Lecture 10: Giraud's Theorem

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In the previous lecture, we showed that every topos $\mathcal{X}$ is a pretopos. Our goal in this lecture is to prove a converse to this assertion, due to Giraud. Roughly speaking, Giraud's theorem asserts that a pretopos $\mathcal{X}$ is a topos if and only if it has infinite coproducts which are compatible with pullback, and satisfies a mild set-theoretic condition guaranteeing that it is not "too big" to arise as a category of sheaves on a small category.

Definition 1. Let $\mathcal{X}$ be a pretopos which admits infinite coproducts. We will say that a collection of morphisms $\{f_i : U_i \to X\}_{i \in I}$ is a *covering* if it induces an effective epimorphism $\amalg U_i \to X$. Equivalently, $\{f_i : U_i \to X\}$ is a covering if, for every subobject $X_0 \subseteq X$ such that each f_i factors through X_0, we must have $X_0 = X$ (as subobjects of X).

Warning 2. In Lecture 8, we defined a Grothendieck topology on any coherent category $\mathcal{C}$ using a very similar notion of covering. However, Definition 1 is different because we do not require that every covering family admits a finite subcover.

Theorem 3. *Let $\mathcal{X}$ be a category. The following conditions are equivalent:*

(1) *The category $\mathcal{X}$ is a topos: that is, it can be realized as a category of sheaves* $\mathrm{Shv}(\mathcal{C})$*, where $\mathcal{C}$ is a small category with finite limits which is equipped with a Grothendieck topology.*

(2) *There exists a small category $\mathcal{C}$ and a fully faithful embedding $\mathcal{X} \hookrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ which admits a left adjoint $L : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \to \mathcal{X}$ for which L preserves finite limits.*

(3) *The category $\mathcal{X}$ satisfies Giraud's axioms:*

- $(G1)$ *The category $\mathcal{X}$ admits finite limits.*
- $(G2)$ *Every equivalence relation in $\mathcal{X}$ is effective.*
- $(G3)$ *The category $\mathcal{X}$ admits small coproducts, and coproducts in $\mathcal{X}$ are disjoint.*
- $(G4)$ *The collection of effective epimorphisms in $\mathcal{X}$ is closed under pullback.*
- $(G5)$ *The formation of coproducts commutes with pullback: that is, for every morphism $f : X \to Y$ in $\mathcal{X}$, the pullback functor $f^* : \mathcal{X}_{/Y} \to \mathcal{X}_{/X}$ preserves coproducts.*
- $(G6)$ *There exists a* set *of objects $\mathcal{U}$ of $\mathcal{X}$ which generate $\mathcal{X}$ in the following sense: for every object $X \in \mathcal{X}$, there exists a covering $\{U_i \to X\}$, where each U_i belongs to $\mathcal{U}$.*

Remark 4. In the statement of Theorem 3, we do not assume that the category $\mathcal{X}$ is small (in practice, this will only happen in trivial cases). However, we always implicitly assume that it is *locally small*: that is, the collection of morphisms $\mathrm{Hom}_{\mathcal{X}}(X, Y)$ forms a set, for every pair of objects $X, Y \in \mathcal{X}$. Note that axiom $(G6)$ would be vacuous if the category $\mathcal{X}$ were small (because we could take $\mathcal{U}$ to consist of *all* the objects of $\mathcal{X}$).

Remark 5. In Lecture 8, we defined the notion of Grothendieck topology only under the assumption that $\mathcal{C}$ admits finite limits. If we had given the definition more generally, we could add the following equivalent characterization to Theorem 3:

(1.5) The category $\mathcal{X}$ is equivalent to $\mathrm{Shv}(\mathcal{C})$, where $\mathcal{C}$ is a small category with a Grothendieck topology.

The implication (1) $\Rightarrow$ (1.5) would then be vacuous, and the implication (1.5) $\Rightarrow$ (2) follows from the sheafification construction of Lecture 9.

In the previous lecture, we showed that (1) implies (2) and that (2) implies axioms $(G1)$ through $(G5)$. Let us show that it also implies $(G6)$. Assume for simplicity that $\mathcal{X}$ is given as a full subcategory of $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}et)$, where $\mathcal{C}$ is a small category, and that the inclusion functor admits a left exact left adjoint $L : \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}et) \to \mathcal{X}$. For each object $X \in \mathcal{C}$, let $h_X : \mathcal{C}^{op} \to \mathcal{S}et$ denote the functor represented by X, given by the formula $h_X(Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$. We claim that the collection $\{Lh_X\}_{X \in \mathcal{C}}$ is a set of generators for $\mathcal{X}$. To prove this, we note that every presheaf $\mathscr{F} : \mathcal{C}^{op} \to \mathcal{S}et$ admits an effective epimorphism

$$\amalg_{X \in \mathcal{C}, \eta \in \mathscr{F}(X)} h_X \to \mathscr{F}.$$

If $\mathscr{F}$ belongs to $\mathcal{X}$, then the induced maps $\{Lh_X \to \mathscr{F}\}_{X \in \mathcal{C}, \eta \in \mathscr{F}(X)}$ are a covering of $\mathscr{F}$ in the category $\mathcal{X}$.

Our goal for the rest of this lecture is to prove the implication (3) $\Rightarrow$ (1) of Theorem 3. Let $\mathcal{X}$ be a category satisfying $(G1)$ through $(G6)$. Using $(G6)$, we can choose a small full subcategory $\mathcal{C} \subseteq \mathcal{X}$ whose objects generate $\mathcal{X}$, in the sense of $(G6)$. Enlarging $\mathcal{C}$ if necessary, we can assume that $\mathcal{C}$ is closed under finite limits (meaning that every finite diagram in $\mathcal{C}$ admits a limit in $\mathcal{X}$ which also belongs to $\mathcal{C}$). We will prove the following:

Theorem 6. *Let $\mathcal{X}$ be a category satisfying axioms $(G1)$ through $(G5)$ and let $\mathcal{C} \subseteq \mathcal{X}$ be a full subcategory of $\mathcal{X}$ which is closed under finite limits and generates $\mathcal{X}$ (in the sense of $(G6)$). Let us say that a family of morphisms $\{U_i \to X\}$ in $\mathcal{C}$ is a covering if it is a covering in $\mathcal{X}$ (in the sense of Definition 1). Then:*

- (a) *The collection of coverings families determines a Grothendieck topology on $\mathcal{C}$.*
- (b) *For every object $Y \in \mathcal{X}$, let $h_Y : \mathcal{C}^{op} \to \mathcal{S}et$ denote the functor represented by Y* on the subcategory $\mathcal{C}$, *given by the formula $h_Y(X) = \mathrm{Hom}_{\mathcal{X}}(X, Y)$. Then h_Y is a sheaf with respect to the Grothendieck topology of (a).*
- (c) *The construction $Y \mapsto h_Y$ induces an equivalence of categories $\mathcal{X} \to \mathrm{Shv}(\mathcal{C})$.*

Proof. We first prove (a). Suppose first that we are given a collection of morphisms $\{U_i \to X\}$ in $\mathcal{C}$ and a morphism $Y \to X$ in $\mathcal{C}$; we wish to show that the collection of projection maps $\{U_i \times_X Y \to Y\}$ is also a covering. In other words, we are given that the map $\amalg U_i \to X$ is an effective epimorphism in $\mathcal{X}$, and we wish to show that the induced map $\amalg(U_i \times_X Y) \to Y$ is also an effective epimorphism in $\mathcal{X}$. This is clear, since axioms $(G4)$ and $(G5)$ guarantee that pullback along the map $Y \to X$ preserves the formation of coproducts and the property of being an effective epimorphism.

Now suppose that we are given a covering $\{U_i \to X\}$ in $\mathcal{C}$ and, for each index i, a covering $\{V_{i,j} \to U_i\}$ in $\mathcal{C}$. In this case, we would like to show that the composite maps $\{V_{ij} \to X\}$ are also a covering. We proceed as in proof of the analogous statement for a coherent category. Suppose we are given a subobject $X_0 \subseteq X$ (not necessarily in $\mathcal{C}$) such that each of the composite maps $V_{i,j} \to X$ factors through X_0. Then, for each i, the maps $V_{i,j} \to U_i$ factor through $X_0 \times_X U_i$. Since the $V_{i,j}$ cover U_i, we have $X_0 \times_X U_i = U_i$: that is, each of the maps $U_i \to X$ factors through X_0. Since the maps $\{U_i \to X\}$ are a cover of X, we deduce that $X_0 = X$.

To complete the proof that the collection of coverings determines a Grothendieck topology, we should verify that if a collection of maps $\{f_i : U_i \to X\}$ is a covering whenever some f_i admits a section. This is clear (in this case, the morphism f_i itself is an effective epimorphism in $\mathcal{C}$).

We now prove (b). Fix an object $Y \in \mathcal{X}$; we wish to show that the functor $h_Y : \mathcal{C}^{op} \to \mathcal{S}et$ is a sheaf. Choose any covering $\{U_i \to X\}$ of an object $Y \in \mathcal{C}$; we wish to show that the diagram of sets

$$h_Y(X) \longrightarrow \prod h_Y(U_i) \rightrightarrows \prod h_Y(U_i \times_X U_j)$$

is an equalizer. Unwinding the definitions, we can rewrite this diagram as

$$\mathrm{Hom}_{\mathcal{X}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{X}}(\amalg_i U_i, Y) \rightrightarrows \mathrm{Hom}_{\mathcal{X}}(\amalg_{i,j} U_i \times_X U_j, Y)$$

Our assumption that $\{U_i \to X\}$ is a covering guarantees that the map $\amalg_i U_i \to X$ is an effective epimorphism. We are therefore reduced to proving that the natural map

$$\amalg_{i,j}(U_i \times_X U_j) \to (\amalg_i U_i) \times_X (\amalg_j U_j)$$

is an isomorphism, which follows from axiom $(G5)$.

To verify (c), we will need the following:

Lemma 7. *Let $\{U_i \to X\}_{i\in I}$ be a covering in $\mathcal{X}$ (not necessarily between objects of $\mathcal{C}$). Then the induced map $\amalg h_{U_i} \to h_X$ is an effective epimorphism in* $\mathrm{Shv}(\mathcal{C})$

Proof. Suppose we are given a section $s \in h_X(C)$, for some $C \in \mathcal{C}$, given by a morphism $C \to X$ in the category $\mathcal{X}$. Since $\{U_i \to X\}$ is a covering, the induced map $\amalg U_i \to X$ is an effective epimorphism in the category $\mathcal{X}$. Using $(G4)$ and $(G5)$, we deduce that the collection of maps $\{U_i \times_X C \to C\}$ is also a covering in $\mathcal{X}$. Since the objects of $\mathcal{C}$ generate $\mathcal{X}$, each $U_i \times_X C$ admits a covering $\{V_{i,j} \to U_i \times_X C\}$, where $V_{i,j} \in \mathcal{C}$. Then the collection of composite maps $\{V_{i,j} \to C\}$ are a covering in the category $\mathcal{C}$. By construction, for each pair (i,j), the image $s_{i,j} \in h_X(V_{i,j})$ of s belongs to the image of the map $h_{U_i}(V_{i,j}) \to h_X(V_{i,j})$. Allowing C and s to vary, we conclude that the maps $\{h_{U_i} \to h_X\}$ are a covering of h_X in the topos $\mathrm{Shv}(\mathcal{C})$. □

Our next goal is to show that the functor $h : \mathcal{X} \to \mathrm{Shv}(\mathcal{C})$ is fully faithful: that is, for every pair of objects $X, Y \in \mathcal{C}$, the natural map

$$\theta_X : \mathrm{Hom}_{\mathcal{X}}(X,Y) \to \mathrm{Hom}_{\mathrm{Shv}(\mathcal{C})}(h_X, h_Y)$$

is bijective. Let us regard Y as fixed. We will say that X is *good* if θ_X is bijective. The argument now proceeds in several steps:

(i) Every object $X \in \mathcal{C}$ is good: this follows from Yoneda's lemma.

(ii) Suppose that $X \in \mathcal{X}$ admits a covering $\{U_i \to X\}_{i\in I}$. If each U_i and each fiber product $U_i \times_X U_j$ is good, then X is good. To prove this, we observe that we have effective epimorphisms

$$\amalg_{i\in I} U_i \to X \qquad \amalg_{i\in I}\, h_{U_i} \to h_X$$

in the categories $\mathcal{X}$ and $\mathrm{Shv}(\mathcal{C})$, respectively (Lemma 7). Using the fact that both categories satisfy $(G5)$ (and the observation that h preserves finite limits), we obtain coequalizer diagrams

$$\amalg_{i,j\in I} U_i \times_X U_j \rightrightarrows \amalg_{i\in I} U_i \longrightarrow X$$

$$\amalg_{i,j\in I} h_{U_i \times_X U_j} \rightrightarrows \amalg_{i\in I} h_{U_i} \longrightarrow h_X.$$

It follows that the map θ_X fits into a commutative diagram of sets

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathcal{X}}(X,Y) & \longrightarrow & \prod_i \mathrm{Hom}_{\mathcal{X}}(U_i,Y) & \rightrightarrows & \prod_{i,j} \mathrm{Hom}_{\mathcal{X}}(U_i \times_X U_j, Y) \\
\downarrow{\scriptstyle \theta_X} & & \downarrow & & \downarrow \\
\mathrm{Hom}_{\mathrm{Shv}(\mathcal{C})}(h_X,h_Y) & \longrightarrow & \prod_i \mathrm{Hom}_{\mathcal{X}}(h_{U_i},h_Y) & \rightrightarrows & \prod_{i,j} \mathrm{Hom}_{\mathrm{Shv}(\mathcal{C})}(h_{U_i\times_X U_j}, h_Y)
\end{array}$$

where the rows are equalizer diagrams. Since the vertical maps in the middle and on the right are isomorphisms, it follows that θ_X is also an isomorphism.

(*iii*) Let X be an object of $\mathcal{C}$ and let $U \subseteq X$ be a subobject. Then U is good. To prove this, choose a covering $\{U_i \to U\}$, where each U_i belongs to $\mathcal{C}$. Since $\mathcal{C}$ is closed under finite limits, each of the fiber products $U_i \times_U U_j \simeq U_i \times_X U_j$ belongs to $\mathcal{C}$. It follows from (*i*) that the objects U_i and $U_i \times_X U_j$ are good, so that U is good by (*ii*).

(*iv*) Let X be an arbitrary object of $\mathcal{C}$. Choose a covering $\{U_i \to X\}$, where each U_i belongs to $\mathcal{C}$. Then each fiber product $U_i \times_X U_j$ is a subobject of $U_i \times U_j \in \mathcal{C}$, and is therefore good by (*iii*). Using (*ii*), we deduce that X is good.

This completes the proof that the functor $X \mapsto h_X$ is fully faithful. We wish to show that it is essentially surjective. Note that we have not yet used the full strength of our axioms: specifically, we have not used the fact that coproducts in $\mathcal{X}$ are disjoint, or that equivalence relations in $\mathcal{X}$ are effective.

Applying Lemma 7 in the case where $I = \emptyset$, we deduce that h carries the initial object of $\mathcal{X}$ to an initial object of $\mathrm{Shv}(\mathcal{C})$. We claim that h preserves coproducts. Fix a collection of objects $\{X_i\} \in \mathcal{X}$ with coproduct X; we wish to show that the canonical map

$$\theta : \amalg h_{X_i} \to h_X$$

is an isomorphism in $\mathcal{C}$. By virtue of Lemma 7, it is an effective epimorphism. It will therefore suffice to show that it is also a monomorphism: that is, that the diagonal map

$$\amalg h_{X_i} \to (\amalg h_{X_i} \times_{h_X} \amalg h_{X_j})$$

is an isomorphism. Using the fact that $\mathrm{Shv}(\mathcal{C})$ satisfies ($G5$) and that h is right exact, we can rewrite the codomain of this map as $\amalg_{i,j} h_{X_i \times_X X_j}$. We are therefore reduced to showing that diagonal maps $h_{X_i} \to h_{X_i \times_X X_i}$ are isomorphisms and that $h_{X_i \times_X X_j}$ is an initial object of $\mathrm{Shv}(\mathcal{C})$ for $i \neq j$. This follows from our assumption that coproducts in $\mathcal{X}$ are disjoint (axiom ($G3$)).

We now show that the functor h is essentially surjective. Choose an object $\mathscr{F} \in \mathrm{Shv}(\mathcal{C})$; we wish to show that $\mathscr{F}$ belongs to the essential image of h. We first treat the special case where $\mathscr{F} \subseteq h_X$ for some object $X \in \mathcal{X}$. Choose an effective epimorphism $\amalg h_{C_i} \to \mathscr{F}$, where each C_i belongs to $\mathcal{C}$. Setting $U = \amalg_i C_i$, we obtain an effective epimorphism $h_U \to \mathscr{F}$ for some $U \in \mathcal{X}$. In this case, the composite map $h_U \to \mathscr{F} \hookrightarrow h_X$ arises from some morphism $u : U \to X$ in $\mathcal{X}$. Since $\mathcal{X}$ is a pretopos, the morphism u factors as a composition $U \xrightarrow{u'} Y \xrightarrow{u''} X$, where u' is an effective epimorphism and u'' is a monomorphism. Note that the induced map $h_U \xrightarrow{u'} h_Y$ is an effective epimorphism in $\mathrm{Shv}(\mathcal{C})$ (by virtue of Lemma 7), and the map $h_Y \xrightarrow{u''} X$ is a monomorphism in $\mathrm{Shv}(\mathcal{C})$ (since the functor h is left exact). From the uniqueness of images in the pretopos $\mathrm{Shv}(\mathcal{C})$, we conclude that $\mathscr{F}$ is isomorphic to h_Y.

We now treat the general case. Let $\mathscr{F}$ be any sheaf on $\mathcal{C}$. As before, we can choose an effective epimorphism $h_U \to \mathscr{F}$, for some $U \in \mathcal{X}$. In this case, the fiber product $h_U \times_{\mathscr{F}} h_U$ is a sheaf on $\mathcal{C}$ which can be viewed as a subobject of $h_U \times h_U = h_{U \times U}$. It follows from the analysis above that we can choose an isomorphism $h_U \times_{\mathscr{F}} h_U \simeq h_R$ for some object $R \in \mathcal{X}$. Note that for any object $Y \in \mathcal{X}$, we have a canonical bijection

$$\mathrm{Hom}_{\mathcal{X}}(Y, R) \simeq \mathrm{Hom}_{\mathrm{Shv}(\mathcal{C})}(h_Y, h_U \times_{\mathscr{F}} h_U).$$

From this description, we see that R can be viewed as an equivalence relation on U in the pretopos $\mathcal{X}$. It follows from ($G2$) that this equivalence relation is effective: that is, there exists an effective epimorphism $U \to X$ in $\mathcal{X}$ with $R = U \times_X U$ (as subobjects of $U \times U$). Applying Lemma 7 to the covering $\{U \to X\}$, we obtain an isomorphism

$$h_X \simeq \mathrm{Coeq}(h_R \rightrightarrows h_U) \simeq \mathrm{Coeq}(h_U \times_{\mathscr{F}} h_U \rightrightarrows h_U) \simeq \mathscr{F}.$$

□