Math 155, Problem Set 6 (due October 24)

October 19, 2011

(1) Let X be a finite set containing subsets $X_1, \ldots, X_n \subseteq X$. For $J \subseteq \{1, \ldots, n\}$, let $X_J = \bigcap_{i \in J} X_i$. Show that if $k \geq 0$ is an even integer, then

$$|X - \bigcup_{1 \leq i \leq n} X| \leq \sum_{J \subseteq \{1,\ldots,n\}, |J| \leq k} (-1)^{|J|} |X_J|.$$

Show that if $k \geq 0$ is odd, then

$$|X - \bigcup_{1 \leq i \leq n} X| \geq \sum_{J \subseteq \{1,\ldots,n\}, |J| \leq k} (-1)^{|J|} |X_J|.$$

(2) Let G denote the cyclic group $\mathbf{Z}/n\mathbf{Z}$, acting on itself by translation. Show that the cycle index of G is given by the formula

$$Z_G(s_1, s_2, \ldots) = \frac{\sum_{d|n} \phi(d) s_d^{n/d}}{n},$$

where ϕ denotes the Euler's ϕ-function.

(3) Let G be a finite group, and let $\mathrm{Burn}[G]$ denote its Burnside ring. Let $\{H_i\}_{1 \leq i \leq m}$ be a collection of representatives for the conjugacy classes of subgroups of G. Show that the construction

$$[X] \mapsto (|X^{H_1}|, |X^{H_2}|, \ldots, |X^{H_m}|)$$

determines a ring homomorphism

$$\psi : \mathrm{Burn}[G] \to \mathbf{Z}^m.$$

Show that ψ is injective, and that its image is a subgroup of $\mathbf{Z}^m$ having finite index. (Hint: use Problem 3 from Problem sets 4 and 5).