Lecture 16: Converging to a Zero

November 8, 2018

Throughout this lecture, we fix an algebraically closed perfectoid field $C^b$ of characteristic $p$. Let $f$ be an element of the ring $A_{\inf}$ which is primitive of degree $d$: that is, an element which admits a Teichmüller expansion $\sum_{n \geq 0} [c_n] p^n$ satisfying

\[ c_0 \neq 0 \quad |c_i|_{C^0} < 1 \quad \text{for } i \neq d \quad |c_d|_{C^0} = 1. \]

Assume that $d > 0$, and let $\lambda \in (0, 1)$ be the largest element for which the function $s \mapsto v_s(f)$ fails to be differentiable at $-\log(\lambda)$; that is, $\lambda$ satisfies

\[ |c_i| \lambda^i \leq \lambda^d \quad \text{for all } i \]

\[ |c_i| \lambda^i = \lambda^d \quad \text{for some } i < d \]

Our goal in this lecture is to complete the proof of the following result:

**Proposition 1.** Then there exists a point $y \in Y$ satisfying $d(0, y) = \lambda$ and $f(y) = 0$.

Note that we have $|f|_\lambda = \lambda^d$. Consequently, for each point $y \in Y$ satisfying $d(0, y) = \lambda$, we automatically have

\[ |f(y)| \leq |f|_\lambda = \lambda^d. \]

Moreover, we expect the inequality to be strict if and only if $y$ is “close” to a root of $f$. More precisely, if $f$ factors as a product of distinguished elements of $A_{\inf}$ (which will follow once Proposition 1 has been proved), then we expect

\[ |f(y)| = \lambda^d \cdot \prod \frac{d(y', y)}{\lambda}, \]

where the product is taken over the collection of all $y'$ satisfying $d(0, y') = \lambda$ and $f(y') = 0$ (counted with multiplicity!); here at most $d$ factors appear. In particular, we should be able to choose at least one such point $y'$ satisfying

\[ \frac{d(y', y)}{\lambda} \leq \left( \frac{|f(y)|}{\lambda^d} \right)^{1/d}. \]

We now show that this is the case.

**Lemma 2.** Let $y$ be a point of $Y$ satisfying $d(0, y) = \lambda$, and suppose that $|f(y)| = \lambda^d \cdot \alpha$ for some $\alpha < 1$. Then there exists a point $y' \in Y$ satisfying $d(y, y') \leq \lambda \cdot \alpha^{1/d}$ and $f(y') \leq \lambda^{d+1} \cdot \alpha$.

**Proof of Proposition 1 from Lemma 2.** We proved in Lecture 15 that there exists a point $y_1 \in Y$ satisfying $d(0, y_1) = \lambda$ and $|f(y_1)| \leq \lambda^{d+1}$. Applying Lemma 2, we can choose a point $y_2 \in Y$ satisfying $d(y_1, y_2) \leq \lambda^{1+\frac{d}{2}}$ and $|f(y_2)| \leq \lambda^{d+2}$. Note that we then also have $d(0, y_2) = \lambda$, so we can apply Lemma 2 again to choose a point $y_3 \in Y$ satisfying $d(y_2, y_3) \leq \lambda^{1+\frac{d}{2}}$ and $|f(y_3)| \leq \lambda^{d+3}$. Continuing in this way, we obtain a sequence of points $\{y_n\}$ on the circle $Y_{[\lambda, \lambda]}$ satisfying

\[ d(y_n, y_{n+1}) \leq \lambda^{1+\frac{d}{2}} \quad \text{and} \quad |f(y_n)| \leq \lambda^{d+n}. \]
The first inequality implies that the sequence \( \{y_n\} \) is Cauchy, and therefore converges to a point \( y \in Y_{[\lambda, \xi]} \). The second inequality implies that \( |f(y)| = \lim_{n \to \infty} |f(y_n)| = 0 \), so that \( f(y) = 0 \).

**Proof of Lemma 2.** Fix a point \( y = (K, \iota) \in Y \) satisfying \( d(0, y) = \lambda \) and \( |f(y)|_K \leq \lambda^d \cdot \alpha \). Let \( \xi \) be a distinguished element of \( A_{\text{inf}} \) satisfying \( \xi(y) = 0 \). Since \( A_{\text{inf}} \) is \( \alpha \)-adically complete and every element of \( A_{\text{inf}}/\xi \) belongs to the image of \( \xi : \mathcal{O}_K \rightarrow \mathcal{O}_K \), we can write \( f \) as a sum

\[
\sum_{n \geq 0} [c_n]\xi^n
\]

(beware that this representation is **not** unique, because the map \( \xi : \mathcal{O}_K \rightarrow \mathcal{O}_K \) is not bijective). Note that under the reduction map

\[
A_{\text{inf}} = W(\mathcal{O}^\wedge_C) \rightarrow W(\mathcal{O}_C^\wedge / \mathfrak{m}_C^\wedge) = W(K),
\]

the image of \( \xi \) is a unit multiple of \( p \) (since \( \xi \) is distinguished) and the image of \( f \) is a unit multiple of \( p^d \) (since \( f \) is primitive of degree \( d \)). It follows that \( |c_i|_{\mathcal{O}^\wedge} < 1 \) for \( i < d \) and that \( |c_d|_{\mathcal{O}^\wedge} = 1 \). Replacing \( f \) by \( f/[c_d] \), we may assume without loss of generality that \( c_d = 1 \). Note that we have

\[
|c_0|_{\mathcal{O}^\wedge} = |[c_0](y)|_K = |f(y)|_K = \lambda^d \cdot \alpha.
\]

We will assume that \( c_0 \neq 0 \) (otherwise, we can take \( y' = y \)).

Consider the polynomial

\[
F(x) = c_0 + c_1 x + \cdots + c_{d-1} x^{d-1} + x^d \in C[y].
\]

Since \( C \) is algebraically closed, we can factor \( F(x) \) as a product of linear factors

\[
F(x) = (x - r_1)(x - r_2) \cdots (x - r_d)
\]

for some elements \( r_1, r_2, \ldots, r_d \in C \). Choose \( r \in \{r_1, \ldots, r_d\} \) so that the absolute value of \( r \) is as small as possible. Note that, for \( 0 \leq m \leq d \), we have \( c_m = \pm c_{d-m}(r_1, \ldots, r_d) \), where \( c_{d-m} \) denotes the \((d-m)\)th elementary symmetric polynomial. We therefore have

\[
|r|_{\mathcal{O}^\wedge}^m c_m \leq \sup_{J \subseteq \{1, \ldots, d\}, |J| = d - m} \prod_{j \in J} |r_j|_{\mathcal{O}^\wedge}
\]

\[
\leq \prod_{j=1}^d |r_j|_{\mathcal{O}^\wedge}^m = |c_0|_{\mathcal{O}^\wedge}^m = \lambda^d \cdot \alpha.
\]

In the special case \( m = d \), we have \( |r|_{\mathcal{O}^\wedge}^d \leq \lambda^d \cdot \alpha \), or \( |r|_{\mathcal{O}^\wedge} \leq \lambda \cdot \alpha^{1/d} \). Set \( \xi' = \xi - [r] \). Then \( \xi' \) is also a distinguished element, vanishing at a point \( y' \in Y \). We have

\[
d(y, y') = |\xi'(y)|_K = | - [r](y)|_K = |r|_{\mathcal{O}^\wedge} \leq \lambda \cdot \alpha^{1/d}.
\]

It follows that \( d(y, y') < \lambda = d(0, y) \), so that we have \( d(0, y') = \lambda \). Let \( K' \) be the untilt of \( C \) corresponding to the point \( y' \) and let \( \xi : C \rightarrow K' \) be the usual map (given by \( x^\xi = [x](y') \)). Then \( \xi(y') = (\xi' + [r])(y') = r^\xi \).

We therefore have

\[
\frac{f(y')}{|c_0|} = \sum_{n \geq 0} c_n^2 \xi(y')^n = \sum_{n \geq 0} \left( \frac{c_n r^n}{c_0} \right)^{\xi}.
\]
Note that the ration $\frac{c_n r^n}{c_0}$ belongs to $O_C$, for $n \leq d$ (by virtue of the inequality $|c_n r^n|_{C'} \leq |c_0|$ established above). For $n > d$, we have

$$|\frac{c_n r^n}{c_0}|_{C'} \leq \frac{|r^n|_{C'}}{|c_0|} \leq \frac{\lambda^n \cdot \alpha^{n/d}}{\lambda^d \cdot \alpha} \leq \lambda = |p|_{K'}.$$ 

It follows that each $(\frac{c_n r^n}{c_0})^{\sharp}$ belongs to the valuation ring $O_{K'}^b$, and is divisible by $p$ (in $O_{K'}^b$) when $n > d$. We therefore compute

$$\frac{f(y')}{|c_0|} = \sum_{n \geq 0} (\frac{c_n r^n}{c_0})^{\sharp}$$
$$= \sum_{n=0}^{d} (\frac{c_n r^n}{c_0})^{\sharp} \pmod{p}$$
$$= (\sum_{n=0}^{d} (\frac{c_n r^n}{c_0})^{\sharp}) \pmod{p}$$
$$= F(r)^{\sharp}$$
$$= 0.$$

We therefore have

$$|f(y')|_{K'} \leq |[c_0]|_{K'} \cdot |p|_{K'} = \lambda^d \cdot \alpha \cdot \lambda = \lambda^{d+1} \cdot \alpha.$$