Lecture 6: Definition of the Fargues-Fontaine Curve

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Throughout this lecture, we fix a perfectoid field $C^\circ$ of characteristic $p$, with valuation ring $\mathcal{O}_C^\circ$. Fix an element $\pi \in C^\circ$ with $0 < |\pi|_{C^\circ} < 1$. We let $\mathbf{A}_{inf}$ denote the ring of Witt vectors $W(\mathcal{O}_C^\circ)$. In the previous lecture, we defined the Gauss norm $|\bullet|_p : \mathbf{A}_{inf}[\frac{1}{p}, \frac{1}{|\pi|}] \to \mathbb{R}_{\geq 0}$, for every real number $\rho \in (0, 1)$. By definition, it is given by the formula $|\sum c_n p^n|_\rho = \sup \{|c_n|_{C^\circ} \cdot \rho^n\}$. For every pair of real numbers $0 < a \leq b < 1$, we let $B_{[a, b]}$ denote the completion of $\mathbf{A}_{inf}[\frac{1}{p}, \frac{1}{|\pi|}]$ with respect to the pair of norms $|\bullet|_a$ and $|\bullet|_b$.

**Exercise 1.** Show that, for $0 < a \leq c < b < 1$, we have $|f|_c \leq \sup \{|f|_a, |f|_b\}$. Consequently, the completion of $\mathbf{A}_{inf}[\frac{1}{p}, \frac{1}{|\pi|}]$ with respect to any finite collection of Gauss norms $|\bullet|_{\rho_0}, \cdots, |\bullet|_{\rho_n}$ is given $B_{[a, b]}$, where $a = \min\{\rho_i\}$ and $b = \max\{\rho_i\}$.

Recall that the ring $B$ is defined as the inverse limit $\lim_{\leftarrow} B_{[a, b]}$, where $[a, b]$ ranges over the collection of all closed intervals contained in $(0, 1)$. Equivalently, we can describe $B$ as the completion of $\mathbf{A}_{inf}[\frac{1}{p}, \frac{1}{|\pi|}]$ with respect to all of the Gauss norms $|\bullet|_\rho$ (for $0 < \rho < 1$). This inverse limit inherits a topology, and each of the norms $|\bullet|_\rho$ on $\mathbf{A}_{inf}[\frac{1}{p}, \frac{1}{|\pi|}]$ admits a unique continuous extension to $B$ (which we will also denote by $|\bullet|_\rho$).

Moreover, a sequence $\{f_n\}_{n \geq 0}$ converges to $f \in B$ if and only if $\lim_{n \to \infty} |f - f_n|_\rho = 0$ for all $\rho \in (0, 1)$. By virtue of Exercise 1, the collection of real numbers $\rho$ which satisfies this condition is convex (so it suffices to check convergence for real numbers of the form $\frac{n}{N}$ and $\frac{N}{n}$, for example).

**Warning 2.** The ring $B$ is a topological vector space over $\mathbb{Q}_p$, but it is not a $p$-adic Banach space: its topology cannot be defined by a single norm. It is instead an example of a $p$-adic Frechet space. However, it can still be regarded as a completion of $\mathbf{A}_{inf}[\frac{1}{p}, \frac{1}{|\pi|}]$ in the following sense: every element $f \in B$ can be realized as the limit of a sequence $\{f_n\}$, where each $f_n$ belongs to $\mathbf{A}_{inf}[\frac{1}{p}, \frac{1}{|\pi|}]$. For example, we can take any sequence satisfying

$$|f - f_n|_{\frac{1}{n}} \leq \frac{1}{n} \quad |f - f_n|_{1 - \frac{1}{n}} \leq \frac{1}{n}$$

for $n > 1$.

Let us describe these completion a little bit more concretely. Let $V$ be a $\mathbb{Q}_p$-vector space equipped with a non-archimedean norm $|\bullet|_V$. Suppose we are given a collection of vectors $\{v_i\}_{i \in I}$ in $V$ with the property that, for every real number $\epsilon > 0$, we have $|v_i|_V \leq \epsilon$ for all but finitely many $i \in I$. In this case, the sum $\sum_{i \in I} v_i$ converges (absolutely) in the completion $\widehat{V}$ of $V$ with respect to the norm $|\bullet|_V$.

**Exercise 3.** Let $V$ be a $\mathbb{Q}_p$-vector space equipped with a norm $|\bullet|_V$, and suppose we are given a sequence of points $v_0, v_1, v_2, \ldots \in V$. Show that the following conditions are equivalent:

- The sequence $\{v_n\}_{n \geq 0}$ is a Cauchy sequence (with respect to the metric $d(v, w) = |v - w|_V$).
- $\lim_{n \to \infty} |v_n - v_{n-1}|_V = 0$.
- The sum $v_0 + \sum_{n>0}(v_n - v_{n-1})$ is (absolutely) convergent in the completion $\widehat{V}$ of $V$. 

If these conditions are satisfied, then the limit \( \lim_{n \to \infty} v_n \) (in the completion \( \hat{V} \) of \( V \)) coincides with \( v_0 + \sum_{n>0} (v_n - v_{n-1}) \). Consequently, any element of \( \hat{V} \) can be written as an (absolutely convergent) sum of elements of \( V \).

**Variant 4.** Let \( \hat{V} \) be the completion of a \( \mathbb{Q}_p \)-vector space \( V \) with respect to a pair of norms \( | \cdot |_V \) and \( | \cdot |_{V'} \). In this case, a sum \( \sum_{i \in I} v_i \) converges in \( \hat{V} \) provided that \( \lim |v_i|_V = \lim |v_i|_{V'} = 0 \).

Let us now specialize to the case of interest to us.

**Example 5 (Teichmüller Expansions).** Suppose we are given a formal sum

\[
\sum_{n \in \mathbb{Z}} [c_n] p^n,
\]

where each \( c_n \) is an element of \( C^\flat \). Then:

- The sum converges for the Gauss norm \( | \cdot |_\rho \) if and only if
  \[
  \lim_{n \to \infty} |c_n|_{C^\flat} p^n = 0 \quad \text{and} \quad \lim_{n \to \infty} |c_{-n}|_{C^\flat} p^{-n} = 0.
  \]

- The sum converges in \( B_{[a,b]} \) if and only if it converges for the Gauss norms \( | \cdot |_a \) and \( | \cdot |_b \). That is, if and only if we have
  \[
  \lim_{n \to \infty} |c_n|_{C^\flat} b^n = 0 \quad \text{and} \quad \lim_{n \to \infty} |c_{-n}|_{C^\flat} a^n = 0.
  \]

- The sum converges in \( B \) if and only if it converges with respect to the Gauss norm \( | \cdot |_\rho \) for every \( \rho \in (0,1) \). This is equivalent to the statement
  \[
  \limsup_{n>0} |c_n|_{C^\flat}^{1/n} \leq 1 \quad \text{and} \quad \lim_{n \to \infty} |c_{-n}|_{C^\flat}^{1/n} = 0.
  \]

**Remark 6 (Complex-Analytic Analogue).** Let \( f \) be a holomorphic function defined on the punctured unit disk \( D^\times = \{ z \in \mathbb{C} : 0 < |z| < 1 \} \). Then \( f \) admits a Laurent series expansion

\[
f(z) = \sum c_n z^n,
\]

where the coefficients \( c_n \) are complex numbers satisfying the conditions

\[
\limsup_{n>0} |c_n|^{1/n} \leq 1 \quad \text{and} \quad \lim_{n \to \infty} |c_{-n}|^{1/n} = 0.
\]

Conversely, for any sequence \( \{c_n\}_{n \in \mathbb{Z}} \) of complex numbers satisfying these conditions, the sum \( \sum c_n z^n \) determines a holomorphic function on \( D^\times \).

**Warning 7.** It follows from Example 5 that every collection \( \{c_n\}_{n \in \mathbb{Z}} \) of elements of \( C^\flat \) satisfying the conditions

\[
\limsup_{n>0} |c_n|_{C^\flat}^{1/n} \leq 1 \quad \text{and} \quad \lim_{n \to \infty} |c_{-n}|_{C^\flat}^{1/n} = 0
\]

determines an element of the ring \( B \), given by \( \sum_{n \in \mathbb{Z}} [c_n] p^n \). However, it is not clear that every element of \( B \) can be represented in this way, or that such representations are unique when they exist.

Recall that \( C^\flat \) is a perfect field of characteristic \( p \), so the Frobenius map

\[
\varphi : C^\flat \to C^\flat \quad \varphi(c) = e^p
\]
is an automorphism of $C^p$. This automorphism restricts to an automorphism of the valuation ring $O_C^p$, and therefore induces an automorphism of the ring of Witt vectors $\mathbf{A}_{\inf} = W(O_C^p)$. We will denote both of these automorphisms also by $\varphi$. Note that $\varphi([\pi]) = [\pi]^p$, so that inverting $[\pi]$ has the same effect as inverting $\varphi([\pi])$. Consequently, the Frobenius automorphism of $\mathbf{A}_{\inf}$ extends to an automorphism of $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{p}]$, which we will again denote by $\varphi$. On Teichmüller expansions, it is given by the formula

$$\varphi\left( \sum_{n, \gg -\infty} [c_n]p^n \right) = \sum_{n, \gg -\infty} [c_n^p]p^n.$$ 

In particular, we have

$$|\varphi\left( \sum_{n, \gg -\infty} [c_n]p^n \right)|_p = \sup\{|c_n|_{C^p}^p p^n \} = (\sup\{|c_n|_{C^p} p^{n/p}\})^p = | \sum_{n, \gg -\infty} [c_n]p^n |_{\rho^{1/p}},$$

which we can write more simply as

$$|\varphi(f)|_{\rho^p} = (|f|_{\rho})^p.$$ 

It follows that the automorphism $\varphi$ of $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{p}]$ extends to an isomorphism $B_{[a, b]} \simeq B_{[\rho, \rho^p]}$. Passing to the inverse limit over all intervals $[a, b] \subseteq (0, 1)$, we obtain an automorphism of the ring $B$, which we will (once again) denote by $\varphi$.

**Notation 8.** For every integer $n$, we let $B^{\varphi=p^n}$ denote the subset of $B$ consisting of those elements $x$ satisfying $\varphi(x) = p^n x$.

Note that if $f$ belongs to $B^{\varphi=p^n}$ and $g$ belongs to $B^{\varphi=p^m}$, then we have $\varphi(fg) = \varphi(f) \cdot \varphi(g) = (p^m f) \cdot (p^n g) = p^{n+m} fg$, so that $fg$ belongs to $B^{\varphi=p^{n+m}}$. It follows that we can regard the sum

$$\bigoplus_{n \in \mathbb{Z}} B^{\varphi=p^n}$$

as a graded ring. We can now finally define the main object of study in this course:

**Definition 9.** The Fargues-Fontaine curve is the scheme $\text{Proj}(\bigoplus_{n \geq 0} B^{\varphi=p^n})$.

To get a feeling for what is going on, let’s try to write down some elements of the graded ring $\bigoplus_{n \in \mathbb{Z}} B^{\varphi=p^n}$. Suppose that $f$ is an element of $B$ which admits a convergent Teichmüller expansion

$$f = \sum_{n \in \mathbb{Z}} [c_n]p^n,$$

so that

$$\lim_{n \to \infty} |c_n|^{1/n} = 1 \quad \lim_{n \to -\infty} |c_{-n}|^{1/n} = 0.$$ 

In this case, the elements $p^k f$ and $\varphi(f)$ also admits convergent Teichmüller expansions

$$p^k f = \sum_{n \in \mathbb{Z}} [c_n]p^{n+k} = \sum_{n \in \mathbb{Z}} [c_{n-k}]p^n,$$

$$\varphi(f) = \sum_{n \in \mathbb{Z}} [c_n^p]p^n.$$

Consequently, to satisfy the equation $\varphi(f) = p^k f$, it is sufficient (but perhaps not necessary) to have a termwise equality of Teichmüller expansions $c_{n-k} = c_n^p$. 


Example 10. Suppose that \( k < 0 \). Then, for each \( n \in \mathbb{Z} \), the sequence

\[
    c_{n+k} = c_n^{1/p}, \quad c_{n+2k} = c_n^{1/p^2}, \quad c_{n+3k} = c_n^{1/p^3}, \quad \ldots
\]
is required to converge to zero. It follows that \( c_n = 0 \) for all \( n \). In other words, there are no “obvious” nonzero elements of \( B^e=p^k \) for \( k < 0 \). (We will see in Lecture 11 that there are no nonzero elements at all: that is, the ring \( \bigoplus_{n \in \mathbb{Z}} B^e=p^k \) is nonnegatively graded.)

Example 11. Suppose that \( k = 0 \). In this case, for a Teichmüller expansion \( \sum_{n \in \mathbb{Z}} [c_n]p^n \) to represent an element of \( B^e=p^k \), it is sufficient to have \( c_n = c_n^p \) for all \( n \): that is, each coefficient belongs to the subfield \( F_p \subseteq C^p \). In this case, the convergence condition on the coefficients \( c_n \) just demands that \( c_n = 0 \) for \( n \ll 0 \). These are exactly the Teichmüller expansions of elements of \( Q_p = W(F_p)[\frac{1}{p}] \). We therefore obtain a map \( Q_p \to B^e=p^0 \). We will see later that this map is an isomorphism.

Example 12. Suppose that \( k > 0 \). In this case, the condition \( c_{n-k} = c_n^p \) shows that the entire sequence is determined by a finite number of terms \( c_0, c_1, \ldots, c_{k-1} \). Moreover, for the Teichmüller expansion to converge, each of these coefficients must belong to \( m_C^e \). Via this procedure, we can write down a large number of elements of \( B^e=p^k \) (beware that it is not clear if these elements are distinct, or if all elements of \( B^e=p^k \) can be obtained in this way).

Example 13. In the case \( k = 1 \), we see that every element \( c \in m_C^e \) determines an element of \( B^e=p^k \), given by the formula \( \sum_{n \in \mathbb{Z}} [c_n^{1/p^n}]p^n \). We will study these elements in the next lecture.

Remark 14. Note that every element of the ring \( A_{\text{inf}}[\frac{1}{p}, \frac{1}{\pi}] \) admits a unique Teichmüller expansion \( \sum_{n \gg -\infty} [c_n]p^n \), and therefore belongs to \( A_{\text{inf}}[\frac{1}{p}, \frac{1}{\pi}]^{e=p^k} \) if and only if \( c_{n-k} = c_n^p \) for all \( n \). If \( k \neq 0 \), the vanishing of \( c_n \) for \( n \ll 0 \) implies the vanishing of \( c_n \) all \( n \). In other words, the graded ring

\[
    \bigoplus_n A_{\text{inf}}[\frac{1}{p}, \frac{1}{\pi}]^{e=p^n}
\]
is just the field \( Q_p \). To obtain interesting elements of the ring \( \bigoplus_{n \in \mathbb{Z}} B^e=p^n \), it is important to complete the vector space \( A_{\text{inf}}[\frac{1}{p}, \frac{1}{\pi}] \) by allowing “essential singularities at \( p = 0 \)."