Math 155 (Lecture 29)

November 12, 2011

For every pair of integers m and n, let $R(m,n)$ denote the corresponding Ramsey number: that is, $R(m,n)$ is the least integer k such that every graph with k vertices either contains a clique of size m or an anticlique of size n. Several lectures ago, we proved Ramsey's theorem, which asserts the existence of the integer $R(m,n)$. Moreover, the proof was constructive and gave an upper bound

$$R(m,n) \leq \binom{m+n-2}{m-1}.$$

In particular, we have $R(m,n) \leq 2^{m+n}$, so that $R(n,n) \leq 4^n$.

In this lecture, we'll discuss how to obtain some lower bounds on the Ramsey numbers $R(m,n)$. For this, we want to produce graphs which do not admit large cliques or anticliques: in other words, graphs G which do not exhibit any "patterned behavior." For this purpose, Paul Erdös introduced the idea that G should be chosen at random.

Question 1. Let G be a randomly chosen graph with vertex set $\{1, \ldots, k\}$ (where each graph has the same probability of occurring: in particular, this means that a pair of vertices $i \neq j$ are adjacent with probability $\frac{1}{2}$). What is the probability p that G has a clique or anticlique of size n?

For each subset $S \subseteq \{1, \ldots, k\}$ of size n, the probability that S is a clique of G is equal to $2^{-\binom{n}{2}}$, and the probability that S is an anticlique of G is also equal to $2^{-\binom{n}{2}}$. There are $\binom{k}{n}$ choices for the set S. We therefore have

$$p \leq 2\binom{k}{n}2^{-\binom{n}{2}}.$$

If $p < 1$, then there is at least one graph of size k which does not contain a clique or anticlique of size n. Consequently, if

$$2\binom{k}{n}2^{-\binom{n}{2}} < 1,$$

we must have $k < R(n,n)$. We can rewrite the above inequality as

$$\binom{k}{n} < 2^{\binom{n}{2}-1}.$$

Using the rough estimate

$$\binom{k}{n} \leq \frac{k^n}{n!},$$

we see that $R(n,n) > k$ whenever

$$k^n < n! 2^{\binom{n}{2}-1}.$$

Note that $n! \geq 2^{n-1}$. It therefore suffices to check that

$$k^n < 2^{\binom{n}{2}+n-2} = 2^{\frac{n^2}{2}+\frac{n}{2}-2}.$$

or that $k < 2^{\frac{n}{2}+\frac{1}{2}-\frac{2}{n}}$. Assuming $n \geq 4$, it suffices to check that $k < 2^{\frac{n}{2}}$. We have therefore proven:

Theorem 2 (Erdös)**.** *For every integer $n \geq 4$ have $R(n,n) \geq 2^{\frac{n}{2}}$ (this is also true for $n = 2$ and $n = 3$, as we see from the equalities $R(2,2) = 2$, $R(3,3) = 6$).*

Consequently, we have

$$2^{\frac{n}{2}} \leq R(n,n) \leq 2^{2n}.$$

In particular, we see that $R(n,n)$ grows roughly as an exponential function of n.

Remark 3. Theorem 2 asserts that for every $k < 2^{\frac{n}{2}}$, we can find a graph of size k with no cliques or anticliques of size n. The proof is nonconstructive: it does not give us an explicit procedure for building such a graph. However, it does suggest a practical procedure. Note that our estimates on the probability p were quite rough: for $k < 2^{\frac{n}{2}}$, we should expect not only that $p < 1$ but also that p is quite small. Consequently, a randomly chosen graph will have a very high probability of not containing any large cliques or anticliques.

We can use the above methods to find lower bounds for many other Ramsey-type theorems.

Question 4. For each integer n, let $W(n)$ denote the smallest integer k such that every coloring of the set $\{1, 2, \ldots, k\}$ using two colors has a monochromatic arithmetic progression of size n (such an integer exists, by van der Waerden's theorem). How big are the integers $W(n)$?

Our proof of van der Waerden's theorem gives in principle an upper bound for the integers $W(n)$, though in practice these bounds are incredibly large. We can use a probabilistic argument to get a lower bound:

Question 5. Given a randomly chosen coloring c of the set $\{1, \ldots, k\}$ with two colors, what is the probability p that there is a monochromatic arithmetic progression of size n?

For every arithmetic progression $S \subseteq \{1, 2, \ldots, k\}$ of size n, the probability that S is monochromatic (for a randomly chosen coloring c) is equal to 2^{-n}. Consequently, we have an inequality

$$p \leq C 2^{-n},$$

where C is the number of arithmetic progressions of $\{1, 2, \ldots, k\}$ of size n. Since an arithmetic progression is determined by its first two terms, we have $C \leq k^2$. Thus $p \leq k^2 2^{-n}$. If $p < 1$, then there must exist a coloring which has no monochromatic arithmetic progressions of length n. Thus $W(n) > k$ whenever $k^2 2^{-n} < 1$. This proves the following lower bound:

Proposition 6. *For every integer n, we have $W(n) \geq 2^{\frac{n}{2}}$.*

Let's now study a slightly different question. Let $m \geq 2$ be a fixed integer, and let us ask how the Ramsey number $R(m,n)$ varies as a function of n. For the sake of concreteness, let's take $m = 4$. Our upper bound gives

$$R(4,n) \leq \binom{n+4-2}{4-1} = \binom{n+2}{3} \leq \frac{1}{6}(n+2)^3.$$

That is, $R(4,n)$ is bounded above by a cubic polynomial in n.

Let's try to get a lower bound using probabilistic reasoning. We might first try choosing a graph with vertex set $\{1, 2, \ldots, k\}$ at random, as before. But this does not give us very much information. The probability that a subset $S \subseteq \{1, \ldots, k\}$ of size 4 is a clique is given by $2^{-6} = \frac{1}{64}$, which does not depend on n. So our earlier strategy will not yield a lower bound which depends on n.

We therefore introduce a slight variation. Let X denote the set of all graphs with vertex set $\{1, \ldots, k\}$. We would like to have a way of choosing graphs in X "randomly" so that a randomly chosen graph G is not likely to have either a clique of size 4 or an anticlique of size n. The former condition suggests that we should bias our choice of graphs in favor of those which do not have many edges. To this end, let us fix a real number $0 \leq q \leq 1$. We now randomly select a graph G so that every edge $\{i, j\} \subseteq \{1, \ldots, k\}$ has probability q of appearing in our graph. That is, we *weight* our choice of graph G in X, so that the graph G is chosen with probability

$$q^e (1-q)^{\binom{k}{2} - e},$$

where e is the number of edges of G.

Now suppose we've chosen k and q, and let's try to estimate the probability that our randomly chosen graph G contains either a clique of size 4 or an anticlique of size n. For every given subset $S \subseteq \{1, \ldots, k\}$, the probability of S being a clique is given by q^6. There are $\binom{k}{4} \leq \frac{1}{24}k^4$ choices for the subset S, so that the probability of finding a clique of size 4 is $\leq \frac{1}{24}k^4 q^6$. If we take $q = k^{-\frac{2}{3}}$, then the probability of finding a clique of size 4 is $\leq \frac{1}{24}$.

What about the probability of finding an anticlique of size n? Fix a real number $\delta < 0$, and suppose that $k \leq n^{3/2-\delta}$. For a given set $T \subseteq \{1, \ldots, k\}$ of size n, the probability that T is an anticlique is given by $(1-q)^{\binom{n}{2}}$. Note that

$$e^q = 1 + q + \frac{1}{2}q^2 + \cdots < 1 + q + q^2 + q^3 = \frac{1}{1-q},$$

so that $1 - q < e^{-q}$. Consequently, the probability that T is an anticlique is

$$< (e^{-q})^{\binom{n}{2}} = e^{-(n-1)/2 \times nk^{-2/3}} \leq e^{-(n-1)/2 \times n^{2\delta/3}}$$

Moreover, there are $\binom{k}{n}$ choices for the set T. The probability of finding such an anticlique is therefore bounded above by

$$\binom{k}{n} e^{-(n-1)/2 \times n^{2\delta/3}} \leq k^n e^{-(n-1)/2 \times n^{2\delta/3}} \leq n^{3n/2} e^{-(n-1)/2 \times n^{2\delta/3}}$$

We would like to guarantee that this is $< \frac{23}{24}$ (so that the total probability of having a clique of size 4 or an anticlique of size n will be < 1). Taking logarithms, we want

$$\frac{3}{2}n \log n - \frac{n-1}{2}n^{2\delta/3} < \log \frac{23}{24}$$

or

$$\frac{n-1}{2}n^{2\delta/3} - \frac{3}{2}n \log n > \log \frac{24}{23}.$$

Note that this condition is automatically satisfied for large enough values of n (since $\log n$ grows more slowly than $n^{2\delta/3}$). We have proven:

Theorem 7. *Let $\delta > 0$ be a positive real number. Then, for every sufficiently large number n, we have*

$$R(4, n) \geq n^{3/2-\delta}.$$