Math 155 (Lecture 23)

October 28, 2011

Definition 1. Let S be a finite set. We let $\mathrm{Part}(S)$ denote the set of all *partitions* of S: that is, the set of all decompositions of S into nonempty disjoint subsets. In other words, $\mathrm{Part}(S)$ is the collection of all equivalence relations E on S. If E is an equivalence relation on S, we denote the set of equivalence classes by S/E.

We regard S as a partially ordered set as follows: we let $E \leq E'$ if xEy implies $xE'y$. In other words, $E \leq E'$ if every equivalence class of E is contained in an equivalence class of E'.

Remark 2. The partially ordered set $\mathrm{Part}(S)$ has a least element $E_\perp$, given by the *discrete* equivalence relation where $xE_\perp y$ if and only if $x = y$. It also has a greatest element $E_\top$, given by the *indiscrete* equivalence relation with $xE_\top y$ for all $x, y \in S$.

Example 3. If S has cardinality 1, then $\mathrm{Part}(S)$ has exactly one element. If $S = \{1, 2\}$, then $\mathrm{Part}(S)$ has two elements: the greatest element and the least element described in Remark 2. If $S = \{1, 2, 3\}$, then the partially ordered set $\mathrm{Part}(S)$ is depicted in the diagram

$$\begin{tikzcd}
& \{1,2,3\} & \\
\{1\}\{2,3\} \arrow[ur] & \{1,3\}\{2\} \arrow[u] & \{1,2\}\{3\} \arrow[ul] \\
& \{1\}\{2\}\{3\} \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

We would like to study the Möbius function $\mu_{\mathrm{Part}(S)}$ of the partially ordered set $\mathrm{Part}(S)$ of partitions of a finite set S. Suppose that E is equivalence relation on S. We let $\mathrm{Part}(S)_{\leq E} = \{E' \in \mathrm{Part}(S) : E' \leq E\}$. Write S as a disjoint union $S_1 \cup S_2 \cup \cdots \cup S_m$ of E-equivalence classes. Note that to give an equivalence relation E' on S with $E' \leq E$, we just need to specify the restriction of E' to each of the sets S_i. In other words, we have a canonical isomorphism of partially ordered sets

$$\mathrm{Part}(S)_{\leq E} \simeq \prod \mathrm{Part}(S_i)$$

. This isomorphism carries E to the greatest element of the product $\prod_{1 \leq i \leq m} \mathrm{Part}(S_i)$. Using our product formula for Möbius functions, we get

$$\mu_{\mathrm{Part}(S)}(E_\perp, E) = \prod_{1 \leq i \leq m} \mu_{\mathrm{Part}(S_i)}(E_\perp, E_\top),$$

where $E_\perp$ and $E_\top$ denote the discrete and indiscrete equivalence relations of Remark 2 (note that underlying set on which these equivalence relations reside depends on i).

Question 4. Let S be a set with n elements. What is the integer $\mu_{\mathrm{Part}(S)}(E_\perp, E_\top)$?

Example 5. If $n = 1$, then $E_\perp = E_\top$ so the answer to Question 4 is 1. If $n = 2$, then $E_\perp < E_\top$ with nothing in between, so the answer to Question 4 is -1. If $n = 3$, then an inspection of the diagram of Example 3 shows that there are three chains of length 2 from $E_\perp$ to $E_\top$, and a chain of length 1. The answer is therefore $3 - 1 = 2$.

Let us do one more example. Let $S = \{1, 2, 3, 4\}$. Let's count the chains from $E_\perp$ to $E_\top$ in $\mathrm{Part}(S)$:

(a) There is exactly one chain of length 1, given by $\{E_\perp, E_\top\}$.

(b) The chains of length 2 are exactly those of the form $\{E_\perp < E < E_\top\}$, where E is some element of $\mathrm{Part}(S)$ distinct from $E_\perp$ and $E_\top$. The number of chains is therefore $b_4 - 2$, where b_4 is the 4*th Bell number* from Lecture 4. We have $b_4 = 15$, so there are 13 such chains.

(c) The chains of length 3 have the form $\{E_\perp < E < E' < E_\top\}$. Here E is necessarily an equivalence relation which partitions S into a two element subset $\{i, j\}$ and two singletons. There are $\binom{4}{2} = 6$ choices for E. Given E, there are three ways to build a larger equivalence relation E' distinct from $E_\top$: we can enlarge the equivalence class $\{i, j\}$ by adding either of the two other elements, or we could combine those two elements into another equivalence class. The number of such chains is therefore $3 \times 6 = 18$.

(d) There are no chains of length ≥ 4.

It follows that $\mu_{\mathrm{Part}(S)}(E_\perp, E_\top) = -1 + 13 - 18 = -6$.

Motivated by the calculations

$$1, -1, 2, -6, \ldots$$

of Example 5, we can make the following conjecture:

Guess 6. If S has n elements, then

$$\mu_{\mathrm{Part}(S)}(E_\perp, E_\top) = (-1)^{n-1}(n-1)!.$$

Let's prove that this guess is correct. We will use induction on n. We have already handled the case $n = 1$, so assume that $n > 1$. The inductive hypothesis tells us the following: for every set T having cardinality $m < n$, we have

$$\mu_{\mathrm{Part}(T)}(E_\perp, E_\top) = (-1)^m(m-1)!.$$

In particular, if $E < E_\top$ in $\mathrm{Part}(S)$, we have

$$\mu_{\mathrm{Part}(S)}(E_\perp, E) = \prod_{T \in S/E} \mu_{\mathrm{Part}(T)}(E_\perp, E_\top) = \prod_{T \in S/E} (-1)^{|T|-1}(|T|-1)!.$$

Since $n > 1$, we have $E_\perp \neq E_\top$, so that

$$\sum_{E \in \mathrm{Part}(S)} \mu_{\mathrm{Part}(S)}(E_\perp, E) = 0.$$

We can write this sum as

$$\mu_{\mathrm{Part}(S)}(E_\perp, E_\top) + \sum_{E \neq E_\top} \prod_{T \in S/E} (-1)^{|T|-1}(|T|-1)!.$$

To prove that $\mu\mathrm{Part}(S)(E_\perp, E_\top) = (-1)^{n-1}(n-1)!$, it will suffice to show that the sum

$$\sum_{E \in \mathrm{Part}(S)} \prod_{T \in S/E} (-1)^{|T|-1}(|T|-1)!$$

is equal to zero.

Rather than proving this identity separately for each n, let us try to prove it for all n simultaneously. Define

$$C_n = \sum_{E \in \mathrm{Part}(\langle n \rangle)} \prod_{T \in \langle n \rangle / E} (-1)^{|T|-1} (|T|-1)!$$

and let $f(x)$ denote the generating function

$$\sum_{n \geq 0} \frac{(-1)^n C_n}{n!} x^n.$$

We wish to prove that $C_n = 0$ for $n \geq 2$: that is, that $f(x)$ is a linear function.

Let's decompose C_n into pieces. Fix integers $k_1, k_2, \ldots$ with $k_1 + 2k_2 + 3k_3 + \cdots = n$, and let $C_n^{\vec{k}}$ denote the sum

$$\sum_{E} \prod_{T \in \langle n \rangle / E} (-1)^{|T|-1} (|T|-1)!$$

where the sum is taken over all equivalence relations with k_1 equivalences classes of size 1, k_2 equivalence classes of size 2, and so forth. Each term in the sum is identical, given by $\prod_{i \geq 1} ((-1)^{i-1}(i-1)^!)^{k_i}$. The number of terms is given by the quotient

$$\frac{n!}{\prod (i!)^{k_i} k_i!}.$$

It follows that we can write

$$\begin{aligned}
f(x) &= \sum_{n \geq 0} \frac{(-1)^n x^n}{n!} \sum_{n = k_1 + 2k_2 + \cdots} \frac{n!}{\prod_{i \geq 1} (i!^{k_i} k_i!)} \prod_{i \geq 1} (i-1)!^{k_i} (-1)^{(i-1)k_i} \\
&= \sum_{n \geq 0} \sum_{n = k_1 + 2k_2 + \cdots} \prod_{i \geq 1} \frac{x^{ik_i} (-1)^{k_i}}{i^{k_i} k_i!} \\
&= \prod_{i \geq 1} \sum_{k \geq 0} \left(\frac{-x^i}{i}\right)^k \frac{1}{k!} \\
&= \prod_{i \geq 1} e^{\frac{-x^i}{i}} \\
&= e^{\sum_{i \geq 1} - \frac{x^i}{i}} \\
&= e^{\log(1-x)} \\
&= 1 - x.
\end{aligned}$$

which is a linear function, as desired.