Lecture 6: Definition of the Fargues-Fontaine Curve

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Throughout this lecture, we fix a perfectoid field $C^\flat$ of characteristic p, with valuation ring $\mathcal{O}_C^\flat$. Fix an element $\pi \in C^\flat$ with $0 < |\pi|_{C^\flat} < 1$. We let $\mathbf{A}_{\mathrm{inf}}$ denote the ring of Witt vectors $W(\mathcal{O}_C^\flat)$. In the previous lecture, we defined the *Gauss norm* $| \bullet |_\rho : \mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}] \to \mathbf{R}_{\geq 0}$, for every real number $\rho \in (0,1)$. By definition, it is given by the formula $|\sum [c_n] p^n|_\rho = \sup\{|c_n|_{C^\flat} \cdot \rho^n\}$. For every pair of real numbers $0 < a \leq b < 1$, we let $B_{[a,b]}$ denote the completion of $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$ with respect to the pair of norms $| \bullet |_a$ and $| \bullet |_b$.

Exercise 1. Show that, for $0 < a \leq c \leq b < 1$, we have $|f|_c \leq \sup\{|f|_a, |f|_b\}$. Consequently, the completion of $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$ with respect to any finite collection of Gauss norms $| \bullet |_{\rho_0}, \cdots, | \bullet |_{\rho_n}$ is given $B_{[a,b]}$, where $a = \min\{\rho_i\}$ and $b = \max\{\rho_i\}$.

Recall that the ring B is defined as the inverse limit $\varprojlim B_{[a,b]}$, where $[a,b]$ ranges over the collection of all closed intervals contained in $(0,1)$. Equivalently, we can describe B as the completion of $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$ with respect to *all* of the Gauss norms $| \bullet |_\rho$ (for $0 < \rho < 1$). This inverse limit inherits a topology, and each of the norms $| \bullet |_\rho$ on $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$ admits a unique continuous extension to B (which we will also denote by $| \bullet |_\rho$). Moreover, a sequence $\{f_n\}_{n \geq 0}$ converges to $f \in B$ if and only if $\lim_{n \to \infty} |f - f_n|_\rho = 0$ for all $\rho \in (0,1)$. By virtue of Exercise 1, the collection of real numbers ρ which satisfies this condition is convex (so it suffices to check convergence for real numbers of the form $\frac{1}{N}$ and $\frac{N-1}{N}$, for example).

Warning 2. The ring B is a topological vector space over $\mathbf{Q}_p$, but it is *not* a p-adic Banach space: its topology cannot be defined by a single norm. It is instead an example of a *p-adic Frechet space*. However, it can still be regarded as a completion of $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$ in the following sense: every element $f \in B$ can be realized as the limit of a sequence $\{f_n\}$, where each f_n belongs to $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$. For example, we can take any sequence satisfying

$$|f - f_n|_{\frac{1}{n}} \leq \frac{1}{n} \qquad |f - f_n|_{1-\frac{1}{n}} \leq \frac{1}{n}$$

for $n > 1$.

Let us describe these completion a little bit more concretely. Let V be a $\mathbf{Q}_p$-vector space equipped with a non-archimedean norm $| \bullet |_V$. Suppose we are given a collection of vectors $\{v_i\}_{i \in I}$ in V with the property that, for every real number $\epsilon > 0$, we have $|v_i|_V < \epsilon$ for all but finitely many $i \in I$. In this case, the sum $\sum_{i \in I} v_i$ converges (absolutely) in the completion $\widehat{V}$ of V with respect to the norm $| \bullet |_V$.

Exercise 3. Let V be a $\mathbf{Q}_p$-vector space equipped with a norm $| \bullet |_V$, and suppose we are given a sequence of points $v_0, v_1, v_2, \ldots \in V$. Show that the following conditions are equivalent:

- The sequence $\{v_n\}_{n \geq 0}$ is a Cauchy sequence (with respect to the metric $d(v,w) = |v - w|_V$).
- $\lim_{n \to \infty} |v_n - v_{n-1}|_V = 0$.
- The sum $v_0 + \sum_{n > 0}(v_n - v_{n-1})$ is (absolutely) convergent in the completion $\widehat{V}$ of V.

If these conditions are satisfied, then the limit $\lim_{n\to\infty} v_n$ (in the completion $\widehat{V}$ of V) coincides with $v_0 + \sum_{n>0}(v_n - v_{n-1})$. Consequently, any element of $\widehat{V}$ can be written as an (absolutely convergent) sum of elements of V.

Variant 4. *Let $\widehat{V}$ be the completion of a $\mathbf{Q}_p$-vector space V is a $\mathbf{Q}_p$-vector space with respect to a pair of norms $|\bullet|_V$ and $|\bullet|_{V'}$. In this case, a sum $\sum_{i\in I} v_i$ converges in $\widehat{V}$ provided that* $\lim |v_i|_V = \lim |v_i|'_V = 0$.

Let us now specialize to the case of interest to us.

Example 5 (Teichmüller Expansions)**.** Suppose we are given a formal sum

$$\sum_{n\in\mathbf{Z}} [c_n]p^n,$$

where each c_n is an element of $C^\flat$. Then:

- The sum converges for the Gauss norm $|\bullet|_\rho$ if and only if
$$\lim_{n\to\infty} |c_n|_{C^\flat}\rho^n = 0 \qquad \lim_{n\to\infty} |c_{-n}|_{C^\flat}\rho^{-n} = 0.$$
- The sum converges in $B_{[a,b]}$ if and only if it converges for the Gauss norms $|\bullet|_a$ and $|\bullet|_b$. That is, if and only if we have
$$\lim_{n\to\infty} |c_n|_{C^\flat} b^n = 0 \qquad \lim_{n\to\infty} \frac{|c_{-n}|_{C^\flat}}{a^n} = 0.$$
- The sum converges in B if and only if it converges with respect to the Gauss norm $|\bullet|_\rho$ for *every* $\rho \in (0,1)$. This is equivalent ot the statement
$$\limsup_{n>0} |c_n|_{C^\flat}^{1/n} \leq 1 \qquad \lim_{n\to\infty} |c_{-n}|_{C^\flat}^{1/n} = 0.$$

Remark 6 (Complex-Analytic Analogue)**.** Let f be a holomorphic function defined on the punctured unit disk $D^\times = \{z \in \mathbf{C} : 0 < |z| < 1\}$. Then f admits a Laurent series expansion

$$f(z) = \sum c_n z^n,$$

where the coefficients c_n are complex numbers satisfying the conditions

$$\limsup_{n>0} |c_n|^{1/n} \leq 1 \qquad \lim_{n\to\infty} |c_{-n}|^{1/n} = 0.$$

Conversely, for any sequence $\{c_n\}_{n\in\mathbf{Z}}$ of complex numbers satisfying these conditions, the sum $\sum c_n z^n$ determines a holomorphic function on $D^\times$.

Warning 7. It follows from Example 5 that every collection $\{c_n\}_{n\in\mathbf{Z}}$ of elements of $C^\flat$ satisfying the conditions

$$\limsup_{n>0} |c_n|_{C^\flat}^{1/n} \leq 1 \qquad \lim_{n\to\infty} |c_{-n}|_{C^\flat}^{1/n} = 0$$

determines an element of the ring B, given by $\sum_{n\in\mathbf{Z}}[c_n]p^n$. However, it is not clear that every element of B can be represented in this way, or that such representations are unique when they exist.

Recall that $C^\flat$ is a perfect field of characteristic p, so the Frobenius map

$$\varphi : C^\flat \to C^\flat \qquad \varphi(c) = c^p$$

is an automorphism of $C^\flat$. This automorphism restricts to an automorphism of the valuation ring $\mathcal{O}_C^\flat$, and therefore induces an automorphism of the ring of Witt vectors $\mathbf{A}_{\inf} = W(\mathcal{O}_C^\flat)$. We will denote both of these automorphisms also by φ. Note that $\varphi([\pi]) = [\pi]^p$, so that inverting $[\pi]$ has the same effect as inverting $\varphi([\pi])$. Consequently, the Frobenius automorphism of $\mathbf{A}_{\inf}$ extends to an automorphism of $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$, which we will again denote by φ. On Teichmüller expansions, it is given by the formula

$$\varphi(\sum_{n \gg -\infty} [c_n]p^n) = \sum_{n \gg -\infty} [c_n^p]p^n.$$

In particular, we have

$$|\varphi(\sum_{n \gg -\infty} [c_n]p^n)|_\rho = \sup\{|c_n|_{C^\flat}^p \rho^n\} = (\sup\{|c_n|_{C^\flat} \rho^{n/p}\})^p = |\sum_{n \gg -\infty} [c_n]p^n|_{\rho^{1/p}}^p,$$

which we can write more simply as

$$|\varphi(f)|_{\rho^p} = (|f|_\rho)^p.$$

It follows that the automorphism φ of $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$ extends to an isomorphism $B_{[a,b]} \simeq B_{[a^p,b^p]}$. Passing to the inverse limit over all intervals $[a, b] \subseteq (0, 1)$, we obtain an automorphism of the ring B, which we will (once again) denote by φ.

Notation 8. For every integer n, we let $B^{\varphi = p^n}$ denote the subset of B consisting of those elements x satisfying $\varphi(x) = p^n x$.

Note that if f belongs to $B^{\varphi = p^m}$ and g belongs to $B^{\varphi = p^n}$, then we have $\varphi(fg) = \varphi(f) \cdot \varphi(g) = (p^m f) \cdot (p^n g) = p^{n+m} fg$, so that fg belongs to $B^{\varphi = p^{n+m}}$. It follows that we can regard the sum

$$\bigoplus_{n \in \mathbf{Z}} B^{\varphi = p^n}$$

as a graded ring. We can now finally define the main object of study in this course:

Definition 9. The *Fargues-Fontaine curve* is the scheme $\mathrm{Proj}(\bigoplus_{n \geq 0} B^{\varphi = p^n})$.

To get a feeling for what is going on, let's try to write down some elements of the graded ring $\bigoplus_{n \in \mathbf{Z}} B^{\varphi = p^n}$. Suppose that f is an element of B which admits a convergent Teichmüller expansion

$$f = \sum_{n \in \mathbf{Z}} [c_n]p^n,$$

so that

$$\limsup_{n > 0} |c_n|^{1/n} \leq 1 \qquad \lim_{n \to \infty} |c_{-n}|^{1/n} = 0.$$

In this case, the elements $p^k f$ and $\varphi(f)$ also admits convergent Teichmüller expansions

$$p^k f = \sum_{n \in \mathbf{Z}} [c_n]p^{n+k} = \sum_{n \in \mathbf{Z}} [c_{n-k}]p^n$$

$$\varphi(f) = \sum_{n \in \mathbf{Z}} [c_n^p]p^n.$$

Consequently, to satisfy the equation $\varphi(f) = p^k f$, it is sufficient (but perhaps not necessary) to have a termwise equality of Teichm"uller expansions $c_{n-k} = c_n^p$.

Example 10. Suppose that $k < 0$. Then, for each $n \in \mathbf{Z}$, the sequence

$$c_{n+k} = c_n^{1/p}, c_{n+2k} = c_n^{1/p^2}, c_{n+3k} = c_n^{1/p^3}, \cdots$$

is required to converge to zero. It follows that $c_n = 0$ for all n. In other words, there are no "obvious" nonzero elements of $B^{\varphi=p^k}$ for $k < 0$. (We will see in Lecture 11 that there are no nonzero elements at all: that is, the ring $\bigoplus_{n\in\mathbf{Z}} B^{\varphi=p^n}$ is nonnegatively graded.)

Example 11. Suppose that $k = 0$. In this case, for a Teichmüller expansion $\sum_{n\in\mathbf{Z}}[c_n]p^n$ to represent an element of $B^{\varphi=p^k}$, it is sufficient to have $c_n = c_n^p$ for all n: that is, each coefficient belongs to the subfield $\mathbf{F}_p \subseteq C^\flat$. In this case, the convergence condition on the coefficients c_n just demands that $c_n = 0$ for $n \ll 0$. These are exactly the Teichmüller expansions of elements of $\mathbf{Q}_p = W(\mathbf{F}_p)[\frac{1}{p}]$. We therefore obtain a map $\mathbf{Q}_p \to B^{\varphi=p^0}$. We will see later that this map is an isomorphism.

Example 12. Suppose that $k > 0$. In this case, the condition $c_{n-k} = c_n^p$ shows that the entire sequence is determined by a finite number of terms $c_0, c_1, \ldots, c_{k-1}$. Moreover, for the Teichmüller expansion to converge, each of these coefficients must belong to $\mathfrak{m}_C^\flat$. Via this procedure, we can write down a large number of elements of $B^{\varphi=p^k}$ (beware that it is not clear if these elements are distinct, or if all elements of $B^{\varphi=p^k}$ can be obtained in this way).

Example 13. In the case $k = 1$, we see that every element $c \in \mathfrak{m}_C^\flat$ determines an element of $B^{\varphi=p}$, given by the formula $\sum_{n\in\mathbf{Z}}[c^{1/p^n}]p^n$. We will study these elements in the next lecture.

Remark 14. Note that every element of the ring $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$ admits a *unique* Teichmüller expansion $\sum_{n\gg-\infty}[c_n]p^n$, and therefore belongs to $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]^{\varphi=p^k}$ if and only if $c_{n-k} = c_n^p$ for all n. If $k \neq 0$, the vanishing of c_n for $n \ll 0$ implies the vanishing of c_n all n. In other words, the graded ring

$$\bigoplus_n \mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]^{\varphi=p^n}$$

is just the field $\mathbf{Q}_p$. To obtain interesting elements of the ring $\bigoplus_{n\in\mathbf{Z}} B^{\varphi=p^n}$, it is important to complete the vector space $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$ by allowing "essential singularities at $p = 0$."