Math 155 (Lecture 3)

September 8, 2011

In this lecture, we'll consider the answer to one of the most basic counting problems in combinatorics.

Question 1. How many ways are there to choose a k-element subset of the set $\{1, 2, \ldots, n\}$?

The answer to this question is denoted by $\binom{n}{k}$, which is typically read as "n choose k". To obtain a formula for $\binom{n}{k}$, we first consider a different counting problem: how many ways are there to choose a sequence of k distinct elements of the set $\{1, \ldots, n\}$? In other words, how many injective functions f are there from the set $\{1, \ldots, k\}$ to the set $\{1, 2, \ldots, n\}$? This is easy to determine: there are n possible values for $f(1)$, $(n-1)$ possible values for $f(2)$, and so forth, so the number of such functions is given by

$$n(n-1)(n-2)\cdots(n-k+1) = \frac{n(n-1)(n-2)\cdots(3)(2)(1)}{(n-k)(n-k-1)\cdots(3)(2)(1)} = \frac{n!}{(n-k)!}.$$

Of course, this is different from the answer to Question 1, because we are counting ordered sequences of length k, rather than k element subsets. Every k element subset of $\{1, 2, \ldots, n\}$ can be ordered in precisely $k!$ different ways. We therefore obtain the identity $k!\binom{n}{k} = \frac{n!}{(n-k)!}$, or

$$\binom{n}{k} = \frac{n!}{k!(n-k)!.}$$

Remark 2. The above formula makes sense for $k \leq n$; we obviously have $\binom{n}{k} = 0$ for $k > n$.

Let us now consider another method of determining the integers $\binom{n}{k}$, which proceeds not by counting directly but instead by establishing a recurrence relation. Let $X_{n,k}$ denote the collection of all subsets of $\{1, \ldots, n\}$ having size k, so that $|X_{n,k}| = \binom{n}{k}$. We can partition the set $X_{n,k}$ into two subsets.

- Let $X^+_{n,k}$ denote the collection of all k-element subsets of $\{1, \ldots, n\}$ which contain the integer n. Such a subset is given by the union of $\{n\}$ with a $(k-1)$-element subset of $\{1, \ldots, n-1\}$. It follows that

$$|X^+_{n,k}| = |X_{n-1,k-1}| = \binom{n-1}{k-1}.$$

- Let $X^-_{n,k}$ denote the collection of k-element subsets of $\{1, \ldots, n\}$ which do not contain the integer n. We then have $X^-_{n,k} = X_{n-1,k}$, so that

$$|X^-_{n,k}| = \binom{n-1}{k}.$$

We therefore have

$$\binom{n}{k} = |X_{n,k}| = |X^+_{n,k}| + |X^-_{n,k}| = \binom{n-1}{k-1} + \binom{n-1}{k},$$

at least for $n > 0$.

Exercise 3. Prove the identity $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ directly from the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

It may be instructive to organize the integers $\binom{n}{k}$ into a table

$$\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots,
\end{array}$$

which is called *Pascal's triangle*. The recurrence relation $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n}{k}$ dictates how to fill this table in: each entry is the sum of the two entries diagonally above it. More specifically, we obtain

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots,
\end{array}$$

Let's now see what we can learn using the method of generating functions introduced in the last lecture. Fix an integer $n \geq 0$, and define a power series $F_n(x)$ by the formula

$$F_n(x) = \sum_{k \geq 0} \binom{n}{k} x^k.$$

Since $\binom{n}{k} = 0$ for $k > n$, this is actually a finite sum: that is, $F_n(x)$ is a polynomial function of x. For example, when $n = 0$, we obtain $F_0 = 1$. For $n > 0$, we can use our recurrence relation to obtain

$$\begin{aligned} F_n(x) &= \sum \binom{n}{k} x^k \\ &= \sum \binom{n-1}{k-1} x^k + \sum \binom{n-1}{k} x^k \\ &= x(\sum \binom{n-1}{l} x^l + \sum \binom{n-1}{k} x^k \\ &= (x+1)F_{n-1}(x). \end{aligned}$$

It follows that $F_n(x) = (x+1)^n$.

Now take 2 variables y and z, and write

$$(y+z)^n = y^n(1+\frac{z}{y})^n = y^n F_n(\frac{z}{y}) = y^n \sum_k \binom{n}{k} (\frac{z}{y})^k = \sum_k \binom{n}{k} y^{n-k} z^k.$$

This is an identity of polynomials, and therefore remains valid after substituting any numbers we like for y and z. We have proven:

Theorem 4 (Binomial Theorem)**.** *For any quantities y and z (belonging to the integers, or to the real numbers, or more generally any commutative ring) and any $n \geq 0$, we have*

$$(y+z)^n = \sum \binom{n}{k} y^{n-k} z^k.$$

Because of Theorem 4, the integers $\binom{n}{k}$ are often called *binomial coefficients*.

In the last lecture, we mentioned that generating functions should generally be viewed as *formal* power series: that is, we generally do not care whether or not they converge. However, we can sometimes get information by evaluating a generating function at a point. Let us close this lecture by giving an illustration of this principle.

Question 5. Fix an integer $n \geq 0$. Compute the sum $\sum_{k \geq 0} k\binom{n}{k}$.

We now describe a few different approaches to this question. First, let's use the method of generating functions. Let $F_n(x)$ be defined as above, so that $\binom{n}{k}$ is the coefficient of x^k in $F_n(x)$. Let $F'_n(x)$ denote the derivative of $F_n(x)$, so that the coefficient of x^{k-1} in $F'_n(x)$ is $k\binom{n}{k}$. It follows that $\sum_{k \geq 0} k\binom{n}{k}$ is the sum of the coefficients of the polynomial $F'_n(x)$: that is, it is the integer $F'_n(1)$. We saw above that $F_n(x) = (x+1)^n$. It follows that $F'_n(x) = n(x+1)^{n-1}$, so that $F'_n(1) = n2^{n-1}$.

Let us now try to arrive at the same answer by combinatorial means. The idea is to interpret $\sum_{k \geq 0} k\binom{n}{k}$ as the solution to a counting problem.

Question 6. Suppose we are given a group of n people. How many ways are there to select a committee (that is, a subset of the collection of people) together with a leader of that committee (who we require belongs to the committee).

Let's try answer Question 6 in two different ways. Suppose first that we are interested in committees of size k. In this case, there are $\binom{n}{k}$ choices for the members of the committee, and k choices for its leader: a total of $k\binom{n}{k}$ choices in all. The answer to Question 6 is then given by summing over k: that is, by $\sum_{k \geq 0} k\binom{n}{k}$.

Here is another way to solve the counting problem posed in Question 6. First, there are n choices for the leader of the committee. Once the leader is fixed, the remaining $n-1$ people can each be assigned to the committee or not, for a total of 2^{n-1} choices. We can therefore give $n2^{n-1}$ as an answer to Question 6. We conclude that

$$\sum_{k \geq 0} k\binom{n}{k} = n2^{n-1}.$$

Let us describe one more way of arriving at the answer. The idea is to interpret the sum appearing in Question 5 not the solution to a counting problem, but as the solution to an expected value problem.

Question 7. Suppose that a subset S of $\{1, \ldots, n\}$ is chosen at random. What is the expected number of elements of S?

The answer is evidently $\frac{n}{2}$. On the other hand, we can compute the answer as

$$\frac{\sum_{S \subseteq \{1,\ldots,n\}} |S|}{2^n}.$$

Here there $\binom{n}{k}$ terms in the numerator where the associated summand is k, so we can rewrite the numerator as $\sum_{k \geq 0} k\binom{n}{k}$. We therefore obtain

$$\sum_{k \geq 0} k\binom{n}{k} = 2^n \frac{n}{2} = n2^{n-1}.$$

Recall that a *derangement* of the set $\{1, \ldots, n\}$ is a permutation of $\{1, \ldots, n\}$ with no fixed points: that is, a permutation π such that $\pi(i) \neq i$ for all i. Let's begin with the following question:

Question 8. How many derangements are there of the set $\{1, \ldots, n\}$? What is the probability that a permutation chosen at random is a derangement?

To fix ideas, let D_n denote the number of derangements of the set $\{1, \ldots, n\}$. We have $D_0 = 1$ (the identity permutation of the empty set has no fixed points), $D_1 = 0$ (the identity permutation of $\{1\}$ does have a fixed point), $D_2 = 1$ (there is a unique derangement of $\{1, 2\}$, given by the non-identity permutation), $D_3 = 2$ (the derangements of the set $\{1, 2, 3\}$ are precisely the cyclic permutations).

Our goal now is to obtain a formula for the integers D_n. As a starting point, we know that the *total* number of permutations of the set $\{1, \ldots, n\}$ is $n!$. We therefore have

$$n! = D_n + |X|,$$

where X is the set of permutations of $\{1, \ldots, n\}$ which have at least one fixed point. In fact, we can say something more refined. For $k > 0$, let X_k denote the collection of all permutations of $\{1, \ldots n\}$ having exactly k fixed points. Then X can be written as the disjoint union of the subsets $X_1, X_2, \ldots, X_n$. We therefore have

$$n! = D_n + |X_1| + |X_2| + |X_3| + \cdots + |X_n|.$$

Remark 9. We can write this equation more naturally as

$$n! = |X_0| + |X_1| + \cdots + |X_n|,$$

where X_0 denotes the set of permutations with no fixed points: that is, the set of derangements.

To determine D_n from the above formula, we need to know how big the sets X_k are. How can we describe a permutation π with exactly k fixed points? First, we need to specify which elements of $\{1, \ldots, n\}$ are fixed points of π: there are $\binom{n}{k}$ possibilities for this in all. Second, we need to specify what our permutation does on the remaining $n - k$ elements. This restricted permutation must be fixed point free (otherwise, π would have more than k fixed points), so the number of choices is given by D_{n-k}. We therefore obtain the formula

$$|X_k| = \binom{n}{k} D_{n-k}$$

so that our earlier equation reads

$$n! = \sum_{0 \leq k \leq n} \binom{n}{k} D_{n-k}.$$

Using our formula for the binomial coefficients $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, we see that both sides of this expression are divisible by $n!$. Dividing out, we get

$$1 = \sum_{0 \leq k \leq n} \frac{D_{n-k}}{k!(n-k)!}.$$

The analysis above gives a series of equations (one for each $n \geq 0$) which can be used to successively solve for the integers D_n. Rather than treating all of these equations separately, it will be convenient to think about them all at once, using the method of generating functions. For this, let us introduce a formal variable x. Multiplying our previous equation by x^n, we get

$$x^n = \sum_{0 \leq k \leq n} \frac{D_{n-k} x^n}{k!(n-k)!}.$$

Summing over all n, we obtain an identity of formal power series

$$\sum_{n \geq 0} x^n = \sum_{n \geq 0} \sum_{0 \leq k \leq n} \frac{D_{n-k} x^n}{k!(n-k)!}.$$

It is now convenient to rearrange the sum on the right hand side: note that giving an integer $n \geq 0$ and another integer k between 0 and n is equivalent to giving a pair of nonnegative integers k and l, with $n = k+l$. We therefore obtain

$$\sum_{n \geq 0} x^n = \sum_{k,l \geq 0} \frac{D_l x^{k+l}}{k!l!} = \sum_{k,l \geq 0} \frac{x^k}{k!} \frac{D_l x^l}{l!}.$$

The right hand side of this expression factors as a product

$$(\sum_{k \geq 0} \frac{x^k}{k!})(\sum_{l \geq 0} \frac{D_l x^l}{l!}).$$

The first factor should be familiar: it the power series expansion for the exponential function e^x. Let us introduce a notation for the second factor:

$$F(x) = \sum_{l \geq 0} \frac{D_l x^l}{l!} = 1 + \frac{x^2}{2} + \frac{x^3}{3} + \cdots.$$

The power series $F(x)$ is called the *exponential generating function* for the sequence of integers $\{D_l\}_{l \geq 0}$. It differs from the generating functions we have met so far because of the addition of an auxiliary factor of $\frac{1}{l!}$ on the coefficient of x^l. We can now rewrite our equation as

$$\frac{1}{1-x} = e^x F(x),$$

which is easy to solve: we get

$$F(x) = \frac{e^{-x}}{1-x}.$$