Throughout this lecture, we let $\mathcal{C}$ denote an essentially small coherent category with disjoint coproducts (for example, a small pretopos). In the previous lecture, we proved that $\text{Pro}(\mathcal{C})$ is also a coherent category with disjoint coproducts. In particular, we can endow $\text{Pro}(\mathcal{C})$ with a finitary Grothendieck topology, where a finite collection of morphisms $\{U_i \to X\}$ is a covering if the induced map $\coprod U_i \to X$ is an effective epimorphism. We let $\text{Shv}(\text{Pro}(\mathcal{C}))$ denote the category of sheaves with respect to this topology.

**Warning 1.** The category $\text{Shv}(\text{Pro}(\mathcal{C}))$ is not a topos (note that $\text{Pro}(\mathcal{C})$ is not small).

**Example 2.** Let $X$ be a quasi-compact and quasi-separated scheme, and let $\text{Sch}^\text{et}_X$ denote the category of quasi-compact, quasi-separated schemes $U$ equipped with an étale map $U \to X$. Then $\text{Sch}^\text{et}_X$ is an essentially small coherent category, and $\text{Shv}(\text{Pro}(\text{Sch}^\text{et}_X))$ can be identified with the category of pro-étale sheaves on $X$ introduced by Bhatt-Scholze.

Similarly, Scholze’s category of pro-étale sheaves on a (quasi-compact, quasi-separated) perfectoid space $X$ can be realized as $\text{Shv}(\text{Pro}(\mathcal{C}))$, where $\mathcal{C}$ is the category of (quasi-compact, quasi-separated) perfectoid spaces which are étale over $X$.

Our first goal is to understand the relationship of $\text{Shv}(\text{Pro}(\mathcal{C}))$ with the topos $\text{Shv}(\mathcal{C})$.

**Proposition 3.** Let $\mathcal{C}$ be as above and let $\mathcal{F}: \text{Pro}(\mathcal{C})^{\text{op}} \to \text{Set}$ be a functor. Then:

1. If $\mathcal{F}$ is a sheaf on the category $\text{Pro}(\mathcal{C})$, then the restriction $\mathcal{F}|_{\mathcal{C}}^{\text{op}}$ is a sheaf on $\mathcal{C}$.
2. If $\mathcal{F}|_{\mathcal{C}}^{\text{op}}$ is a sheaf on $\mathcal{C}$ and the functor $\mathcal{F}$ commutes with filtered colimits, then $\mathcal{F}$ is a sheaf on $\text{Pro}(\mathcal{C})$.

**Proof.** We will prove (2) and leave (1) as an exercise for the reader. Assume that $\mathcal{F}|_{\mathcal{C}}^{\text{op}}$ is a sheaf and that $\mathcal{F}$ commutes with filtered colimits; we wish to show that $\mathcal{F}$ is a sheaf. For this, we must prove the following:

(a) The functor $\mathcal{F}$ carries finite coproducts in $\text{Pro}(\mathcal{C})$ to products of sets.

(b) For each effective epimorphism $U \to X$ in $\text{Pro}(\mathcal{C})$, the diagram

$$\mathcal{F}(X) \to \mathcal{F}(U) \rightrightarrows \mathcal{F}(U \times_X U)$$

is an equalizer.

We begin with (a). Suppose we are given a finite collection of objects $C_1, \ldots, C_n \in \text{Pro}(\mathcal{C})$, each of which is the limit of a pro-system $\{C_{i,\alpha}\}$ in $\mathcal{C}$; without loss of generality, we may assume that each of these pro-systems is indexed by the same category. Then the coproduct $C_1 \amalg \cdots \amalg C_n$ is given by the limit of the pro-system $\{C_{1,\alpha} \amalg \cdots \amalg C_{n,\alpha}\}$. Since $\mathcal{F}$ carries filtered limits in $\text{Pro}(\mathcal{C})$ to filtered colimits of sets, we are reduced to showing that the canonical map

$$\lim_{\alpha} \mathcal{F}(C_{1,\alpha} \amalg \cdots \amalg C_{n,\alpha}) \to \prod_{1 \leq i \leq n} \lim_{\alpha} \mathcal{F}(C_{i,\alpha})$$

is an isomorphism.
is an isomorphism, which follows from the fact that filtered colimits of sets commute with products and our assumption that $\mathcal{F}|_{\mathcal{C}^{op}}$ is a sheaf.

We now prove (b). Let $f : U \to X$ be an effective epimorphism in $\text{Pro}(\mathcal{C})$. Then we can write $f$ as the limit of a diagram $\{f_\alpha : U_\alpha \to X_\alpha\}$ of effective epimorphisms in $\mathcal{C}$. Using our assumption that $\mathcal{F}$ is compatible with filtered limits in $\text{Pro}(\mathcal{C})$, we are reduced to showing that the diagram

$$\lim_{\alpha} \mathcal{F}(X_\alpha) \to \lim_{\alpha} \mathcal{F}(U_\alpha) \Rightarrow \lim_{\alpha} \mathcal{F}(U_\alpha \times_{X_\alpha} U_\alpha).$$

This follows from our assumption that $\mathcal{F}|_{\mathcal{C}^{op}}$ is a sheaf, since the collection of equalizer diagrams in $\text{Set}$ is closed under filtered colimits. \qed

The universal property of $\text{Pro}(\mathcal{C})$ implies that any presheaf $\mathcal{F}_0 \in \text{Fun}(\mathcal{C}^{op}, \text{Set})$ admits an essentially unique extension to a presheaf $\mathcal{F} \in \text{Fun}(\text{Pro}(\mathcal{C})^{op}, \text{Set})$ which preserves filtered colimits. It follows from Proposition 3 that $\mathcal{F}_0$ is a sheaf if and only if $\mathcal{F}$ is a sheaf. This proves the following:

**Proposition 4.** Let $\text{Shv}_c(\text{Pro}(\mathcal{C}))$ denote the full subcategory of $\text{Shv}(\text{Pro}(\mathcal{C}))$ consisting of those sheaves $\mathcal{F} : \text{Pro}(\mathcal{C})^{op} \to \text{Set}$ which preserve filtered colimits. Then the restriction functor $\mathcal{F} \mapsto \mathcal{F}|_{\mathcal{C}^{op}}$ induces an equivalence of categories $\text{Shv}_c(\text{Pro}(\mathcal{C})) \to \text{Shv}(\mathcal{C})$.

Proposition 4 is the starting point of a strategy for understanding the topos $\text{Shv}(\mathcal{C})$: its objects can also be understood as sheaves on the larger coherent category $\text{Pro}(\mathcal{C})$, satisfying a certain continuity condition. This is convenient because $\text{Pro}(\mathcal{C})$ contains many useful objects that do not belong to $\mathcal{C}$.

**Definition 5.** Recall that a model of $\mathcal{C}$ is a morphism of coherent categories $M : \mathcal{C} \to \text{Set}$: that is, a functor which satisfies the following axioms:

1. The functor $M$ commutes with finite limits.
2. The functor $M$ carries effective epimorphisms in $\mathcal{C}$ to surjections of sets.
3. The functor $M$ preserves finite coproducts.

Let $\text{Mod}(\mathcal{C})$ denote the full subcategory of $\text{Fun}(\mathcal{C}, \text{Set})$ spanned by the models of $\mathcal{C}$. By definition, $\text{Pro}(\mathcal{C})$ is the opposite of the full subcategory of $\text{Fun}(\mathcal{C}, \text{Set})$ spanned by those functors which satisfy condition (1). We can therefore identify $\text{Mod}(\mathcal{C})^{op}$ with a full subcategory of $\text{Pro}(\mathcal{C})$. Note that objects of $\text{Mod}(\mathcal{C})^{op}$ very rarely belong to $\mathcal{C}$ itself (regarded as a full subcategory of $\text{Pro}(\mathcal{C})$ via the Yoneda embedding).

We will say that an object $M \in \text{Pro}(\mathcal{C})$ is weakly projective if it satisfies conditions (1) and (2). We let $\text{Pro}^{wp}(\mathcal{C})$ denote the full subcategory of $\text{Pro}(\mathcal{C})$ spanned by the weakly projective objects.

**Example 6.** Any model of $\mathcal{C}$ is weakly projective when viewed as an object of $\text{Pro}(\mathcal{C})$. That is, we have inclusions

$$\text{Mod}(\mathcal{C}) \subseteq \text{Pro}^{wp}(\mathcal{C})^{op} \subseteq \text{Pro}(\mathcal{C})^{op} \subseteq \text{Fun}(\mathcal{C}, \text{Set}).$$

**Example 7.** Suppose that $\mathcal{C}$ is the category of finite sets. Then every effective epimorphism in $\mathcal{C}$ admits a section, so condition (2) of Definition 5 is automatic: that is, we have $\text{Pro}^{wp}(\mathcal{C}) = \text{Pro}(\mathcal{C})$.

**Remark 8.** By definition, an object $X \in \text{Pro}(\mathcal{C})$ is weakly projective if and only if, for every effective epimorphism $C \to D$ in $\mathcal{C}$, the map $\text{Hom}_{\text{Pro}(\mathcal{C})}(X, C) \to \text{Hom}_{\text{Pro}(\mathcal{C})}(X, D)$ is surjective: that is, every map from $X$ to $D$ factors through $C$. It follows that $\text{Pro}^{wp}(\mathcal{C})$ is closed under (possibly infinite) coproducts in $\text{Pro}(\mathcal{C})$.

Beware that the map $\text{Hom}_{\text{Pro}(\mathcal{C})}(X, C) \to \text{Hom}_{\text{Pro}(\mathcal{C})}(X, D)$ is generally not surjective if we assume only that $C \to D$ is an effective epimorphism in $\mathcal{C}$ (this is the motivation for the using the modifier “weakly” to describe the condition of Definition 4).

**Remark 9.** The full subcategory $\text{Pro}^{wp}(\mathcal{C}) \subseteq \text{Pro}(\mathcal{C})$ is closed under filtered inverse limits (since the collection of surjections in $\text{Set}$ is closed under filtered direct limits).
The following result allows us to “resolve” any object of Pro(ℂ) by weakly projective objects:

**Proposition 10.** For every object \( X \in \text{Pro}(ℂ) \), there exists an effective epimorphism \( ρ_X : λ(X) \to X \) in \( \text{Pro}(ℂ) \) where \( λ(X) \) is weakly projective. Moreover, we can arrange that \( λ(X) \) is a functor of \( X \), that \( ρ_X \) is a natural transformation of functors, and that the functor \( λ \) commutes with filtered limits.

**Proof.** We use the small object argument of Quillen. Let \( λ \) be a natural transformation of functors, and that the functor \( λ \) admits finite limits. In general, the category \( \text{Pro}(ℂ) \) need not admit finite limits. In such cases, we must replace condition \((T1)\) appearing in Lecture 8 with the following:

\((T1')\) For every covering \( \{ U_i \to X \}_{i \in I} \) in \( \text{Pro}(ℂ) \) and every morphism \( Y \to X \) in \( ℂ \), there exists a covering \( \{ V_j \to Y \} \) for which each of the maps \( V_j \to Y \) factors through some \( U_i \).

We also need to revise the notion of sheaf. A functor \( ℱ : ℂ^{\text{op}} \to \text{Set} \) is said to be a sheaf if, for every covering \( \{ U_i \to X \} \) in \( ℂ \), the canonical map

\[ ℱ(X) \to \lim_{(U_i)} ℱ(U) \]

is a bijection, where the limit is taken over the sieve on \( X \) generated by the objects \( U_i \) (see Definition 13 of Lecture 9).

**Example 12.** Let \( ℂ \) be the category of finite sets. Then \( \text{Pro}^{\text{wp}}(ℂ) = \text{Pro}(ℂ) \) can be identified with the category of Stone spaces. The preceding topology can be described as follows: a finite collection of maps of Stone spaces \( \{ Y_i \to X \} \) is a covering if and only if the induced map \( \coprod Y_i \to X \) is surjective.

**Proposition 13.** The construction \( ℱ \mapsto ℱ|_{\text{Pro}^{\text{wp}}(ℂ)} \) induces an equivalence of categories \( \text{Shv}(\text{Pro}(ℂ)) \to \text{Shv}(\text{Pro}^{\text{wp}}(ℂ)) \). Moreover, a sheaf \( ℱ : \text{Pro}(ℂ)^{\text{op}} \to \text{Set} \) commutes with filtered colimits if and only if \( ℱ|_{\text{Pro}^{\text{wp}}(ℂ)} \) commutes with filtered colimits.

**Proof.** Let \( ℱ \in \text{Shv}(\text{Pro}(ℂ)) \). For each object \( X \in \text{Pro}(ℂ) \), let \( λ(X) \) be defined as in Proposition 11, and set \( µ(X) = λ(λ(X) \times_X λ(X)) \). We then have an equalizer diagram

\[ ℱ(X) \to ℱ(λ(X)) \rightrightarrows ℱ(µ(X)) \]
so that we can functorially recover \( F(X) \) from the values of \( F \) on weakly projective objects. This gives an explicit left inverse to the restriction functor

\[
\text{Shv}(\text{Pro}(\mathcal{C})) \to \text{Shv}(\text{Pro}^{\text{wp}}(\mathcal{C})) \quad F \mapsto F|_{\text{Pro}^{\text{wp}}(\mathcal{C})^{\text{op}}};
\]

we leave it to the reader to verify that it is a right inverse as well.

It is clear that if \( F \) commutes with filtered colimits, then so does the restriction \( F|_{\text{Pro}^{\text{wp}}(\mathcal{C})^{\text{op}}} \). The converse follows from the formula

\[
F(X) = \text{Eq}(F(\lambda(X)) \Rightarrow F(\mu(X))),
\]

since the constructions \( X \mapsto \lambda(X) \) and \( X \mapsto \mu(X) \) both preserve filtered inverse limits (as functors from \( \text{Pro}(\mathcal{C}) \) to itself).

**Corollary 14.** Let \( \text{Shv}_c(\text{Pro}^{\text{wp}}(\mathcal{C})) \) be the full subcategory of \( \text{Shv}(\text{Pro}^{\text{wp}}(\mathcal{C})) \) spanned by those sheaves \( F : \text{Pro}^{\text{wp}}(\mathcal{C})^{\text{op}} \to \text{Set} \) which preserve filtered colimits. Then there is a canonical equivalence of categories \( \text{Shv}(\mathcal{C}) \simeq \text{Shv}_c(\text{Pro}^{\text{wp}}(\mathcal{C})) \).

**Proof.** Combine Propositions 14 and 4.