Lecture 14: Locales and Topoi

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Recall that, if $X$ is an object of a coherent category $\mathcal{C}$, then the poset $\text{Sub}(X)$ is a distributive lattice. If $\mathcal{C}$ is a topos, we can say more.

**Definition 1.** A **locale** is a poset $\mathcal{U}$ with the following properties:

(a) Every subset $S \subseteq \mathcal{U}$ has a least upper bound $\bigvee S$.

It follows from (a) that every subset $S \subseteq \mathcal{U}$ also has a greatest lower bound $\bigwedge S$, given by the least upper bound of the set $\{U \in \mathcal{U} : (\forall V \in S) U \leq V\}$ of all lower bounds for $S$. In particular, every pair of elements $U, V \in \mathcal{U}$ have a meet $U \land V$.

(b) For each element $V \in \mathcal{U}$ and every set of elements $\{U_\alpha\}$, we have a distributive law

$$\bigvee_{\alpha} (U_\alpha \land V) = \bigvee_{\alpha} (U_\alpha \land V).$$

**Remark 2.** Every locale is a distributive lattice.

**Exercise 3.** Let $\mathcal{U}$ be a poset satisfying condition (a) of Definition 1. Show that $\mathcal{U}$ is a locale if and only if it is a **Heyting algebra**: that is, if and only if for every pair of elements $U, V \in \mathcal{U}$, there is an element $(U \Rightarrow V) \in \mathcal{U}$ such that $W \leq (U \Rightarrow V)$ if and only if $U \land W \leq V$.

**Example 4.** Let $B$ be a complete Boolean algebra (that is, a Boolean algebra satisfying condition (a) of Definition 1). Then $B$ is a locale.

**Example 5.** Let $X$ be a topological space and let $\mathcal{U}(X)$ be the collection of open subsets of $X$, partially ordered with respect to inclusion. Then $\mathcal{U}(X)$ is a locale. Moreover, the join $\bigvee U_\alpha$ of a collection of elements $U_\alpha \in \mathcal{U}(X)$ coincides with the set-theoretic union $\bigcup U_\alpha$, and the meet of a pair $U, V \in \mathcal{U}(X)$ is given by the set-theoretic intersection $U \cap V$.

Beware that the meet of an infinite set of elements $U_\alpha \in \mathcal{U}(X)$ usually does not coincide with the intersection $\bigcap U_\alpha$, because the intersection $\bigcap U_\alpha$ need not be open; instead, $\bigwedge U_\alpha$ is given by the interior of $\bigcap U_\alpha$. In particular, we generally have

$$\bigwedge_{\alpha} (U_\alpha \lor V) \neq \bigwedge_{\alpha} (U_\alpha \lor V).$$

**Proposition 6.** Let $X$ be a topos and let $X$ be an object of $X$. Then the poset $\text{Sub}(X)$ is a locale.

**Proof.** Every collection of objects $\{U_i \subseteq X\}_{i \in I}$ has a join, given by the image of the map $\bigsqcup_{i \in I} U_i \to X$. For
\( V \subseteq X \), we compute
\[
(\bigvee U_i) \wedge V = (\bigvee U_i) \times_X V \\
= \text{Im}(\prod_{i \in I} U_i \to X) \times_X V \\
= \text{Im}(\prod_{i \in I} (U_i \times_X V) \to V) \\
= \bigvee_{i \in I} U_i \wedge V.
\]

**Definition 7.** Let \( X \) be a topos and let \( 1 \) be the final object of \( X \). Then \( \text{Sub}(1) \) is a locale. We will refer to \( \text{Sub}(1) \) as the *underlying locale* of \( X \).

In the situation of Definition 7, the poset \( \text{Sub}(1) \) can be regarded as a full subcategory of \( X \).

**Definition 8.** Let \( X \) be a topos. We say that \( X \) is *localic* if it is generated by \( \text{Sub}(1) \): that is, if every object \( X \in X \) admits a covering \( \{U_i \to X\} \), where each \( U_i \) is a subobject of 1.

**Example 9.** Let \( \mathcal{C} \) be a category which admits finite limits, equipped with a Grothendieck topology. Suppose that \( \mathcal{C} \) is a poset (that is, every object of \( \mathcal{C} \) can be identified with a subobject of the final object). Then the topos \( \text{Shv}(\mathcal{C}) \) is localic: it is generated by objects of the form \( Lh_C \), each of which is a subobject of the final object of \( \text{Shv}(\mathcal{C}) \).

**Example 10.** Let \( X \) be a topological space. Then the topos \( \text{Shv}(X) \) is localic (this is a special case of Example 9).

We now prove a converse to Example 9.

**Exercise 11.** Let \( \mathcal{U} \) be a locale. Show that \( \mathcal{U} \) admits a Grothendieck topology, where a collection of maps \( \{U_i \to X\} \) is a covering if \( X = \bigvee U_i \).

**Proposition 12.** Let \( X \) be a localic topos, and regard the underlying locale \( \mathcal{U} = \text{Sub}(1) \) as equipped with the Grothendieck topology of Exercise 11. Then we have a canonical equivalence \( X \simeq \text{Shv}(\mathcal{U}) \).

**Proof.** We can regard \( \mathcal{U} \) as an essentially small full subcategory of \( X \) which is closed under finite limits. If \( X \) is localic, then \( \mathcal{U} \) generates \( X \), so the desired result follows as in the proof of Giraud’s theorem.

We now proceed in the reverse direction.

**Proposition 13.** Let \( \mathcal{U} \) be a locale. Then the Yoneda embedding \( h : \mathcal{U} \to \text{Fun}(\mathcal{U}^{\text{op}}, \text{Set}) \) induces an equivalence from \( \mathcal{U} \) to the poset of subobjects of 1 in \( \text{Shv}(\mathcal{U}) \).

**Proof.** We first show that, for each \( U \in \mathcal{U} \), the presheaf \( h_U \) is a sheaf. Suppose we are given a covering \( \{V_i \to V\}_{i \in I} \) in \( \mathcal{U} \); we wish to show that the canonical map
\[
\forall_{i \in I} h_U(V_i) \to \prod_{i \in I} h_U(V_i) = \prod_{i,j} h_U(V_i \wedge V_j)
\]
is an equalizer diagram. Equivalently, we wish to show that \( V \leq U \) if and only if each \( V_i \leq U \), which follows from the identity \( V = \bigvee_{i \in I} V_i \).
It is clear that each \( h_U \) is a subobject of the final object of \( \text{Shv}(\mathcal{U}) \) (note that \( h_U(V) \) is a singleton for \( V \leq U \), and empty otherwise). Conversely, let \( \mathcal{F} \in \text{Shv}(\mathcal{U}) \) be a subobject of the final object, so that \( \mathcal{F}(V) \) has at most one element for each \( V \in \mathcal{U} \). Set \( U = \bigvee_{\mathcal{F}(V) \neq \emptyset} V \). Then we have a covering \( \{V \to U\}_{\mathcal{F}(V) \neq \emptyset} \). Invoking the assumption that \( \mathcal{F} \) is a sheaf, we conclude that \( \mathcal{F}(U) \neq \emptyset \). We therefore have \( \mathcal{F}(V) = \begin{cases} * & \text{if } V \leq U \\ \emptyset & \text{otherwise.} \end{cases} \), so that \( \mathcal{F} \simeq h_U \).

We can summarize Propositions 12 and 13 more informally by saying that we have an equivalence

\[
\{ \text{Localic topoi} \} \simeq \{ \text{Locales} \}.
\]

To every localic topos \( X \), we can associate the locale \( \text{Sub}(1) \) of subobjects of the final object; to any locale \( \mathcal{U} \), we can associate a topos \( \text{Shv}(\mathcal{U}) \), and these constructions are mutually inverse (up to equivalence). In fact, we can be a bit more precise.

**Definition 14.** Let \( \mathcal{U} \) and \( \mathcal{V} \) be locales. A *morphism of locales* from \( \mathcal{V} \) to \( \mathcal{U} \) is an order-preserving map \( f^* : \mathcal{U} \to \mathcal{V} \) such that \( f^* \) preserves finite meets and arbitrary joins (equivalently, it preserves finite limits and small colimits, if we view \( \mathcal{U} \) and \( \mathcal{V} \) as categories). We let \( \text{Fun}^*(\mathcal{U}, \mathcal{V}) \) denote the full subcategory of \( \text{Fun}(\mathcal{U}, \mathcal{V}) \) spanned by the morphisms of locales from \( \mathcal{V} \) to \( \mathcal{U} \) (note that \( \text{Fun}^*(\mathcal{U}, \mathcal{V}) \) is a poset).

**Proposition 15.** Let \( \mathcal{U} \) be a locale and let \( X \) be a topos with underlying locale \( \text{Sub}(1) \). Then composition with the Yoneda embedding \( h : \mathcal{U} \to \text{Shv}(\mathcal{U}) \) induces an equivalence of categories

\[
\text{Fun}^*(\text{Shv}(\mathcal{U}), X) \to \text{Fun}^*(\mathcal{U}, \text{Sub}(1)).
\]

In other words, the category of geometric morphisms from \( X \) to \( \text{Shv}(\mathcal{U}) \) is equivalent to the poset of morphisms of locales from \( \text{Sub}(1) \) to \( \mathcal{U} \).

**Proof.** We proved in Lecture 12 that composition with \( h \) induces an equivalence of categories \( \text{Fun}^*(\text{Shv}(\mathcal{U}), X) \to \text{Fun}^*(\mathcal{U}, X) \), where \( \text{Fun}^*(\mathcal{U}, X) \) is the full subcategory of \( \text{Fun}(\mathcal{U}, X) \) spanned by those functors \( f : \mathcal{U} \to X \) which preserve finite limits and coverings. Since every object of \( \mathcal{U} \) is a subobject of the final object, any functor \( f : \mathcal{U} \to X \) which preserves finite limits automatically carries each element of \( \mathcal{U} \) to a subobject of the final object \( 1 \in X \), and can therefore be identified with a map of posets \( g : \mathcal{U} \to \text{Sub}(1) \). In this case, the assumption that \( f \) preserves finite limits translates into the assumption that \( g \) preserves finite meets, and the assumption that \( f \) preserves coverings translates into the assumption that \( g \) preserves infinite joins.

We can summarize the situation as follows: there are adjoint functors (of 2-categories)

\[
\begin{array}{ccc}
\mathcal{U} & \xrightarrow{\text{Sub}(1)} & \text{Locales} \\
\text{Topoi} & \xleftarrow{\text{Shv}(\mathcal{U})} & \text{Locales} \\
\end{array}
\]

where the construction \( \mathcal{U} \to \text{Shv}(\mathcal{U}) \) is fully faithful by virtue of Proposition 13; its essential image is the 2-category of localic topoi. It follows that for every topos \( X \), there is a universal example of a localic topos which admits a geometric morphism from \( X \), given by \( \text{Shv}(\text{Sub}(1)) \). We refer to this topos as the localic reflection of \( X \).

**Example 16.** Let \( X \) be a topological space equipped with an action of a (discrete) group \( G \). Then the category \( \text{Shv}_G(X) \) of \( G \)-equivariant sheaves on \( X \) is a topos. The subobjects of the final object of \( \text{Shv}(X) \) can be identified with open subsets of \( X \). It follows that subobjects of the final object of \( \text{Shv}_G(X) \) can be identified with \( G \)-equivariant open subsets of \( X \), or equivalently with open subsets of the quotient \( X/G \) (where we endow \( X/G \) with the quotient topology). It follows that there is a canonical map \( \text{Shv}_G(X) \to \text{Shv}(X/G) \) which exhibits \( \text{Shv}(X/G) \) as the localic reflection of \( \text{Shv}_G(X) \).