Math 114, Problem Set 1 (due Monday, September 16)

September 9, 2013

(1) Let $f : [0, \pi] \to \mathbb{R}$ be a continuous function satisfying $f(0) = f(\pi) = 0$, and define real numbers $a_1, a_2, \ldots$ by the formula

$$a_n = \frac{2}{\pi} \int_0^\pi \sin(nx) f(x) dx.$$

Show that the sum $\sum_{n>0} a_n^2$ converges (hint: compare the sum with the integral $\int_0^\pi f(x)^2 dx$).

(2) Let $f : [0, \pi] \to \mathbb{R}$ be the discontinuous function given by the formula

$$f(x) = \begin{cases} 1 & \text{if } \frac{\pi}{4} < x < \frac{3\pi}{4} \\ 0 & \text{otherwise.} \end{cases}$$

Determine the real numbers $a_n = \frac{2}{\pi} \int_0^\pi \sin(nx) f(x) dx$. Using the identity

$$\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \tan^{-1}(1) = \frac{\pi}{4},$$

compute the value of the infinite sums

$$g(x) = \sum_{n>0} a_n \sin(nx)$$

when $x = \frac{\pi}{4}$ and $\frac{3\pi}{4}$.

(3) Let $V \subseteq \mathbb{R}^n$ be a linear subspace of dimension $< n$. Show that the outer measure $\mu^*(V)$ is equal to zero.