Math 155 (Lecture 23)

October 30, 2011

Let S be a finite set, and let $\text{Part}(S)$ denote the collection of all equivalence relations on S. Recall that $\text{Part}(S)$ has a least element $E_\perp$ and a largest element $E_\top$. In the last lecture, we proved the following formula for the Möbius function of $\text{Part}(S)$:

Theorem 1. *If S is a finite set with n elements, we have*

$$\mu(E_\perp, E_\top = (-1)^{n-1}(n-1)!.$$

Let us now describe a typical application of this formula.

Question 2. Let S be a finite set. How many connected graphs are there with vertex set S?

There is an analogous question which is much easier to answer: the total number of (possibly disconnected) graphs with vertex set S is given by $2^{\binom{n}{2}}$, where n is the number of elements of S. To turn this into an answer to Question 2, we need to analyze the difference between connected and disconnected graphs.

Note that if G is a graph with vertex set S, then G determines an equivalence relation E_G on S. Here E_G is the equivalence relation of "being in the same connected component of G": that is, xE_Gy if and only if there is a path in G joining x with y. Note that a graph G is connected if and only if $E_G = E_\top$ is the largest element of $\text{Part}(S)$.

Let X be the set of all graphs with vertex set S. For each equivalence relation $E \in \text{Part}(S)$, define

$$X_E = \{G \in X : E_G \leq E\} \qquad X(E) = \{G \in X : E_G = E\}.$$

Note that $X_E = \bigcup_{E' \leq E} X(E')$, so that

$$|X_E| = \sum_{E' \leq E} |X(E')|.$$

Applying Möbius inversion, we get

$$|X(E')| = \sum_{E \leq E'} \mu(E, E')|X_E|.$$

In particular, the number of connected graphs is given by

$$\sum_{E \in \text{Part}(S)} \mu(E, E_\top)|X_E|.$$

Let us now evaluate each individual summand. Note that $\{E' \in \text{Part}(S) : E \leq E'\}$ is isomorphic to the set of equivalence relations on the set S/E. Using Theorem 1, we deduce

$$\mu(E, E_\top) = (-1)^{|S/E|-1}(|S/E| - 1)!.$$

The size of the set $|X_E|$ is easy to determine: an element of X_E is just a graph, each of whose connected components is contained in an equivalence class of E. The number of such graphs is given by

$$\prod_{K \in S/E} 2^{\binom{|K|}{2}}.$$

We can therefore write the answer to Question 2 as

$$\sum_{E \in \mathrm{Part}(S)} (-1)^{|S/E|-1}(|S/E|-1)! 2^{\sum_{K \in S/E} \binom{|K|}{2}}$$

We can do a little better by writing this as a sum not over equivalence relations, but over partitions $n = k_1 + 2k_2 + \cdots$, where n denotes the number of elements of S. Recall that the number of equivalence relations with exactly k_i equivalence classes of cardinality i is given by

$$\frac{n!}{\prod_{i \geq 1}(i!^{k_i} k_i!)}$$

We may therefore rewrite our answer as

$$n! \sum_{n=k_1+2k_2+\cdots} (-1)^{k-1}(k-1)! \prod_{i \geq 1} \frac{(i! 2^{\binom{i}{2}})^{k_i}}{k_i!}$$

where k denotes the sum $k_1 + k_2 + \cdots$.

Remark 3. We have already studied another method of obtaining formulas like this. Let Y denote the species of graphs and Y_0 denote the species of connected graphs. Since every graph can be written uniquely as a union of connected components, we have $Y = \exp(Y_0)$. It follows that the exponential generating functions of Y and Y_0 are related by the formula

$$F_Y(x) = e^{F_{Y_0}(x)}.$$

We can write this as

$$F_{Y_0}(x) = \log(F_Y(x)) = \log \sum_{n \geq 0} \frac{2^{\binom{n}{2}}}{n!} x^n.$$

Applying the power series expansion for the logarithm to this, we can recover the same answer to Question 2.

Let's now study another example of a Möbius function.

Example 4. Let $\mathbf{Z}_{>0}$ be the set of positive integers, partially ordered by divisibility. Then $\mathbf{Z}_{>0}$ is locally finite (any divisor of n is $\leq n$, so every positive integer has only finitely many divisors). Let us compute the Möbiusfunction $\mu_{\mathbf{Z}_{>0}}$.

Fix an integer $n > 0$ with prime factorization $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$. Let $X \subseteq \mathbf{Z}_{>0}$ be the set of divisors of n: namely, those integers of the form

$$p_1^{f_1} p_2^{f_2} \cdots p_k^{f_k}$$

where $f_i \leq e_i$ for $1 \leq i \leq k$. As a partially ordered set, X can be identified with the product

$$\prod_{1 \leq i \leq k} \{0, 1, \ldots, e_k\}.$$

Combining this with our understanding of the Möbius function of the factors, we see that the Möbius function μ_X of X is given by

$$\mu_X(\prod p_i^{f_i}, \prod p_i^{g_i}) \quad = \quad \prod_{1 \le i \le k} \begin{cases} 1 & \text{if } f_i = g_i \\ -1 & \text{if } f_i = g_i - 1 \\ 0 & \text{otherwise.} \end{cases}$$

In other words, we ahve

$$\mu_X(m, m') = \begin{cases} (-1)^j & \text{if } \frac{m'}{m} \text{ is a product of } j \text{ distinct primes.} \\ 0 & \text{otherwise.} \end{cases}$$

Since μ_X is just given by the restriction of $\mu_{\mathbf{Z}_{>0}}$ to X, we get

$$\mu_{\mathbf{Z}_{>0}}(m, n) = \begin{cases} (-1)^j & \text{if } \frac{n}{m} \text{ is a product of } j \text{ distinct primes.} \\ 0 & \text{otherwise.} \end{cases}$$

Note that the integer $\mu_{\mathbf{Z}_{>0}}(m, n)$ depends only on the quotient $\frac{n}{m}$. It is therefore traditional to rewrite $\mu_{\mathbf{Z}_{>0}}$ as a function one variable. Let us say that an integer n is *square-free* if it is not divisible by the square of any prime. Define

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n = p_1 \cdots p_k \text{ is square-free} \\ 0 & \text{otherwise.} \end{cases}$$

We can then write

$$\mu_{\mathbf{Z}_{>0}}(m, n) = \begin{cases} \mu(\frac{n}{m}) & \text{if } m | n \\ 0 & \text{otherwise.} \end{cases}$$

The function $\mu : \mathbf{Z}_{>0} \to \mathbf{Z}$ constructed above often simply referred to as the *Möbiusfunction*. Applying Möbius inversion in this context gives the following:

Proposition 5. *Let $f : \mathbf{Z}_{>0} \to \mathbf{Z}$ be an arbitrary function, and define $g : \mathbf{Z}_{>0} \to \mathbf{Z}$ by the formula*

$$g(n) = \sum_{d|n} f(d).$$

Then we can recover f by the formula

$$f(n) = \sum_{d|n} g(d)\mu(\frac{n}{d}).$$

Example 6. Recall that Euler's ϕ-function $\phi : \mathbf{Z}_{>0} \to \mathbf{Z}$ assigns to each integer n the number of elements of the set $\{1, 2, \ldots, n\}$ which are relatively prime to n. Set $X = \{1, 2, \ldots, n\}$. For each $d|n$, let X_d denote the set of all elements $m \in X$ such that the greatest common divisor of m and n is d. The function $m \mapsto \frac{m}{d}$ induces a bijection from X_d to the subset of $\{1, 2, \ldots, \frac{n}{d}\}$ consisting of elements which are relatively prime to $\frac{n}{d}$. We therefore have $|X_d| = \phi(\frac{n}{d})$. Since X is given by the disjoint union of the X_d's, we obtain $n = \sum_{d|n} \phi(\frac{n}{d}) = \sum_{d|n} \phi(d)$. Applying Proposition 5, we get

$$\phi(n) = \sum_{d|n} d\mu(\frac{n}{d}),$$

which recovers the formula for ϕ that we deduced from the inclusion-exclusion principle.