

Math 261y: von Neumann Algebras (Lecture 15)

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In the last lecture, we saw that an abelian von Neumann algebra A is determined by the Boolean algebra $P(A)$ of projections of A . Our first goal in this lecture is to describe those Boolean algebras which arise in this way. We have already seen that the Boolean algebra $P(A)$ is always complete.

Definition 1. Let B be a Boolean algebra. A *finitely additive probability measure* on B is a map $\mu : B \rightarrow [0, 1]$ such that $\mu(0) = 0$, $\mu(1) = 1$, and $\mu(x + y) = \mu(x) + \mu(y)$ whenever x and y are orthogonal. We will say that μ is *faithful* if $\mu(x) = 0$ implies that $x = 0$. If B is a complete Boolean algebra, we say that μ is *completely additive* if

$$\mu(\bigvee S) = \sum_{x \in S} \mu(x)$$

for every set S of mutually orthogonal elements of B .

Remark 2. If μ is a finitely additive probability measure on a complete Boolean algebra, we always have

$$\mu(\bigvee S) \geq \sum_{x \in S} \mu(x)$$

for every collection S of mutually orthogonal elements of B . If μ is faithful, we deduce that S is countable. Consequently, complete additivity is equivalent to countable additivity.

Conversely, suppose that μ is completely additive. Let $S = \{x \in B : \mu(x) = 0\}$. Then S is downward closed, so we can choose a subset $T \subseteq S$ consisting of mutually orthogonal elements with $\bigvee T = \bigvee S$. Complete additivity implies that $\mu(\bigvee T) = 0$, so that $\bigvee T$ is a maximal element of S . We may therefore decompose B as a product $B' \times B''$, where μ vanishes on B' and is faithful (and countably additive) on B'' .

Example 3. Let X be a set equipped with a probability measure μ (defined on some σ -algebra of subsets of X). We let $B(X)$ denote the collection of all equivalence classes of measurable subsets of X , where we call two such sets equivalent if they differ on a set of measure zero. Then B is a Boolean algebra. The construction $K \mapsto \chi_K$ determines an isomorphism of B with $P(A)$, where A is the von Neumann algebra $L^\infty(X)$. It follows that B is complete. Moreover, we can regard μ as a faithful completely additive measure on B .

Proposition 4. *Let B be a complete Boolean algebra. Then B decomposes (uniquely) as a product $B' \times B''$, where B' is a product of complete Boolean algebras which admit faithful completely additive measures, and B'' is a complete Boolean algebra which admits no nontrivial completely additive measures.*

Proof. For each $x \in B$, let $B_x = \{y \in B : y \leq x\}$. Then B_x can be regarded as a Boolean algebra in its own right. Let S be maximal among mutually orthogonal subsets of B such that B_x admits a faithful completely additive measure for each $x \in S$, and let $e = \bigvee S$. Then B_e is isomorphic to the product $\prod_{x \in S} B_x$. Let e^c denote the complement of e in B , so that $B \simeq B_e \times B_{e^c}$. To complete the proof, it suffices to observe that B_{e^c} does not admit any nontrivial completely additive measures. Otherwise, we can use Remark 2 to choose a nonzero element $y \leq e^c$ such that B_y admits a faithful completely additive measure, contradicting the maximality of S . \square

There is a converse to Example 3:

Proposition 5. *Let B be a complete Boolean algebra, and let μ be a faithful completely additive probability measure on B . Then there exists a measure space X such that B can be identified with the Boolean algebra of measurable subsets of X (modulo sets of measure zero), and μ agrees with the measure on X .*

Proof. Set $X = \text{Spec } B$. We have an order-preserving bijection of B with the set of closed and open subsets of X , which we will denote by $e \mapsto U_e$. Let Σ denote the collection of Borel subsets of X (that is, the σ -algebra generated by sets of measure zero). Recall that a subset $K \subseteq X$ is said to be *meager* if K is contained in a countable union of closed sets having empty interiors. We will prove the following:

(*) Let $Y \subseteq X$ be a Borel set. Then there exists an element $e \in B$ such that $Y - U_e$ and $U_e - Y$ are meager.

Let us say that a subset $Y \subseteq X$ is *good* if it satisfies condition (*). We wish to show that every Borel set is good. We first observe that every open set is good: if $U \subseteq X$ is open, then its closure \bar{U} is closed and open. Moreover, the difference $\bar{U} - U$ is closed and has empty interior, and is therefore meager.

To prove that every Borel set is good, it will suffice to show that the good sets form a σ -algebra. It is clear that if Y is good, then the complement of Y is good (replace U_e by U_{1-e}). It therefore suffices to prove that the collection of good sets is closed under countable unions. Suppose we are given a countable sequence of good sets Y_i . Then each Y_i differs from a closed and open set Y'_i by a meager set. Let $U = \bigcup Y'_i$; then U is open, and therefore good. Moreover, $\bigcup Y_i$ differs from U by a countable union of meager sets, which is again meager. Since U is good, we deduce that $\bigcup Y_i$ is good. This completes the proof of (*).

Note that the element $e \in B$ appearing in assertion (*) is unique. Otherwise, we could find distinct $e, e' \in B$ such that the differences $U_e - U_{e'}$ and $U_{e'} - U_e$ are meager. This is impossible: since X is compact, the Baire category theorem implies that meager sets have empty interior.

We now define an equivalence relation on Σ : let us write $Y \sim Y'$ if both $Y - Y'$ and $Y' - Y$ are meager. It follows from the above argument that the composite map

$$B \xrightarrow{e \mapsto U_e} \Sigma \rightarrow \Sigma / \sim$$

is an isomorphism of Boolean algebras. It follows that Σ / \sim is a complete Boolean algebra, and that we can identify μ with a completely additive measure on Σ / \sim . In particular, μ determines a countably additive measure on Σ with the desired properties. \square

Combining the above analysis with the previous lecture, we obtain the following characterization of those complete Boolean algebras which arise as the algebras of projections of an abelian von Neumann algebra:

Proposition 6. *Let B be a complete Boolean algebra. Then B is isomorphic to $P(A)$ for an abelian von Neumann algebra A if and only if B can be written as a product $\prod B_\alpha$, where each factor B_α admits a faithful completely additive measure (in other words, if and only if the factor B'' of Proposition 4 is trivial).*

To get a clearer picture of the structure of abelian von Neumann algebras, we need to introduce some countability assumptions.

Proposition 7. *Let A be a von Neumann algebra. The following conditions are equivalent:*

- (1) *The predual of A is a separable Banach space.*
- (2) *The unit ball $A_{\leq 1}$ is metrizable (in the ultraweak topology).*
- (3) *There exists a countable subset of A which is ultraweakly dense in A , and a countable collection of cyclic representations of A which are mutually faithful.*
- (4) *The von Neumann algebra A has a faithful representation on a separable Hilbert space.*

Proof. We first show that (1) is satisfied. Let E be the predual of A , and suppose that E has a countable dense subset $\{x_0, x_1, x_2, \dots\} \subseteq E$. Evaluation on the (x_i) induces a continuous map from E^\vee to \mathbb{R}^∞ . We claim that this map is an embedding on $E_{\leq t}^\vee$ for all real numbers t (that is, it is injective and $E_{\leq t}^\vee$ inherits the subspace topology). To prove this, we must show that for every point $\mu \in E_{\leq t}^\vee$ and every open set U containing μ , there exists an open set $V \subseteq \mathbb{R}^\infty$ whose inverse image is contained in U . We may assume without loss of generality that $\mu = 0$. Rescaling, we can assume that $t = 1$. Then U contains an open set of the form

$$\{\nu \in E_{\leq 1}^\vee : |\nu(v_i)| < \epsilon\}$$

for some $\epsilon > 0$ and some finite sequence of vectors $v_1, v_2, \dots, v_n \in E$. Since the x_i are dense, we may assume (renumbering if necessary) that $\|x_i - v_i\| < \frac{\epsilon}{2}$ for $1 \leq i \leq n$. Then

$$|\nu(v_i)| \leq \|\nu(x_i)\| + \|\nu(v_i - x_i)\| \leq \|\nu(x_i)\| + \|\nu\| \|v_i - x_i\| \leq \|\nu(x_i)\| + \frac{\epsilon}{2}.$$

It follows that U contains the inverse image of

$$\left(-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right)^n \times \mathbb{R}^\infty \subseteq \mathbb{R}^\infty.$$

Suppose that (2) is satisfied; we will prove (3). Then $A_{\leq 1}$ is a compact metric space (in the ultraweak topology) and therefore has a countable dense subset. It follows by rescaling that $A_{\leq n}$ has a countable dense subset for every integer n . Taking the union of these, we obtain a countable set which is ultraweakly dense in A . To complete the proof of (3), consider a maximal collection of mutually orthogonal central projections $\{e_\alpha\}$ in A such that each Ae_α admits a cyclic representation. Set $e = \sum e_\alpha$. If $e \neq 1$, then $A(1 - e)$ admits a cyclic representation, contradicting maximality. It follows that $\sum e_\alpha = 1$, so that A is a von Neumann algebra product of the Ae_α . Then $A_{\leq 1} = \prod_\alpha (Ae_\alpha)_{\leq 1}$. If this product is metrizable, only countably many factors can appear; this proves (3).

Now assume (3); we prove (4). First, suppose that V is a representation of A which admits a cyclic vector v . Let $S \subseteq A$ be a countable subset which is ultraweakly dense, and let $V_0 \subseteq V$ be the closed subspace generated by Sv . If $w \in V_0^\perp$, then $(sv, w) = 0$ for all $s \in S$. Since S is ultraweakly dense, we deduce that $(sv, w) = 0$ for all $s \in A$. Since Av is dense in V , we conclude that $w = 0$; thus $V_0^\perp = 0$ so that $V = V_0$ is separable. Since A admits a faithful representation on a direct sum of countably many cyclic representations, we conclude that A admits a faithful representation on a separably Hilbert space.

We complete the proof by showing that (4) implies (1). Suppose that $A \subseteq B(V)$ where V is a separable Hilbert space. The predual of A is a quotient of $B^{\text{tc}}(V)$. It will therefore suffice to show that $B^{\text{tc}}(V)$ is separable Banach space. By construction, the collection of all finite linear combinations of operators of the form

$$v \mapsto (v, w)u$$

is dense in $B^{\text{tc}}(V)$ (in the trace class norm). To obtain a countable dense subset, we can take all $\mathbf{Q}[i]$ -linear combinations of such operators where w and u range over a countable dense subset of V . \square

Definition 8. A von Neumann algebra A is said to be *separable* if it satisfies the equivalent conditions of Proposition 7.

Warning 9. A separable von Neumann algebra is generally *not* separable when regarded as a metric space in the norm topology. For example, $B(V)$ is separable as a Banach space (in the norm topology) only when the Hilbert space V is finite dimensional.