Math 155, Problem Set 8 (due November 7)

October 30, 2011

(1) Let A be a finite partially ordered set containing elements $a, b \in A$. Suppose there exists a third element $c \in A$ such that $a < c < b$ with the following property: for every $d \in A$, if $a \leq d \leq b$, then either $d \leq c$ or $d \geq c$. Prove that $\mu(a, b) = 0$, where μ denotes the Möbius function of A.

(2) Let S be a finite set and let $\mathrm{Part}(S)$ denote the partially ordered set of equivalence relations on S. Give general formula for $\mu(E, E')$, where μ denotes the Möbius function of $\mathrm{Part}(S)$ (hint: reduce the problem to the special case treated in class).

(3) Let G be a finite group, and let $\{H_1, \ldots, H_m\}$ be a set of representatives for the conjugacy classes of subgroups of G. For $1 \leq i \leq j \leq m$, let $c_{i,j}$ denote the number of subgroups of G which contain H_i and are conjugate to H_j. Show that the matrix $[c_{i,j}]_{1 \leq i \leq j \leq n}$ has an inverse $[\mu_{i,j}]_{1 \leq i,j \leq m}$, where

$$\mu_{i,j} = \sum_{K_0 \subsetneq K_1 \subsetneq K_2 \subsetneq \cdots \subsetneq K_n} (-1)^n.$$

Here the sum is taken over all chains of subgroups

$$K_0 \subsetneq K_1 \subsetneq K_2 \subsetneq \cdots \subsetneq K_n$$

where $K_0 = H_i$ and K_n is conjugate to H_j.