Lecture 17: Algebraic Closure of Untilts

November 9, 2018

Our goal in this lecture is to prove the following result, which we have used several times without proof:

**Theorem 1.** Let $K$ be a perfectoid field. If the tilt $K^\flat$ is algebraically closed, then $K$ is algebraically closed.

We will prove Theorem 1 using an approximation argument which is similar to (but much easier than) the strategy of the last two lectures. The key point is to prove the following:

**Proposition 2.** Let $K$ be a perfectoid field such that the tilt $K^\flat$ is algebraically closed, and let $f(x) = x^n + a_1 x^{n-1} + \cdots + a_n \in K[x]$ be a non-constant irreducible polynomial. Let $y$ be an element of $K$. Then there exists an element $y' \in K$ satisfying

$$|y - y'|_K \leq |f(y)|_K^{1/n} \quad |f(y')|_K \leq |pf(y)|_K.$$

**Proof of Theorem 1 from Proposition 2.** Let $K$ be a perfectoid field such that $K^\flat$ is algebraically closed. We assume that $K$ has characteristic zero (otherwise there is nothing to prove). We wish to show that every non-constant polynomial $f(x) \in K[x]$ has a root in $K$. Without loss of generality, we may assume that $f(x)$ is monic and irreducible of degree $n > 0$. Replacing $f(x)$ by $p^nf(x)$ for $d > 0$, we may assume that the coefficients of $f$ belong to $\mathcal{O}_K$. Setting $y_0 = 0$, it follows that $f(y_0) \in \mathcal{O}_K$, or equivalently that $|f(y_0)|_K \leq |p|^n|_K$. Applying Proposition 2, we deduce that there exists $y_1 \in K$ satisfying $|y_0 - y_1|_K \leq |f(y_0)|_K^{1/n} \leq |p|^n|_K$ and $|f(y_1)|_K \leq |pf(y_0)|_K \leq |p|^n|_K$. Applying Proposition 2 to the element $y_1$, we obtain an element $y_2 \in \mathcal{O}_K$ satisfying $|y_1 - y_2|_K \leq |f(y_1)|_K^{1/n} \leq |p|^n|_K$ and $|f(y_2)|_K \leq |pf(y_1)|_K \leq |p|^2|_K$. Proceeding in this way, we obtain a sequence of elements $y_0 = 0, y_1, y_2, \ldots \in K$ satisfying

$$|y_m - y_{m+1}|_K \leq |p|^m|_K^{1/n} \quad |f(y_m)|_K \leq |p|^m|_K.$$

It follows from the first inequality (and the completeness of $K$) that the sequence $\{y_m\}$ converges to an element $y \in K$. Then

$$|f(y)|_K = \lim_{m \to \infty} |f(y_m)|_K = 0,$$

so that $y$ is a root of $f$. \qed

For the proof of Proposition 2, we will use the following result from the theory of valued fields:

**Theorem 3.** Let $K$ be a field which is complete with respect to a non-archimedean absolute value $|\cdot|_K$, and let $L$ be a finite extension field of $K$. Then $|\cdot|_K$ can be extended uniquely to an absolute value on the field $|\cdot|_L$.

**Remark 4.** In the situation of Theorem 3, the absolute value $|\cdot|_L$ is given concretely by the formula

$$|x|_L = |N_{L/K}(x)|^{1/\deg(L/K)}_K,$$
where \( N_{L/K} : L \rightarrow K \) denotes the norm map and \( \deg(L/K) \) denotes the degree of the field extension \( K \hookrightarrow L \). To prove this, we are free to enlarge \( L \) and may thereby assume that \( L \) is a normal extension of \( K \). In this case, we can write

\[
N_{L/K}(x) = \prod_{\gamma \in \text{Gal}(L/K)} \gamma(x)^{d_0},
\]

where \( d_0 \) is the inseparable degree of \( L \) over \( K \). We therefore have

\[
|N_{L/K}(x)|_{K}^{1/\deg(L/K)} = \prod_{\gamma \in \text{Gal}(L/K)} |\gamma(x)|_{L}^{1/|\text{Gal}(L/K)|}.
\]

The desired identity then follows from formula \( |x|_{L} = |\gamma(x)|_{L} \) for \( \gamma \in \text{Gal}(L/K) \) (by virtue of the uniqueness asserted in Theorem 3).

**Warning 5.** In the situation of Theorem 3, one cannot drop the assumption that \( K \) is complete. If \( K \) is not complete, then the norm \( | \bullet |_{K} \) can generally be extended in many different ways to extension fields \( L \) over \( K \), and the formula \( |x|_{L} = |N_{L/K}(x)|_{K}^{1/\deg(L/K)} \) of Remark 4 need not define an absolute value on \( L \).

**Corollary 6.** Let \( L \) be a field which is complete with respect to a non-archimedean absolute value \( | \bullet |_{L} \), and let \( f(x) = x^n + a_1 x^{n-1} + \cdots + a_n \) be an irreducible polynomial with coefficients in \( K \). If \( a_n \) belongs to \( \mathcal{O}_K \), then each \( a_i \) belongs to \( \mathcal{O}_L \).

**Proof.** Let \( L \) be a finite normal extension of \( K \) over which the polynomial \( f(x) \) factors as a product \( f(x) = (x - r_1) \cdot (x - r_2) \cdots (x - r_n) \). Equip \( L \) with the absolute value \( | \bullet |_{L} \) of Theorem 3. Since the roots \( r_i \) are conjugate by the action of the Galois group \( \text{Gal}(L/K) \), they must all have the same absolute value; that is, there exists a real number \( \lambda \) satisfying \( |r_i|_{L} = \lambda \) for all \( i \). Then \( a_n = (-1)^n \prod_{i=1}^{n} r_i \). Consequently, if \( a_n \) belongs to \( \mathcal{O}_K \), then each \( r_i \) belongs to \( \mathcal{O}_L \). It follows that the polynomial

\[
f(x) = \prod_{i=1}^{n} (x - r_i)
\]

has coefficients in \( \mathcal{O}_L \), so that each \( a_i \) belongs to \( \mathcal{O}_L \cap K = \mathcal{O}_K \) as desired. \( \square \)

**Proof of Proposition 2.** Let \( K \) be a perfectoid field such that the tilt \( K^b \) is algebraically closed, and let \( f(x) = x^n + a_1 x^{n-1} + \cdots + a_n \in K[x] \) be a non-constant irreducible polynomial. We wish to show that, for each element \( y \in K \), we can find another point \( y' \in K \) satisfying

\[
|y - y'|_K \leq |f(y)|_K^{1/n} \quad |f(y')|_K \leq |pf(y)|_K.
\]

Replacing \( f(x) \) by the polynomial \( f(x + y) \), we can reduce to the case \( y = 0 \); in this case, we wish to show that there exists \( y' \in K \) satisfying

\[
|y'|_K \leq |f(0)|_K^{1/n} \quad |f(y')|_K \leq |pf(0)|_K.
\]

Let us assume that \( f(0) \neq 0 \) (otherwise, we can take \( y' = 0 \) and there is nothing to prove). Note that the value group of \( K \) is the same as the value group of \( K^b \), and is therefore divisible (since \( K^b \) is algebraically closed). We can therefore choose an element \( c \in K \) satisfying \( |c|_K = |f(0)|_K^{1/n} \). In this case, we can rewrite the inequalities above as

\[
\left| \frac{y'}{c} \right|_K \leq 1 \quad \frac{1}{cn} f\left(c \cdot \frac{y'}{c}\right)|_K \leq |p|_K.
\]

Replacing \( f(x) \) by the monic polynomial \( \frac{1}{c^n} f(cx) \) (and \( y' \) by \( \frac{y'}{c} \)), we can reduce to the case where \( |f(0)|_K = 1 \). In this case, we wish to show that there exists \( y' \in K \) satisfying

\[
|y'|_K \leq 1 \quad |f(y')|_K \leq |p|_K.
\]

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Write \( f(x) = x^n + a_1 x^{n-1} + \cdots + a_n \). Our assumption that \(|f(0)|_K = 1\) guarantees that \(a_n\) belongs to \(\mathcal{O}_K\). Applying Corollary 6, we see that each of the coefficients \(a_i\) belongs to \(\mathcal{O}_K\). We can therefore choose elements \(b_i \in \mathcal{O}_K^\circ\) satisfying \(b_i^2 \equiv a_i \pmod{p}\). Set

\[
g(x) = x^n + b_1 x^{n-1} + b_2 x^{n-2} + \cdots + b_n \in K^\circ[x].
\]

Since \(K^\circ\) is algebraically closed, the polynomial \(g(x)\) factors as a product

\[
g(x) = (x - r_1) \cdots (x - r_n)
\]

for some \(r_1, r_2, \ldots, r_n \in K^\circ\). Note that we have

\[
|r_1|_{K^\circ} \cdots |r_n|_{K^\circ} = |(-1)^n b_n|_{K^\circ} \leq 1.
\]

It follows that there must exist \(r \in \{r_1, \ldots, r_n\}\) satisfying \(|r|_{K^\circ} \leq 1\), so that \(r\) belongs to \(\mathcal{O}_K^\circ\). Setting \(y' = r^\#\), we have \(|y'|_K = |r|_{K^\circ} \leq 1\), and

\[
f(y') = y'^n + a_1 y'^{n-1} + \cdots + a_n
\]

\[
\equiv y^n + b_1^\# y^{n-1} + \cdots + b_n^\# \pmod{p}
\]

\[
= (r^\#)^n + b_1^\# (r^\#)^{n-1} + \cdots + b_n^\#
\]

\[
\equiv (g(r))^\# \pmod{p}
\]

\[
= 0
\]

so that \(|f(y')|_K \leq |p|_K\), as desired. \(\square\)