Lecture 26-Isocrystals

December 6, 2018

Throughout this lecture, we fix an algebraically closed perfectoid field $C^\flat$ of characteristic p and let

$$X = \mathrm{Proj}(\bigoplus_{n \geq 0} B^{\varphi = p^n})$$

be the Fargues-Fontaine curve. In Lecture 21, we explained how to construct a semistable vector bundle $\mathcal{E}$ on X of any rank $n > 0$ and any degree m (hence of any rational slope $\frac{m}{n}$). Namely, one can choose a degree n extension $E \supset \mathbf{Q}_p$ and a line bundle $\mathcal{L}$ of degree m on the curve X_E; we can then take $\mathcal{E} = \rho_* \mathcal{L}$, where $\rho : X_E \to X$ is the projection map. Over the last few lectures, we proved that this construction is independent of the choice of $\mathcal{L}$ (since a line bundle on X_E is determined up to isomorphism by its degree). Our next goal is to show that it is also independent of E.

As in the previous lecture, let us write $\mathbf{Q}_p \subseteq E_0 \subseteq E$ where E_0 is an unramified extension of $\mathbf{Q}_p$ of degree d and E is a totally ramified extension of E_0 having degree e (so that $n = d \cdot e$). Then $E_0 = W(\mathbf{F}_{p^d})[\frac{1}{p}]$, and we assume that we have fixed an embedding $\mathbf{F}_{p^d} \hookrightarrow C^\flat$. Let $\pi \in \mathcal{O}_E$ be a uniformizer. Let $U \subseteq X$ be an affine open subset given by the complement of the vanishing locus of some homogeneous element $t \in \bigoplus_{n>0} B^{\varphi = p^n}$. Then the vector bundle $\mathcal{E}$ constructed above can be given by the formula

$$\mathcal{E}(U) = (B[\frac{1}{t}] \otimes_{E_0} E)^{\varphi^d = \pi^m}.$$

We now describe a variant of this construction.

Definition 1. Let k be a perfect field of characteristic p, let $W(k)$ denote the ring of Witt vectors of k, and set $K = W(k)[\frac{1}{p}]$ be its fraction field. Then the Frobenius automorphism of k induces an automorphism of K, which we will denote by φ_K.

An *isocrystal* (over k) is a finite-dimensional vector space V over K equipped with a Frobenius-semilinear automorphism: that is, an isomorphism of abelian groups $\varphi_V : V \to V$ satisfying $\varphi_V(\lambda v) = \varphi_K(\lambda)\varphi_V(v)$ for $\lambda \in K$ and $v \in V$.

Remark 2. Our terminology is not standard; many authors use the term *F-isocrystal* or *Frobenius isocrystal* to refer to the notion of isocrystal that we just defined.

Example 3. Let X be a smooth projective algebraic variety over a perfect field k. Then the (rationalized) *crystalline cohomology groups* $\mathrm{H}^m_{\mathrm{crys}}(X; W(k))[\frac{1}{p}]$ have a Frobenius semilinear automorphism induced by the absolute Frobenius map $\varphi : X \to X$, and can therefore be regarded as isocrystals over k.

Example 4. Let k be a perfect field and let $T(u, v) \in k[[u, v]]$ be a formal group law over k which is not isomorphic to the additive group. Then the associated formal group $\mathbf{G}_T$ is determined (up to isomorphism) by its *Dieudonné module* $\mathbf{D}(\mathbf{G}_F)$: this is a free $W(k)$-module of finite rank equipped with a Frobenius semilinear endomorphism F and an inverse-Frobenius semilinear endomorphism V satisfying $FV = VF = p$. The rationalized Dieudonné module $\mathbf{D}(\mathbf{G}_F)[\frac{1}{p}]$ is then an isocrystal over k (the Frobenius endomorphism F has inverse given by $\frac{V}{p}$).

Example 5. Let m and n be relatively prime integers, with $n > 0$, and let $V_{\frac{m}{n}} = K^n$. Then we can equip V with the structure of an isocrystal by defining

$$\varphi_{V_{\frac{m}{n}}}(x_1, x_2, \ldots, x_n) = (\varphi_K(x_2), \varphi_K(x_3), \cdots, \varphi_K(x_n), p^m \varphi_K(x_1)).$$

This isocrystal is characterized by a universal property: giving a map from V into another isocrystal W (which is K-linear and Frobenius equivariant) is equivalent to giving an element of the eigenspace $W^{\varphi^n = p^m}$.

Theorem 6 (Dieudonné-Manin Classification)**.** *Let k be an algebraically closed field of characteristic p. Then:*

- *The category of isocrystals over k is semisimple. That is, every isocrystal over k can be written as a direct sum of simple objects.*
- *The simple isocrystals over k are exactly those of the form $V_{\frac{m}{n}}$, where m and n are relatively prime integer with $n > 0$.*

Construction 7. Let $k = \overline{\mathbf{F}}_p$ be the algebraic closure of $\mathbf{F}_p$ in the field $C^\flat$. Then the inclusion $\overline{\mathbf{F}}_p \hookrightarrow C^\flat$ extends to a map $W(\overline{\mathbf{F}}_p) \to \mathbf{A}_{\text{inf}}$, hence to a map

$$K = W(\overline{\mathbf{F}}_p)[\frac{1}{p}] \to B.$$

Let V be an isocrystal over K. We let $\mathcal{E}_V$ denote the quasi-coherent sheaf on $X = \text{Proj}(\bigoplus_{n \geq 0} B^{\varphi = p^n})$ associated to the graded module

$$\bigoplus_{n \geq 0} \text{Hom}_K(V, B)^{\varphi = p^n}.$$

In other words, if U is an affine open subset of X given by the complement of the vanishing locus of some homogeneous element $t \in \bigoplus_{n>0} B^{\varphi = p^n}$, then we have

$$\mathcal{E}_V(U) = \{ \ \phi\text{-equivariant } K\text{-linear maps } V \to B[\frac{1}{t}]\}.$$

In the special case where $V = V_{\frac{m}{n}}$ is the isocrystal of Example 5, we will denote the quasi-coherent sheaf $\mathcal{E}_V$ by $\mathcal{O}(\frac{m}{n})$.

Example 8. Fix relatively prime integers m and n with $n > 0$. Let $U \subset X$ be the affine open subset given by the vanishing locus of some homogeneous element $t \in \bigoplus_{n>0} B^{\varphi = p^n}$. We then have

$$\begin{aligned} \mathcal{O}(\frac{m}{n})(U) &\simeq (B[\frac{1}{t}])^{\varphi^n = p^m} \\ &= (\rho_* \mathcal{O}_{X_E}(m))(U) \end{aligned}$$

where E is the *unramified* extension of $\mathbf{Q}_p$ of degree n and $\rho : X_E \to X$ is the projection map. It follows that $\mathcal{O}(\frac{m}{n})$ is a semistable vector bundle of degree m and rank n.

Remark 9. It follows from Example 8 and the Dieudonné-Manin classification that, for every isocrystal V over $\overline{\mathbf{F}}_p$, the quasi-coherent sheaf $\mathcal{E}_V$ of Construction 7 is a vector bundle on X (whose rank is equal to the dimension of V as a vector space over K).

Example 10. Let E be a totally ramified extension of $\mathbf{Q}_p$ with uniformizer $\pi \in \mathcal{O}_E$, let $E^\vee$ denote the dual of E as a $\mathbf{Q}_p$-vector space (which we can identify with E via the trace pairing), and regard $V = E^\vee \otimes_{\mathbf{Q}_p} K$ as an isocrystal via the formula

$$\varphi_V(x \otimes y) = \pi^m x \otimes \varphi_K(y).$$

Unwinding the definitions, we have

$$\mathcal{E}_V(U) = \mathrm{Hom}_K(V, B[\frac{1}{t}])^{\varphi=1} = (E \otimes_{\mathbf{Q}_p} B[\frac{1}{t}])^{\varphi=\pi^m} = \rho_* \mathcal{O}_{X_E}(m)$$

where $\rho : X_E \to X$ is the projection map.

Exercise 11. Let E be a finite extension of $\mathbf{Q}_p$ (not necessarily unramified or totally ramified), let $\rho : X_E \to X$ be the projection map, and consider the vector bundle $\rho_* \mathcal{O}_{X_E}(m)$ (which is semistable of rank n and degree m).

- Show that $\rho_* \mathcal{O}_{X_E}(m)$ can be written as $\mathcal{E}_V$, where V is a suitable isocrystal over $\overline{\mathbf{F}}_p$ (hint: take $V = E^\vee \otimes_{\mathbf{Q}_p} K$, endowed with a suitable Frobenius action which depends on m).
- If m is relatively prime to n, show that V is isomorphic to $V_{\frac{m}{n}}$ (hint: use the Dieudonné-Manin classification).
- Conclude that if m and n are relatively prime, then $\rho_* \mathcal{O}_{X_E}(m)$ is isomorphic to the vector bundle $\mathcal{O}(\frac{m}{n})$ of Construction 7.

We can now state the classification theorem for semistable vector bundles on X in a more precise form.

Definition 12. Let μ be a rational number, which we write as $\mu = \frac{m}{n}$ where m and n are relatively prime and $n > 0$. We say that an isocrystal V over $\overline{\mathbf{F}}_p$ is *isoclinic of slope* μ if it is isomorphic to a direct sum of copies of the isocrystal $V_{\frac{m}{n}}$ of Example 5.

Example 13. An isocrystal over $\overline{\mathbf{F}}_p$ is isoclinic of slope 0 if and only if it is isomorphic to a sum of copies of K (with the usual Frobenius action). In this case, the vector bundle $\mathcal{E}_V$ is a sum of copies of $\mathcal{O}_X$: that is, it is a trivial vector bundle on X.

Remark 14. By the Dieudonné-Manin classification, every isocrystal V over $\overline{\mathbf{F}}_p$ splits uniquely as a direct sum of isoclinic isocrystals (of different slopes).

Theorem 15. (1) *For every vector bundle $\mathcal{E}$ on X, the Harder-Narasimhan filtration of $\mathcal{E}$ splits: that is, $\mathcal{E}$ can be written (non-uniquely) as a sum of semistable vector bundles.*

(2) *For every rational number μ, the construction*

$$V \mapsto \mathcal{E}_V$$

induces an equivalence of categories

$$\{\textit{Isoclinic isocrystals of slope } \mu\}^{\mathrm{op}} \to \{\textit{Semistable vector bundles on } X \textit{ of slope } \mu\}.$$

Corollary 16. *Every vector bundle $\mathcal{E}$ on X can be obtained by applying Construction 7 to some isocrystal V over $\overline{\mathbf{F}}_p$.*

Warning 17. The category of vector bundles on X is not equivalent to the category of isocrystals over $\overline{\mathbf{F}_p}$. The construction $V \mapsto \mathcal{E}_V$ is fully faithful when restricted to isoclinic isocrystals of some fixed slope μ, but is not fully faithful in general. For example, let (K, φ_K) denote the field K regarded as an isocrystal via its usual Frobenius automorphism, and let $(K, p\varphi_K)$ denote the field K regarded as an isocrystal via the map $x \mapsto \frac{\varphi_K(x)}{p}$. Then

$$\mathcal{E}_{(K,\varphi_K)} \simeq \mathcal{O}_X \qquad \mathcal{E}_{(K,p\varphi_K)} \simeq \mathcal{O}_X(1).$$

There are no maps from $(K, p\varphi_K)$ to (K, φ_K) in the category of isocrystals, but there are plenty of maps from $\mathcal{O}_X$ to $\mathcal{O}_X(1)$ in the category of vector bundles on X.

In the next lecture, we will use the following consequence of Theorem 15.

Corollary 18. *Let $\mathcal{E}$ be a vector bundle on X which is semistable of slope 0. Then $\mathcal{E}$ is trivial (that is, it is a sum of copies of $\mathcal{O}_X$).*