Lecture 10: Structure of the Fargues-Fontaine Curve

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Throughout this lecture, we fix an algebraically closed perfectoid field $C$ of characteristic $p$, with valuation ring $O_C$. Let $Y$ denote the set of all isomorphism classes of characteristic zero untilts $y = (K, \iota)$ of $C$. To each nonzero element $f$ of the ring $B$, we associate the “divisor”

$$
\sum_{y \in Y} \text{ord}_y(f) \cdot y
$$

(which is generally an infinite sum, though “locally” finite). We recall three results from the previous lecture (which we have not yet proved):

**Theorem 1.**

1. Every nonzero element $f \in B$, has finite order of vanishing $\text{ord}_y(f)$ at each point $y \in Y$.
2. Another nonzero element $g \in B$ is divisible by $f$ if and only if $\text{Div}(f) \leq \text{Div}(g)$: that is, $\text{ord}_y(f) \leq \text{ord}_y(g)$ for each $y \in Y$.

**Theorem 2.**

For $n < 0$, the eigenspace $B^{p^n}$ vanishes.

**Theorem 3.**

Every untilt of $C$ is algebraically closed.

Let us now collect some consequences.

**Corollary 4.**

The ring $B^{p^1}$ is a field.

In fact, the field $B^{p^1}$ can be identified with $Q_p^\infty$: we stated this without proof in the previous lecture, but will not need it yet.

**Proof of Corollary 4.**

Let $f$ be a nonzero element of $B^{p^1}$; we wish to prove that $f$ is invertible in $B$ (in which case it is clear that the inverse $f^{-1}$ also belongs to $B^{p^1}$. By virtue of Theorem 1, it will suffice to show that the divisor $\text{Div}(f)$ vanishes. Since $f$ is fixed by the Frobenius, the divisor $\text{Div}(f)$ is likewise fixed by the Frobenius. Consequently, if $\text{Div}(f) \neq 0$, then we can write $\text{Div}(f) \geq \sum_{n \in \mathbb{Z}} \varphi^n(y)$ for some $y = (K, \iota) \in Y$. It follows from Theorem 3 that $K$ contains a copy of $Q_p^{\infty}$, so that we can write $\sum_{n \in \mathbb{Z}} \varphi^n(y) = \log([\epsilon])$ for some $\epsilon \in 1 + m_C$. Applying Theorem 1, we can write $f = g \cdot \log([\epsilon])$. It follows that $g \in B^{p^1} = \{0\}$, contradicting our assumption that $f \neq 0$.

**Corollary 5.**

For $n \geq 0$, every nonzero element $f \in B^{p^n}$ factors as a product $\lambda \log([\epsilon_1]) \cdots \log([\epsilon_n])$ for some $\lambda \in B^{p^1}$ and $\epsilon_1, \ldots, \epsilon_n \in 1 + m_C$. Moreover, the factors are uniquely determined up reordering and multiplication by elements of $Q_p^\infty$.

**Proof.** We prove existence by induction on $n$. If $n = 0$, there is nothing to prove. We will therefore assume that $n > 0$. Note that if $\text{Div}(f) = 0$, then $f$ is invertible (Theorem 1) and the inverse $f^{-1}$ belongs to $B^{p^{-n}}$, contradicting Theorem 2. As in the proof of Corollary 4, we learn that $f$ is divisible by $\log([\epsilon])$ for some
equivalently that $p \in B^{e=p^n-1}$. It follows from our inductive hypothesis that we can write $g = \lambda \log([\epsilon_1]) \cdots \log([\epsilon_n])$ for some $\lambda \in B^{e=1}$ and $\epsilon_1, \ldots, \epsilon_n \in 1 + m_C$, so that $f = \lambda \log([\epsilon_1]) \cdots \log([\epsilon_n-1]) \cdot \log([\epsilon_1])$.

To prove uniqueness, it will suffice to show for $1 \neq \epsilon \in 1 + m_C$, the element $\log([\epsilon])$ is a prime element of the graded ring $\bigoplus_{n \geq 0} B^{e=p^n}$: that is, if $\log([\epsilon])$ divides a product $f \cdot g$, then either $\log([\epsilon])$ divides $f$ or $\log([\epsilon])$ divides $g$. Since $\log([\epsilon])$ is homogeneous, it suffices to check this in the case where $f$ and $g$ are homogeneous: that is, we may assume that $f \in B^{e=p^m}$ and $g \in B^{e=p^n}$. Choose a point $y \in Y$ belonging to the vanishing locus of $\log([\epsilon])$. Then either $f$ or $g$ must vanish at the point $y$; without loss of generality, we may assume that $f(y) = 0$. The equation $\varphi(f) = p^m f$ guarantees that the divisor $\text{Div}(f)$ is Frobenius-invariant, so we must have $\text{Div}(f) \geq \sum_{n \in \mathbb{Z}} \varphi^n(y) = \text{Div}(\log([\epsilon]))$. Applying Theorem 1, we conclude that $\log([\epsilon])$ divides $f$.

Let $P$ denote the graded ring $\bigoplus_{n \geq 0} B^{e=p^n}$. Recall that the Fargues-Fontaine curve $X_{FF}$ is defined to be the scheme $\text{Proj}(P)$. By definition, the points of $X_{FF}$ (as a topological space) can be identified with homogeneous prime ideals $p \subseteq P$ which do not contain the “irrelevant” ideal $\bigoplus_{n>1} B^{e=p^n}$. Let us give two examples of such ideals:

- It follows from Theorem 1 that $B$ is an integral domain. Consequently, the graded ring $P$ is also an integral domain, so the zero ideal $(0) \subseteq P$ is prime. This prime ideal corresponds to the generic point of the Fargues-Fontaine curve $X_{FF}$.

- Let $(K, \iota) \in Y$ be a characteristic zero untilt of $C^b$, and choose an element $\epsilon \in 1 + m_C$, such that $\epsilon \neq 1$ and $\log([\epsilon])$ vanishes at $K$. It follows from the proof Corollary 5 that the principal ideal $(\log([\epsilon]))$ is prime, and therefore corresponds to a point of the Fargues-Fontaine curve that we will denote by $x_K$. Note that multiplying $\log([\epsilon])$ by a unit in $Q_p$ does not change the principal ideal $(\log([\epsilon]))$.

Consequently, the point $x_K$ depends only on the untilt $K$. Moreover, we have $x_K = x_{K'}$ if and only if $K$ and $K'$ belong to the same Frobenius orbit of $Y$.

We now show that these are the only points of the Fargues-Fontaine curve:

**Proposition 6.** Let $x$ be a point of the Fargues-Fontaine curve $X_{FF}$ which is not the generic point. Then we have $x = x_K$ for some point $(K, \iota) \in Y$. Moreover, the residue field of $X_{FF}$ at the point $x$ can be identified with $K$.

**Proof.** By construction, the scheme $X_{FF} = \text{Proj}(P)$ can be obtained by gluing together open affine subschemes of the form $P[1/f_1^{e=1}] = B[1/f_1^{e=1}]$, where $f$ is a nonzero homogeneous element of $P$ having positive degree. Let us suppose that $x$ belongs to one of these open subschemes, and therefore corresponds to a nonzero prime ideal $p \subseteq B[1/f_1^{e=1}]$. Choose an element of $p$ and write it as a fraction $g/f$ for some element $g \in B^{e=p^n}$. It follows from Corollary 5 that, after scaling by a unit, we may assume that this element factors as a product $\log([\epsilon_1]) \cdots \log([\epsilon_m])$. Since $p$ is prime, we may assume that it contains one of the factors, which we write as $\log([\epsilon])/f$. Let $y = (K, \iota) \in Y$ be a point at which $\log([\epsilon])$ vanishes. We claim that $x = x_K$, or equivalently that $p$ is generated by $\log([\epsilon])/f$. To prove this (and the last claim of Proposition 6), it will suffice to show that the principal ideal $(\log([\epsilon])/f)$ is maximal, and that the quotient field

$$B[1/f_1^{e=1}]/(\log([\epsilon])/f)$$

can be identified with $K$. Since $f$ does not vanish at $K$, we have a canonical ring homomorphism

$$\rho : B[1/f_1^{e=1}] \subseteq B[1/f_1] \to K.$$
we claim that \( \rho \) is a surjection whose kernel is generated by \( \frac{\log(|z|)}{f} \).

To prove surjectivity, we note that \( \rho \) is already surjective when restricted to \( \frac{1}{f} B^{x=p} \), since every element of \( K \) has the form \( \log(y^p) \) for some \( y \in 1 + m_C^p \) (see Lecture 9). To prove injectivity, we can use Corollary 5 to write every element of \( B[1/f]^{x=1} \) as a product \( \lambda \frac{\log(|w|)}{f} \cdots \frac{\log(|v|)}{f} \). If this point belongs to \( \ker(\rho) \), then some fraction \( \frac{\log(|z|)}{f} \) must be annihilated by \( \rho \). The desired result then follows from the observation that \( \frac{\log(|z|)}{f} \) and \( \frac{\log(|y|)}{f} \) differ by multiplication by some nonzero element of \( Q_p \).

Corollary 7. The construction \( y = (K, \iota) \mapsto x_K \) induces a bijection
\[ Y/\varphi^Z \simeq \{ \text{Closed points of } X_{FF} \} \]

Corollary 8. The Fargues-Fontaine curve \( X_{FF} \) is a Dedekind scheme.

Proof. By definition, we can cover \( X_{FF} \) by open affine subschemes of the form \( R = B[1/f]^{x=1} \). The proof of Proposition 6 shows that every nonzero prime ideal of \( R \) is a maximal ideal generated by a single element. In particular, every prime ideal of \( R \) is finitely generated so, by a theorem of Cohen, \( R \) is Noetherian. Since every nonzero prime ideal in \( R \) is maximal, it has Krull dimension 1. Moreover, since every maximal ideal of \( R \) is generated by a single element, the ring \( R \) is regular. It follows that \( R \) is a Dedekind ring, so that \( X_{FF} \) is a Dedekind scheme. \[ \square \]