Throughout this lecture, we fix an algebraically closed perfectoid field $C^{\flat}$ of characteristic $p$. Let $X$ denote the Fargues-Fontaine curve, given by

$$X = \text{Proj}(\bigoplus_{n \geq 0} B_{\varphi \equiv p^n}).$$

In the previous lecture, we explained a strategy for producing semistable vector bundles of any given slope $\lambda = \frac{m}{n}$ on $X$:

- First, choose a finite degree $n$ extension $E$ of $\mathbb{Q}_p$.
- Then, choose a line bundle $L$ of degree $m$ on the fiber product $X_E = X \times_{\text{Spec}(\mathbb{Q}_p)} \text{Spec}(E)$.

The direct image of $L$ along the map $X_E \to X$ is then a semistable vector bundle of degree $m$ and rank $n$. This vector bundle is a priori dependent on the choice of extension $E$ and line bundle $L$. But it turns out not to matter: up to isomorphism, the resulting vector bundle depends only on the integers $d$ and $n$.

First, independence of $L$ is a consequence of the following:

**Theorem 1.** Let $E$ be a finite extension of $\mathbb{Q}_p$. Then the degree map $\text{deg} : \text{Pic}(X_E) \to \mathbb{Z}$ is an isomorphism.

To prove Theorem 1, it will suffice to show that, for every pair of closed points $x, x' \in X_E$, we have $\mathcal{O}_{X_E}(x) = \mathcal{O}_{X_E}(x')$. We have already seen that this is true when $E = \mathbb{Q}_p$. Essentially, we proved this by observing that $\mathcal{O}_X(x)$ and $\mathcal{O}_X(x')$ can be identified with another line bundle $\mathcal{O}(1)$, whose definition did not depend on a choice of point of $X$. We would like to show that something similar happens for the scheme $X_E$.

**Notation 2.** For the remainder of this lecture, we fix a finite extension field $E$ of $\mathbb{Q}_p$ of degree $n$. Then the inclusion $\mathbb{Q}_p \hookrightarrow E$ admits an essentially unique factorization as $\mathbb{Q}_p \hookrightarrow E_0 \hookrightarrow E$, where $E_0$ is an unramified extension of $\mathbb{Q}_p$, having some degree $d$ (so that $E_0 \simeq W(\mathbb{F}_p^d)[\frac{1}{p}]$) and $E$ is a totally ramified extension of $E_0$ having some degree $e$; we then have $n = d \cdot e$. We let $\mathcal{O}_E$ denote the ring of integers of $E$, and $\pi \in \mathcal{O}_E$ a choice of uniformizer.

**Exercise 3.** Choose an embedding $\mathbb{F}_p^d \hookrightarrow \mathcal{O}_C$, which extends to a map $W(\mathbb{F}_p^d) \to W(\mathcal{O}_C) = \mathbb{A}_{\text{inf}} \to B$ and therefore a map $E_0 \to B$, whose image is stable under the $d$th power of the Frobenius map. Let $t$ be any homogeneous element of the graded ring $\bigoplus_{n \geq 0} B_{\varphi \equiv p^n}$. Show that the canonical map

$$B[\frac{1}{t}] \otimes_{\mathbb{Q}_p} E \to B[\frac{1}{t}] \otimes_{E_0} E$$
induces an isomorphism

\[(B[\frac{1}{t}] \otimes \mathbb{Q}_p, E)^{\varphi=1} \simeq (B[\frac{1}{t}] \otimes_{E_0} E)^{\varphi^d=1}.
\]

In the special case \(E_0 = E\), this recovers the isomorphism \((B[\frac{1}{t}] \otimes \mathbb{Q}_p, E)^{\varphi=1} \simeq B[\frac{1}{t}]\) of the previous lecture.

It follows that, if \(U \subseteq X\) is the complement of the vanishing locus of \(t\) and we set \(U_E = U \times_{\text{Spec}(\mathbb{Q}_p)} E\), then (when \(t\) has positive degree) we can write \(U_E = \text{Spec}(B[\frac{1}{t}] \otimes_{E_0} E)^{\varphi^d=1}\).

**Construction 4.** We attempt to construct a line bundle \(\mathcal{O}_{X_E}(1)\) on \(X_E\) as follows:

- To each affine open subset \(U \subseteq X\) as above (given by the complement of the vanishing locus of \(t\)), we set

  \[\mathcal{O}_{X_E}(1)(U_E) = (B[\frac{1}{t}] \otimes_{E_0} E)^{\varphi^d=1} = \mathcal{O}_{X_E}(1).
  \]

  To simultaneously show that this construction "works" and prove Theorem 1, it will suffice to show that \(\mathcal{O}_{X_E}(1)\) agrees with the line bundle \(\mathcal{O}_{X_E}(x)\), for any choice of point \(x \in X_E\). In other words, we need to find a section of \(\mathcal{O}_{X_E}(1)\) which vanishes exactly at the point \(x\). First, we need some terminology.

**Notation 5.** In the previous lecture, we let \(Y_E\) denote the set of isomorphism classes of triples \((K, \iota, u)\), where \((K, \iota)\) is an untilt of \(C^o\) and \(u : E \to K\) is an embedding of fields. Let \(Y_E^o \subseteq Y_E\) denote the subset consisting of those triples where \(u|_{E_0}\) is given by the composite map \(E_0 \to B \to K\) (corresponding to the embedding \(\mathbb{F}_p \to C^o\) that we have chosen). Then \(Y_E^o\) is not stable under the Frobenius, but is stable under its \(d\)th power; moreover, the inclusion \(Y_E^o \hookrightarrow Y_E\) induces a bijection

  \[Y_E^o/\varphi^d, \xi \simeq Y_E/\varphi^d, \xi.
  \]

  Recall that, for each point \(y = (K, \iota)\) of \(Y\), we have an evaluation map

  \[B \to K \quad f \mapsto f(y).
  \]

  If we promote \(y\) to a point \(\overline{y} = (K, \iota, \epsilon)\) of \(Y_E^o\), then this evaluation map admits an \(E\)-linear extension

  \[B \otimes_{E_0} E \to K,
  \]

  which we will denote by \(f \mapsto f(\overline{y})\).

  In fact, we can do a little bit better. Recall that \(K\) can be identified with the residue field of a discrete valuation ring \(B_{\text{dr}}^+(y)\) (with uniformizer we denote by \(\xi\)) and that the homomorphism \(B \to K\) lifts to a map

  \[B \to B_{\text{dr}}^+(y) \quad f \mapsto \tilde{f}_y,
  \]

  In particular, this allows us to view \(B_{\text{dr}}^+(y)\) as an algebra over the field \(E_0 \subseteq B\). Since \(E\) is a separable extension field of \(E_0\), the \(E_0\)-algebra map

  \[E \to K \simeq B_{\text{dr}}^+(y)/(\xi)
  \]

  lifts uniquely to a homomorphism \(E \to B_{\text{dr}}^+(y)\). Amalgamating, we obtain a homomorphism

  \[B \otimes_{E_0} E \to B_{\text{dr}}^+(y),
  \]

  which we will denote by \(f \mapsto \tilde{f}_\overline{y}\). In particular, this allows us to define an order of vanishing \(\text{ord}_f(f) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}\) for each \(f \in B \otimes_{E_0} E\); namely, the supremum of those integers \(m\) such that \(\tilde{f}_\overline{y}\) is divisible by \(\xi^m\).
We will deduce Theorem 1 from the following result, which we prove in the next lecture:

**Theorem 6.** Let \( x \) be a closed point of \( X_E \), corresponding to a subset \( S \subseteq Y_E^* \) which is an orbit for the action of \( \varphi^\mathbb{Z} \). Then there exists an element \( f \in (B \otimes_{E_0} E)^{\varphi^\mathbb{Z} = \pi} \) satisfying

\[
\text{ord}_\pi(f) = \begin{cases} 1 & \text{if } \overline{y} \in S \\ 0 & \text{otherwise.} \end{cases}
\]

**Example 7.** In the case \( E = \mathbb{Q}_p \) and \( \pi = p \), we can choose \( f \in B^{\varphi = p} \) to be an element of the form \( \log([e]) \) for \( e \in 1 + m_{C_0} \).

**Proof of Theorem 1 from Theorem 6.** Assuming Theorem 6, we show that for each closed point \( x \in X_E \), the line bundle \( \mathcal{O}_{X_E}(x) \) on \( X_E \) is isomorphic to the presheaf \( \mathcal{O}_{X_E}(1) \) described in Construction 4: this will show both that \( \mathcal{O}_{X_E}(1) \) extends to a line bundle and that \( \mathcal{O}_{X_E}(x) \) is independent of \( x \). Choose \( f \in (B \otimes_{E_0} E)^{\varphi^\mathbb{Z} = \pi} \) satisfying the conclusion of Theorem 6. We will show that, for every affine open subset \( U \subseteq X \) (complementary to the vanishing locus of some homogeneous element \( t \in \bigoplus B^{\varphi = p^m} \)), multiplication by \( f \) induces an isomorphism

\[
\mathcal{O}_{X_E}(x)(U_E) \rightarrow \mathcal{O}_{X_E}(1)(U_E) = (B[1/t] \otimes_{E_0} E)^{\varphi^\mathbb{Z} = \pi}.
\]

Note that \( B \otimes_{E_0} E \) is a finite flat ring extension of \( B \) (of degree \( e \)). Let \( N(f) \in B \) denote the norm of \( f \) along this ring extension (that is, the determinant of the \( B \)-module homomorphism of \( B \otimes_{E_0} E \) given by multiplication by \( f \)). Note that, for each point \( y \in Y \), we have \( \overline{N(f)}_y = \prod \overline{f}_y \), where the product is taken over the set of all preimages of \( y \) in \( Y_E^* \). It follows that the vanishing locus of \( N(f) \) is given by a single orbit of \( \varphi^\mathbb{Z} \) on \( Y \) (and that \( N(f) \) has simple zeros at each point where it vanishes). Then the product \( N(f) \varphi(N(f)) \varphi^2(N(f)) \cdots \varphi^{d-1}(N(f)) \in B \) vanishes on a single \( \varphi^\mathbb{Z} \)-orbit of \( Y \) (again with simple zeros), and can therefore be written as a product \( u \log([e]) \) where \( u \) is an invertible element of \( B \) and \( e \in 1 + m_{C_0} \). Here \( \log([e]) \) vanishes at a single point of \( X \), which can be identified with the image of \( x \) under the projection map \( X_E \rightarrow X \). Note that since \( f \) divides the norm \( N(f) \), it divides the product \( N(f) \varphi(N(f)) \varphi^2(N(f)) \cdots \varphi^{d-1}(N(f)) = u \log([e]) \), and therefore also divides \( \log([e]) \).

We now distinguish two cases:

- Suppose that \( x \) does not belong to \( U_E \). Then \( \log([e]) \) is a divisor of \( t \), so \( f \) is a divisor of \( t \) and is therefore invertible in the ring \( B[1/t] \otimes_{E_0} E \). In this case, multiplication by \( f \) induces an isomorphism of

\[
\mathcal{O}_{X_E}(x)(U_E) = (B \otimes_{E_0} E)^{\varphi^d = 1} \xrightarrow{f} (B \otimes_{E_0} E)^{\varphi^\mathbb{Z} = \pi},
\]

with inverse given by multiplication by \( 1/f \).

- Suppose that \( x \) belongs to \( U_E \). Choose some other point \( x' \in X_E \) which does not belong to \( U_E \), and let \( f' \in (B \otimes_{E_0} E)^{\varphi^\mathbb{Z} = \pi} \) satisfy the conclusion of Theorem 6 for the point \( x' \). The preceding argument shows that \( f' \) is invertible in \( B[1/t] \otimes_{E_0} E \). It follows that the ratio \( f'/f \) is a well-defined element of \( (B[1/t] \otimes_{E_0} E)^{\varphi^d = 1} \), which we can identify with a regular function on \( U_E \) with a simple zero at the point \( x \). Consequently, multiplication by \( f'/f \) induces an isomorphism \( \mathcal{O}_{X_E}(U_E) \rightarrow \mathcal{O}_{X_E}(x)(U_E) \). It will therefore suffice to show that the composite map

\[
\mathcal{O}_{X_E}(U_E) \xrightarrow{f'/f} \mathcal{O}_{X_E}(x)(U_E) \xrightarrow{f} (B[1/t] \otimes_{E_0} E)^{\varphi^\mathbb{Z} = \pi}
\]

is an isomorphism. In other words, we may replace \( x \) by \( x' \) and thereby reduce to the case treated above. \(\square\)