Math 114, Problem Set 7 (due Monday, November 4)

November 1, 2013

(1) Let $V_0, V_1, V_2, V_3, \ldots$ be real vector spaces with norms

$$|| \bullet ||_n : V_n \to \mathbb{R}_{\geq 0}.$$

Given an element $\vec{v} = (v_n)_{n \geq 0} \in \prod_{n \geq 0} V_n$, let

$$||\vec{v}|| = \sum_{n \geq 0} ||v_n||_n \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

Let $V \subseteq \prod_{n \geq 0} V_n$ be the subset consisting of those elements $\vec{v}$ such that $||\vec{v}|| < \infty$. Show that V is real vector space and that $\vec{v} \mapsto ||\vec{v}||$ is a norm on V. If each V_n is a Banach space, show that V is a Banach space. We will refer to V as the ℓ^1-*sum* of the Banach spaces $\{V_n\}_{n \geq 0}$.

(2) Suppose we are given a sequence $E_0, E_1, E_2, \ldots \subseteq \mathbb{R}^m$ of pairwise disjoint measurable subsets of $\mathbb{R}^m$. Let $E = \bigcup E_n$. Show that $L^1(E)$ is isomorphic to the ℓ^1-sum of the Banach spaces $L^1(E_n)$.

(3) Let $E \subseteq \mathbb{R}^n$ be a measurable set, let p and q be real numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1$, and suppose that $f \in L^p(E)$, $g \in L^q(E)$ are functions satisfying

$$\int_E fg = ||f||_{L^p} ||g||_{L^q}.$$

Prove that either $f = 0$, or there exists a nonnegative real number λ such that $|g| = \lambda |f|^{p/q}$ almost everywhere.

(4) Let $p, q, r > 1$ be real numbers satisfying $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Let $E \subseteq \mathbb{R}^n$ be measurable, and let $f \in L^p(E)$ and $g \in L^q(E)$. Show that the product function fg belongs to $L^r(E)$, and that

$$||fg||_{L^r} \leq ||f||_{L^p} ||g||_{L^q}.$$