathcal{X}$ from the topos $\mathcal{Y}$ and the object $\Omega_{\mathcal{X}/\mathcal{Y}}$. For this, we need to endow $\Omega_{\mathcal{X}/\mathcal{Y}}$ with a little bit of additional structure.

**Exercise 1.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of topoi. Show that the functor

$$(Y \in \mathcal{Y}) \mapsto \{U, V \in \text{Sub}(f^*Y) : U \subseteq V\}$$

is representable by an object $\Omega_{\mathcal{X}/\mathcal{Y}}$ of $\mathcal{Y}$. Note that $\Omega_{\mathcal{X}/\mathcal{Y}}$ can then be viewed as a subobject of $\Omega_{\mathcal{X}/\mathcal{Y}} \times \Omega_{\mathcal{X}/\mathcal{Y}}$.

We now consider several (essentially equivalent) ways of looking at the object $\Omega_{\mathcal{X}/\mathcal{Y}}$:

(a) We can think of $\Omega_{\mathcal{X}/\mathcal{Y}}$ as a partially ordered object of $\mathcal{Y}$ (with partial order given by $\Omega_{\mathcal{X}/\mathcal{Y}} \subseteq \Omega_{\mathcal{X}/\mathcal{Y}} \times \Omega_{\mathcal{X}/\mathcal{Y}}$). Concretely, this means that the intersection

$$\Omega_{\mathcal{X}/\mathcal{Y}} \cap (\Omega_{\mathcal{X}/\mathcal{Y}})^{\text{op}}$$

coincides with the image of the diagonal map $\Omega_{\mathcal{X}/\mathcal{Y}} \to \Omega_{\mathcal{X}/\mathcal{Y}} \times \Omega_{\mathcal{X}/\mathcal{Y}}$ (here $(\Omega_{\mathcal{X}/\mathcal{Y}})^{\text{op}}$ denote the image of $\Omega_{\mathcal{X}/\mathcal{Y}}$ under the automorphism of $\Omega_{\mathcal{X}/\mathcal{Y}} \times \Omega_{\mathcal{X}/\mathcal{Y}}$ given by swapping the two factors), and that we have

$$\pi_1^*\Omega_{\mathcal{X}/\mathcal{Y}} \cap \pi_2^*\Omega_{\mathcal{X}/\mathcal{Y}} \subseteq \pi_2^*\Omega_{\mathcal{X}/\mathcal{Y}}$$

as subobjects of $\Omega_{\mathcal{X}/\mathcal{Y}} \times \Omega_{\mathcal{X}/\mathcal{Y}} \times \Omega_{\mathcal{X}/\mathcal{Y}}$, where

$$\pi_i : \Omega_{\mathcal{X}/\mathcal{Y}} \times \Omega_{\mathcal{X}/\mathcal{Y}} \times \Omega_{\mathcal{X}/\mathcal{Y}} \to \Omega_{\mathcal{X}/\mathcal{Y}} \times \Omega_{\mathcal{X}/\mathcal{Y}}$$

denotes the projection map which omits the $i$th factor.

The first demand encodes the requirement that the relation $\Omega_{\mathcal{X}/\mathcal{Y}}$ is reflexive and antisymmetric, and the second encodes transitivity.

(b) Rather than viewing $\Omega_{\mathcal{X}/\mathcal{Y}}$ as an object of $\mathcal{Y}$, we can identify it with the functor that it represents, given by

$$Y \mapsto \text{Hom}_Y(Y, \Omega_{\mathcal{X}/\mathcal{Y}}) \simeq \text{Sub}(f^*Y).$$

The axioms of (a) translate to the requirement that, for each $Y \in \mathcal{Y}$, the image of the inclusion map

$$\text{Hom}_Y(Y, \Omega_{\mathcal{X}/\mathcal{Y}}) \to \text{Hom}_Y(Y, \Omega_{\mathcal{X}/\mathcal{Y}} \times \Omega_{\mathcal{X}/\mathcal{Y}}) \simeq \text{Sub}(f^*Y) \times \text{Sub}(f^*Y)$$

determines a partial ordering of $\text{Sub}(f^*Y)$. Of course, this is why those axioms are satisfied in the first place (by construction, this is just the partial order on $\text{Sub}(f^*Y)$ given by inclusions of subobjects).
In other word, we can think of $\Omega_X / y$ as encoding a functor

$$y^{\text{op}} \to \{\text{Partially Ordered Sets}\}.$$  

Such a functor is representable by a partially ordered object of $y$ (in the sense of (a)) if and only if it is a sheaf with respect to the canonical topology on $y$.

(b') In the situation of (b), we can be more specific. Each the partially ordered sets $\text{Sub}(f^* Y)$ is a locale, and each $Y \to Y'$ in $y$ induces a map of posets $\text{Sub}(f^* Y') \to \text{Sub}(f^* Y)$ which preserves finite meets and arbitrary joins; that is, it can be regarded as a locale morphism from $\text{Sub}(f^* Y)$ to $\text{Sub}(f^* Y')$. We can therefore identify $\Omega_X / y$ with a functor

$$y \to \{\text{Locales}\}.$$  

We can do even better: in Lecture 17, we saw that each of the locale morphisms $\text{Sub}(f^* Y) \to \text{Sub}(f^* Y')$ is open. We therefore obtain a functor

$$y \to \{\text{Locales, Open Morphisms of Locales}\}.$$  

(c) Given any category $\mathcal{C}$ and a functor $P : \mathcal{C}^{\text{op}} \to \{\text{Partially Ordered Sets}\}$, we can form a new category $\tilde{\mathcal{C}}$ described as follows:

- The objects of $\tilde{\mathcal{C}}$ are pairs $(C, U)$, where $C \in \mathcal{C}$ and $U \in P(C)$.
- A morphism from $(C, U)$ to $(C', U')$ in $\tilde{\mathcal{C}}$ is a morphism $g : C \to C'$ with the property that $U \leq P(f)(U')$ (in the poset $P(C)$).

Note that the category $\tilde{\mathcal{C}}$ is equipped with a forgetful functor $\pi : \tilde{\mathcal{C}} \to \mathcal{C}$, given on objects by $\pi(C, U) = C$. Moreover, we can recover the original functor $P$ from $\tilde{\mathcal{C}}$ and the functor $\pi$. For each $C \in \mathcal{C}$, we can identify the poset $P(C)$ with the fiber $\pi^{-1}\{C\}$. If $g : C \to C'$ is a morphism in $\tilde{\mathcal{C}}$ and $U'$ is an element of $P(C')$, identified with an object $\tilde{C}' \in \tilde{\mathcal{C}}$ satisfying $\pi(\tilde{C}') = C'$, then we can identify $P(f)(U') \in P(C)$ with the largest element of the poset $\{\tilde{C} \in \pi^{-1}\{C\} : (\exists g : \tilde{C} \to \tilde{C}')\pi(g) = g\}$. In fact, this construction determines an equivalence

$$\{\text{Functors } P : \mathcal{C}^{\text{op}} \to \{\text{Posets}\} \} \simeq \{\text{Functors } \pi : \tilde{\mathcal{C}} \to \mathcal{C} \text{ which are fibered in posets}\}.$$  

In the case of interest, we take $\mathcal{C} = y$ and $P : y^{\text{op}} \to \{\text{Posets}\}$ to be the functor $Y \mapsto \text{Sub}(f^* y)$. In this case, we will denote the category $\tilde{\mathcal{C}}$ by $\text{Loc}(f)$; it can be described concretely as follows:

- The objects of $\text{Loc}(f)$ are pairs $(Y, U)$ where $Y$ is an object of $y$ and $U \subseteq f^* Y$ is a subobject in $\mathcal{X}$. As a mnemonic aide, we will denote such an object by $(U \subseteq f^* Y)$.
- A morphism from $(U \subseteq f^* Y)$ to $(U' \subseteq f^* Y')$ is a morphism $g : Y \to Y'$ in the topos $y$ satisfying $U \subseteq U' \times_{f^* Y'} f^* Y$ (as subobjects of $f^* Y$).

In what follows, it will be convenient to adopt perspective (c).

**Remark 2.** Let $f : \mathcal{X} \to y$ be a geometric morphism of topoi. Then the category $\text{Loc}(f)$ has a forgetful functor to $\mathcal{X}$, given by the construction $(U \subseteq f^* Y) \mapsto U$.

**Remark 3.** Let $f : \mathcal{X} \to y$ be a geometric morphism of topoi. Then the category $\text{Loc}(f)$ admits finite limits. For example, given a pair of morphisms

$$(U_0 \subseteq f^* Y_0) \to (U_{01} \subseteq f^* Y_{01}) \leftarrow (U_1 \subseteq f^* Y_1),$$

the fiber product is given by $(U_0 \times_{U_{01}} U_1 \subseteq f^*(Y_0 \times_{Y_{01}} Y_1))$.  

2
Exercise 4. Let \( f : X \to Y \) be a geometric morphism of topoi. Let us say that a collection of morphisms \( \{ (U_i \subseteq f^*Y_i) \to (U \subseteq f^*Y) \}_{i \in I} \) in the category \( \text{Loc}(f) \) is a covering if the diagram \( \{ U_i \to U \} \) is a covering in \( X \). Show that this determines a Grothendieck topology on the category \( \text{Sub}(f) \). Note that we do not require that the objects \( Y_i \) cover \( Y \).

Our next goal is to prove the following:

Theorem 5. Let \( f : X \to Y \) be a localic geometric morphism. Then there is a canonical equivalence \( X \simeq \text{Shv}(\text{Loc}(f)) \). In particular, the topos \( X \) can be recovered from the fibration \( \text{Loc}(f) \to Y \) (or, equivalently, from \( Y \) together with the partially ordered object \( \Omega_X / Y \)).

The proof of Theorem 5 will require some preliminaries.

Notation 6. Let \( X_0 \subseteq X \) be the full subcategory spanned by those objects \( X \in X \) for which there exists a monomorphism \( X \hookrightarrow f^*Y \), for some \( Y \) in \( Y \). Note that the forgetful functor

\[
\text{Loc}(f) \to X \quad (U \subseteq f^*Y) \mapsto U
\]

factors through \( X_0 \) (in fact, \( X_0 \) is defined as the essential image of this forgetful functor).

Remark 7. The subcategory \( X_0 \subseteq X \) is closed under finite limits. For example, if we are given a fiber product \( X_0 \times_{X_0} X_1 \) where \( X_0 \), \( X_01 \), and \( X_1 \) belong to \( X_0 \), then we can choose monomorphisms \( u : X_0 \hookrightarrow f^*Y_0 \) and \( v : X_1 \hookrightarrow f^*Y_1 \) for some \( Y_0, Y_1 \in Y \). In this case, \( u \) and \( v \) induce a monomorphism \( X_0 \times_{X_0} X_1 \hookrightarrow f^*(Y_0 \times Y_1) \).

Lemma 8. Let \( Z \) be a topos and let \( Z_0 \subseteq Z \) be a full subcategory satisfying the following conditions:

1. The full subcategory \( Z_0 \subseteq Z \) contains a set of generators for \( Z \).
2. The full subcategory \( Z_0 \subseteq Z \) is closed under finite limits.

Then the Yoneda embedding induces an equivalence of categories \( Z \simeq \text{Shv}(Z_0) \), where we equip \( Z_0 \) with the Grothendieck topology given by the covering families in \( Z \).

Proof. When \( Z_0 \) is small, we proved this in Lecture 10. The general case follows by writing \( Z_0 \) as a union of small subcategories satisfying (1) and (2). \( \square \)

Corollary 9. Let \( f : X \to Y \) be a localic geometric morphism of topoi, and let \( X_0 \subseteq X \) be as in Notation ??.

Then the Yoneda embedding induces an equivalence \( X \simeq \text{Shv}(X_0) \), where \( X_0 \) is equipped with the Grothendieck topology given by the covering families in \( X \).

In the situation of Theorem 5, we have a forgetful functor

\[
\pi : \text{Loc}(f) \to X_0 \quad \pi(U \subseteq f^*Y) = U.
\]

This functor preserves finite limits (Remark 3) and coverings, so that composition with \( \pi \) induces a functor \( \text{Shv}(X_0) \to \text{Shv}(\text{Loc}(f)) \). We will complete the proof by establishing the following (which does not require the assumption that \( f \) is localic):

Proposition 10. Composition with \( \pi \) induces an equivalence of categories \( \text{Shv}(X_0) \to \text{Shv}(\text{Loc}(f)) \).

Sketch. Let \( \mathcal{F} : \text{Loc}(f)^{op} \to \text{Set} \) be a sheaf. Essentially, we need to show that for an object \( (U \subseteq f^*Y) \in \text{Loc}(f) \), the set \( \mathcal{F}(U \subseteq f^*Y) \) depends only on \( U \in X_0 \), and not on the particular realization of \( U \) as a subobject of \( f^*Y \). To this end, let us regard the object \( U \in X_0 \) as fixed, and suppose we are given two different monomorphisms \( U \hookrightarrow f^*Y \) and \( U \hookrightarrow f^*Y' \). The product then defines a monomorphism \( U \hookrightarrow f^*(Y \times Y') \). We therefore have a diagram of sets

\[
\mathcal{F}(U \subseteq f^*Y) \to \mathcal{F}(U \subseteq f^*(Y \times Y')) \leftarrow \mathcal{F}(U \subseteq f^*Y').
\]

We claim that these maps are bijective. This is a special case the following:
Let $U$ be a subobject of $f^*Y$, and suppose we are given a morphism $g : Y \to Z$ such that the composite map $U \hookrightarrow f^*Y \to f^*Z$ is still a monomorphism. Then the induced map $\mathcal{F}(U \subseteq f^*Z) \to \mathcal{F}(U \subseteq f^*Y)$ is bijective.

In the situation of $(*)$, the map $(U \subseteq f^*Y) \to (U \subseteq f^*Z)$ is a covering (for the Grothendieck topology of Exercise 4). We therefore have an equalizer diagram of sets

$$\mathcal{F}(U \subseteq f^*Z) \to \mathcal{F}(U \subseteq f^*Y) \Rightarrow \mathcal{F}(U \subseteq f^*(Y \times_Z Y)).$$

Consequently, to prove $(*)$, it will suffice to show that two different projection maps $Y \times_Z Y \to Y$ induce the same map from $\mathcal{F}(U \subseteq f^*Y)$ to $\mathcal{F}(U \subseteq f^*(Y \times Y))$. It is clear that these maps agree after composition with the map

$$\mathcal{F}(U \subseteq f^*(Y \times Y)) \to \mathcal{F}(U \subseteq f^*Y)$$

given by composition with the diagonal map $\delta : Y \to Y \times Y$. Consequently, to prove assertion $(*)$ for the map $g : Y \to Z$, it will suffice to prove $(*)$ for the map $\delta : Y \to Y \times Y$. In particular, assertion $(*)$ is true whenever $g$ is a monomorphism (since in this case $\delta$ is an isomorphism). However, the map $\delta$ is always a monomorphism, and therefore satisfies $(*)$; it follows that $(*)$ is true in general.

Using $(*)$ (and the discussion which precedes it), we see that we can identify $\mathcal{F}(U \subseteq f^*Y)$ with $\mathcal{G}(U)$, for some set $\mathcal{G}(U)$ which is independent of the embedding $U \hookrightarrow f^*Y$. We leave it to the reader to verify that the construction $U \mapsto \mathcal{G}(U)$ is functorial (hint: realize $\mathcal{G}$ as the left Kan extension along the projection $\text{Loc}(f)^\text{op} \to \mathcal{X}_0^\text{op}$). We claim that $\mathcal{G}$ is a sheaf on $\mathcal{X}_0$. To prove this, suppose we are given a covering $\{U_i \to U\}$ in $\mathcal{X}_0$. Choose monomorphisms

$$U \hookrightarrow f^*Y \quad U_i \hookrightarrow f^*Y_i.$$ 

Then the diagram

$$\{(U_i \subseteq f^*(Y_i \times Y)) \to (U \subseteq f^*Y)\}$$

is a covering in $\text{Loc}(f)$. Our assumption that $\mathcal{F}$ is a sheaf then supplies an equalizer diagram

$$\mathcal{F}(U \subseteq f^*Y) \to \prod_i \mathcal{F}(U_i \subseteq f^*(Y_i \times Y)) \Rightarrow \prod_{i,j} \mathcal{F}(U_i \times_U U_j \subseteq f^*(Y_i \times Y_j \times Y)),$$

which we can rewrite as

$$\mathcal{G}(U) \to \prod_{i \in I} \mathcal{G}(U_i) \Rightarrow \prod_{i,j} \mathcal{G}(U_i \times_U U_j).$$

It is now easy to verify that the construction $\mathcal{F} \mapsto \mathcal{G}$ determines an inverse to the functor $\text{Shv}(\mathcal{X}_0) \to \text{Shv}(\text{Loc}(f))$ given by composition with $\pi$. \qed