Math 155 (Lecture 33)

November 29, 2011

If G is a graph and S is a set of vertices of G, we let $G-S$ denote the graph obtained from G by removing the vertex set S. Our first goal in this lecture is to prove the following:

Theorem 1 (Tutte)**.** *Let G be a finite graph with vertex set V. Then G has a perfect matching if and only if the following condition is satisfied, for every subset $S \subseteq V$:*

($\star$) *The number of connected components of $G - S$ having an odd number of vertices is $\leq |S|$.*

We saw in the last lecture that ($\star$) is necessary. Note also that ($\star$) implies that G has an even number of vertices (take $S = \emptyset$).

We now prove the sufficiency. Assume that G satisfies ($\star$); we wish to show that G has a perfect matching. Let us fix the number of vertices of G, and work by reverse induction on the number of edges of G. That is, we will assume that Theorem 1 is valid for any graph G' having the same number of vertices as G but more edges than G. In particular, if x and y are vertices of G which are not connected by an edge, and G' is the graph obtained from G by adjoining an edge joining x to y, then we may assume that Theorem 1 is valid for the graph G'. Note that if G satisfies ($\star$), then G' also satisfies ($\star$): if S is any set of vertices of G', then either $G' - S \simeq G - S$ or $G' - S$ is obtained by adding an edge to $G - S$. In either case, the number of odd components of $G' - S$ must be smaller than the number of odd components of $G - S$.

We now take S to be the set of vertices of G which are connected to every other vertex of G. There are two cases to consider:

(a) Every connected component of $G - S$ is a complete graph. We can then construct a perfect matching for G as follows:

  (i) Choose a perfect matching for each connected component of $G - S$ having even size.

  (ii) For each connected component H of $G-S$ having odd size, choose a vertex $v_H \in H$ and a perfect matching for $H - \{v_H\}$. Choose also a vertex $w_H \in S$, and add to our matching the edge joining v_H to w_H. Since G satisfies ($\star$), we can assume that the vertices w_H are all distinct.

  (iii) Since G has even size, there are an even number of remaining vertices, each of which belongs to S. These vertices are all adjacent to one another, so we can complete our matching by arbitrarily dividing them into pairs.

(b) Suppose that some connected component of $G - S$ is *not* a complete graph: that is, the relation of adjacency is not an equivalence relation on the vertices of $G - S$. Then transitivity must fail: that is, we can find edges x, y, and z of $G - S$ such that x is adjacent to y, y is adjacent to z, but x is not adjacent to z. Since $y \notin S$, we can also choose a vertex w such that y is not adjacent to w.

Let G' be the graph obtained from G by adding an edge from w to y, and G'' the graph obtained from G by adding an edge from x to z. Then G' and G'' both have more vertices than G. Applying the inductive hypothesis, we deduce that G' and G'' have perfect matchings M' and M''. We may assume that M' contains the edge $\{w, y\}$: otherwise, it is a perfect matching for the graph G. Similarly, we may assume that M'' contains the edge $\{x, z\}$.

Let us now consider the graph H with the set of edges $M' \cup M''$. Note that every vertex in this graph belongs to either a single edge (if it belongs to an edge of $M' \cap M''$), or to two edges (one of which belongs to M' and one to M''). It follows that H can be written as a disjoint union of edges belonging to $M' \cap M''$ and cycles of even length, consisting of edges which belong alternatively to M and to M'.

Note that the edges $\{w, y\}$ and $\{x, z\}$ cannot belong to $M' \cap M''$, so they are contained in cycles of $C, D \subseteq H$. Let $C_1, C_2, \ldots, C_m$ be the remaining components of H. Then $C_1, C_2, \ldots, C_m$ are contained in G, and each admits a perfect matching. Let G_0 be the graph obtained from G by removing the vertices of $C_1, C_2, \ldots, C_m$. It will now suffice to show that G_0 admits a perfect matching.

Suppose first that $C \neq D$. Let C_0 be the graph obtained from C by removing the vertex $\{w, y\}$, and define $D_0 \subseteq D$ similarly. Then C_0 and D_0 are chains of even length, and therefore admit perfect matchings. The union of these perfect matchings is then a perfect matching for G_0.

Let us now suppose that $C = D$. Then C is a cycle of even length containing w, y, x, and z, with w adjacent to y and x adjacent to z. Without loss of generality, we may assume that this cycle has the form

$$v_0 = y, v_1 = w, v_2, v_3, \ldots, v_m = x, v_{m+1} = z, v_{m+2}, \ldots, v_n = y$$

for some even number n. Here the edge $\{v_i, v_{i+1}\}$ belongs to M' if i is even, and to M'' if i is odd. In particular, m is an odd number. In this case, we have a perfect matching for G_0 given by the edges

$$\{y, v_2\}, \{v_3, v_4\}, \ldots, \{v_{m-2}, v_{m-1}\}, \{x, y\}, \{z, v_{m+2}\}, \{v_{m+3}, v_{m+4}\}, \ldots, \{v_{n-2}, v_{n-1}\}.$$

This completes the proof of Theorem 1.

Definition 2. Let G be a graph. An *Eulerian cycle* in G is a cycle (possibly self-intersecting)

$$v_0, v_1, \ldots, v_n = v_0$$

which uses each edge of G exactly once: that is, each edge of G has the form $\{v_i, v_{i+1}\}$ for a unique value of i.

Question 3. When does a graph G admit an Eulerian cycle?

There are two obvious constraints:

(1) If G is a graph with an Eulerian cycle, then there must exist a connected component of G which contains all the edges of G (that is, G is the union of a connected graph with a collection of isolated vertices).

(2) If $v_0, v_1, \ldots, v_n = v_0$ is an Eulerian cycle in G, then each vertex of G must have even degree. In fact, the degree of a vertex w is twice the number of occurences of w in the list $v_0, v_1, \ldots, v_{n-1}$.

Theorem 4. *If G is a finite graph satisfying conditions* (1) *and* (2)*, then G admits an Eulerian cycle.*

Theorem 4 is a special case of a more general result about *Eulerian paths*. An *Eulerian path* in G is a path $v_0, v_1, \ldots, v_n$ which uses each edge exactly once. However, we do not assume that $v_0 = v_n$. Any graph G which admits an Eulerian path must satisfy condition (1), together with the following slightly weaker version of condition (2):

(2′) The graph G has at most two vertices of odd degree. In fact, if $v_0, \ldots, v_n$ is an Eulerian path in G, then every vertex of G other than v_0 and v_n must have even degree. Moreover, either $v_0 \neq v_n$ and both have odd degree, or $v_0 = v_n$ has even degree (in which case we have an Eulerian cycle).

Theorem 4 is a consequence of the following:

Theorem 5. *If G is a finite graph satisfying conditions* (1) *and* (2′)*, then G admits an Eulerian path.*

Remark 6. Let G be any finite graph. For each vertex v of G, let $d(v)$ denote the degree of v. Then $\sum_v d(v)$ is twice the number of edges of G (since each edge is counted twice). In particular, $\sum_v d(v)$ is an even number. It follows that G must have an even number of vertices of odd degree. In particular, if $(2')$ is satisfied, then either every vertex of G has even degree, or G has exactly two vertices of odd degree.

Proof of Theorem 5. We may assume without loss of generality that G is connected. If G has two vertices of odd degree, denote them by v and w. Otherwise, choose any vertex $v \in G$ and set $w = v$. We will show that there is an Eulerian path in G starting and v and ending at w. The proof proceeds by induction on the number of edges of G.

Suppose first that there exists an edge $\{v, v'\}$ of G with the following property: the graph H obtained by removing the edge $\{v, v'\}$ is connected. In this case, the inductive hypothesis implies that there is an Eulerian path from v' to w in the graph H. Appending the edge $\{v, v'\}$ to the beginning of this path, we obtain an Eulerian path from v to w in the graph G.

We may therefore assume that $d \geq 2$. The graph $G - \{v\}$ is a union of connected components $G_1, G_2, \ldots, G_d$. Each of these graphs has an even number of vertices having odd degree. Moreover, each G_i has exactly one vertex which is connected to v. It follows that each G_i has an *odd* number of vertices which have odd degree in the graph G. In particular, each G_i has at least one vertex of odd degree in v. Since no vertex of G other than v and w can have odd degree, we conclude that $d \leq 1$. That is, there is a unique edge $\{v, v'\}$ containing v. Then $G - \{v\}$ is connected, and the inductive hypothesis implies that there is an Eulerian path from v' to w in $G - \{v\}$. Appending v to the beginning of this path, we obtain an Eulerian path from v to w in G. $\square$