

Appendix A

Sketch of Mean Width

A.1 Sizes, Dimensions, and Measures

The toolbox of every scientist and engineer is full of devices for measuring the size of objects of interest. Chemists measure the distance between atoms in Å, materials scientists often measure microstructure in μm , and astronomers measure distance in parsecs. These are each linear measures of size that differ only by a constant. Areas and volumes are usually quoted in terms of the same quantities to the power of the dimension of the object being measured.

Clearly, the choice of unit depends on the magnitude of the distance being measured, and the choice of dimensionality of the measure—be it length, area, or volume—depends on the property of interest. For example, consider a fishtank. If we want to know how much water it can hold, we care about its volume. If we want to cover the glass with a reflective coating, we care about its surface area. If we want to know whether it will fit on the bookshelf between the encyclopedias, we care about its length.

We can easily talk about measures of an object in dimensions from one up to the intrinsic dimension of the object. For example, an egg is three-dimensional and DaVinci's Mona Lisa is, nearly, two-dimensional, yet we can still talk about the two-dimensional surface area of an egg or the one-dimensional distance between the tip of the Mona Lisa's nose and her right ear. The application determines the dimension of the measurement.

In many cases, the choice of measure is obvious. For example, the most natural three-dimensional measure of a three-dimensional object is probably volume. A natural one dimensional measure of a two dimensional object might be its perimeter. Sometimes, though, the choice of measure is

not obvious. For example, what is a good one-dimensional measure of a three-dimensional object? Imagine measuring a pencil. If you are designing a high-end pocket protector, you care about the distance from eraser to tip. Yet, if you are designing a pencil sharpener, you probably care more about the pencil's diameter. Both are one-dimensional measures of a three-dimensional object, yet neither, generally speaking, seems more natural or intuitive.

In this chapter, we consider the question of whether some measures are intrinsically better than others. In particular, we show that like volume and surface area, there exists a natural choice for a one-dimensional measure of three-dimensional objects. Surprisingly, most scientists and engineers never use this measure, and as a result, one-dimensional measures of higher dimensional objects have proliferated. To see this, consider how many different ways biologists measure cell sizes or materials scientists measure grain sizes. Clearly, size does matter and how we measure it matters too.

A.2 Steiner's Formula

To see that not all measures of a given dimension are created equal, we consider an interesting mathematical relationship found by the Swiss geometer Jakob Steiner in the 1840's. Steiner considered convex objects growing at constant rates (more specifically, each point on the surface moves normal to the surface at a constant velocity, v). For example, picture an expanding chicken egg or a growing ice cube. How does the volume of the egg or the ice-cube evolve in time? Steiner proved that

$$V(t) = C_0V(0) + C_1S(0)(vt) + C_2\mathcal{L}(0)(vt)^2 + C_3(vt)^3, \quad (\text{A.1})$$

where $V(0)$ is the volume of an object at $t = 0$, $S(0)$ is half of its surface area, and $\mathcal{L}(0)$ is a mysterious linear measure of the object called the *mean width*; the C_n are constants that can be easily determined as shown below [153]. Steiner's formula is exact, beautiful and remarkably simple. It tells us that at any time the volume depends only on vt and three measurements of the initial shape — the volume, surface area, and a linear measure \mathcal{L} we have not yet defined.

We now focus on two simple applications of Steiner's formula. Consider a sphere¹ of radius $r = r_0 + vt$ for all $t \geq 0$. The volume of a sphere is given by $\frac{4}{3}\pi r^3$, so we have:

$$V(t) = \frac{4}{3}\pi(r_0 + vt)^3 = \frac{4}{3}\pi r_0^3 + 4\pi r_0^2(vt) + 4\pi r_0(vt)^2 + \frac{4}{3}\pi(vt)^3 \quad (\text{A.2})$$

Knowing that $V(0) = \frac{4}{3}\pi r_0^3$ and $S(0) = \frac{1}{2}4\pi r_0^2 = 2\pi r_0^2$, we can compare Equation (A.1) with

¹In this chapter we use the term *sphere* to refer to the sphere *and* its interior.

Equation (A.2) to determine that $C_0 = 1$, $C_1 = 2$, and $C_3 = \frac{4}{3}\pi$. Because we have not yet defined \mathcal{L} , we cannot yet determine C_2 . We leave this until the next section.

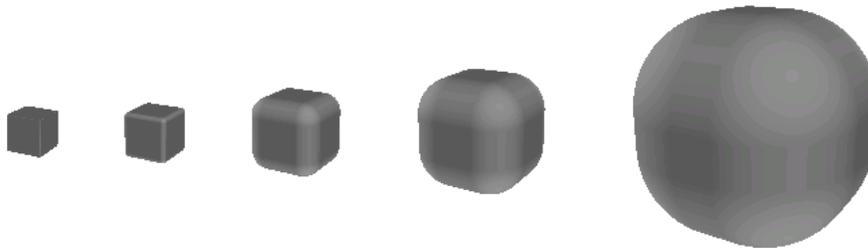


Figure A.1: A growing cube for $vt = 0, 0.25, 1, 2$, and 5 in arbitrary units.

Now, consider a growing cube with initial edge length a , as shown in Figure (A.1) at various times. We can see in Figure (A.2) that the growing cube consists of an initial cube plus 6 parallelepipeds (yellow), 12 quarter cylinders (blue), and 8 eighth spheres (red). Therefore,

$$\begin{aligned} V(t) &= a^3 + (6)a^2(vt) + (12)a\pi(vt)^2/4 + (8)\frac{4}{3}\pi(vt)^3/8 \\ &= a^3 + 6a^2(vt) + 3a\pi(vt)^2 + \frac{4}{3}\pi(vt)^3. \end{aligned} \quad (\text{A.3})$$

Because $V(0) = a^3$ and $S(0) = \frac{1}{2}6a^2 = 3a^2$, we can compare (A.3) with Steiner's formula (A.1) to

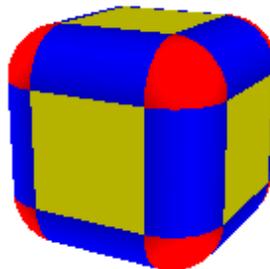


Figure A.2: Dissected growing cube.

see that $C_0 = 1$, $C_1 = 2$, and $C_3 = \frac{4}{3}\pi$. Remarkably, the C_n are the same for the growing cube as they were for the sphere! What is so beautiful about Steiner's formula is that the constants are just that — they are the same for all convex shapes. Note that we can still not determine C_2 because the mean width \mathcal{L} has yet to be defined. However, we can already begin to see that the mean width is as fundamental a measure of three-dimensional objects as volume and surface area. In the following section, we explore the mean width measure \mathcal{L} and then return to Steiner's formula.

A.3 Measures and their Properties

In the previous section, we saw the utility of a special one-dimensional measure of three-dimensional objects called the *mean width*. In this section we rigorously define this measure and discuss a few of its interesting properties. With this we will be able to calculate, among other things, the constant C_2 in Steiner's formula which we were unable to determine above.

By means of introduction, a brief discussion of measures is in order. A *measure*, roughly speaking, is a way of assigning a nonnegative number, or size, to an object. Among other things, defining a measure allows us to say that one object is bigger than another. Measures often have special rules that allow us to measure the combination of two objects in terms of the measures of the individual pieces. For example, we might have a piece of string that is 5 inches long and a second piece that is 8 inches long. If we splice them together, then their total length would be $5 + 8 = 13$ inches minus any overlap of the two pieces. More generally, a measure m is considered to be additive if it satisfies $m(A \cup B) = m(A) + m(B) - m(A \cap B)$, where A and B are two objects being measured, $m(A \cup B)$ is their union, and $m(A \cap B)$ their overlap, or intersection. This is *additivity*.

Many measures that are familiar to us are additive. For example, consider the volumes $V(A)$ and $V(B)$ of two three-dimensional objects A and B . If volume is additive, then the volume of the union of A and B is $V(A \cup B) = V(A) + V(B) - V(A \cap B)$. Figure A.3 shows two spheres with radii 4 (red) and 6 (yellow) separated by a distance of 5; the volume of their intersection is $\frac{565}{12}\pi$. Because volume is additive, we can calculate the volume of the union of the two spheres: $\frac{4}{3}\pi 4^3 + \frac{4}{3}\pi 6^3 - \frac{565}{12}\pi = \frac{1305}{4}\pi$.

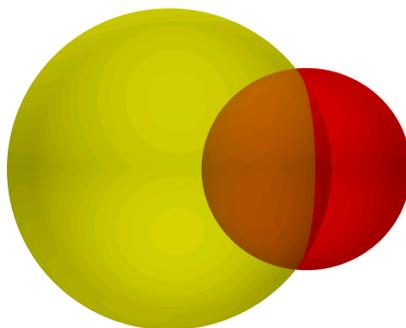


Figure A.3: Two intersecting spheres.

Another example of an additive measure is surface area, S . The surface area of the union of A and B is $S(A \cup B) = S(A) + S(B) - S(A \cap B)$. Returning to Figure A.3 and knowing that the surface area of the intersection is 46π , we can calculate the surface area of the union as $4\pi 4^2 + 4\pi 6^2 - 46\pi = 162\pi$.

Although volume and surface area are additive, not all measures are so well behaved. For example, consider a two-dimensional measure $V^{2/3}$, where V is the conventional volume of the object. Using again the spheres as our objects (Figure A.3), we immediately see that $(\frac{1305}{4}\pi)^{2/3} \neq (\frac{256}{3}\pi)^{2/3} + (\frac{864}{3}\pi)^{2/3} - (\frac{565}{12}\pi)^{2/3}$. Likewise, we might propose a one-dimensional ‘length’ measure of three-dimensional objects, $l = \sqrt[3]{V}$. Again, using our sphere example, it is easy to see that $l(A \cup B) \neq l(A) + l(B) - l(A \cap B)$. And so although we have additive 2- and three-dimensional measures of three-dimensional objects (surface area and volume), we have yet to identify an additive one-dimensional measure. We will later see that the mean width is such a measure.

Two more desirable properties of measures are rotational and translational invariance. Translating or rotating the union of spheres in Figure A.3 does not change its volume. While this may seem trivial, many measures do not possess this type of invariance. To illustrate this point, consider $w(A) = \text{maximal width of } A \text{ along the } x\text{-axis}$. If the union of spheres in Figure A.3 was rotated about an axis coming out of the page by 90° , this measure $w(A \cup B)$ changes from 8 to 6. Clearly, $w(\cdot)$ is not a rotationally invariant measure.

One last useful property of a measure is *scalability*. If we enlarge an object by some factor c , we would like to see the measure of that object increase proportionally. For example, if we double the edge lengths of a cube, then its length increases by a factor of 2^1 , its surface area by a factor of 2^2 , and its volume by a factor of 2^3 . In the more general case, the standard measures (length, area, and volume) are each multiplied by a factor of c^d , where c is the uniform scaling factor and d is the dimensionality of the measure.

A.4 Integrating Mean Curvature

We suggested above that the measure \mathcal{L} , known as the *mean width*, is a natural one-dimensional measure of three-dimensional objects. We first consider a definition of \mathcal{L} as an integral of the mean curvature H over the surface of an object D :

$$\mathcal{L}(D) = \frac{1}{2\pi} \int_{\partial D} H dA. \tag{A.4}$$

where $H = k_1 + k_2$ is the mean curvature, k_1 and k_2 are the principal curvatures (the minimal and maximal curvatures of all curves on the surface passing through a point), ∂D is the surface of D and dA is an element of area of the surface.² Since curvature measures how ‘‘curved’’ a surface is

²In some books the mean curvature is given by $\frac{1}{2}$ of the H we defined here.

at a point, this integral tells you about how “curved” a surface is overall. It can be shown that this measure is both additive and scalable (for the sake of simplicity and brevity we omit the proofs of these statements). The mean width is also rotationally and translationally invariant. Hence, the mean width is a *good* measure. In the 1950’s, the Swiss mathematician Hugo Hadwiger proved that in fact, up to a constant multiple, the mean width is the *only* such measure [154].

Using this definition we can easily calculate the mean width of a sphere and a cube, and determine the constant C_2 . First we consider the mean width of a sphere S of radius r . The surface of a sphere of radius r has uniform mean curvature $H = 1/r + 1/r = 2/r$. Integrating over the surface we get,

$$\mathcal{L}(S) = \frac{1}{2\pi} \int_{\partial S} \frac{2}{r} dA = \frac{1}{2\pi} 4\pi r^2 \frac{2}{r} = 4r. \quad (\text{A.5})$$

Since we now know that the mean width of the sphere of radius r is $4r$, comparing Equations (A.1) and (A.2) shows us that $C_2 = \pi$.

We now return to the cube and calculate its mean width by integrating the mean curvature over its surface. For reasons which will become clear later, we first focus on the “growing cube” of Figure (A.2). The growing cube has 3 types of surfaces elements: 6 flat faces, each with surface area a^2 and constant mean curvature $H = 0$; 12 quarter-cylinder edges, each with surface area $\frac{\pi(vt)a}{2}$ and constant mean curvature $H = 1/(vt)$; and 8 eighth-sphere corners, each with surface area $\frac{\pi(vt)^2}{2}$ and constant mean curvature $H = 2/(vt)$. Hence,

$$\mathcal{L}(\text{growing cube}) = \frac{1}{2\pi} \left[\int_{\text{faces}} H dA + \int_{\text{edges}} H dA + \int_{\text{corners}} H dA \right] \quad (\text{A.6})$$

$$= \frac{1}{2\pi} \left[6(a^2)0 + 12 \left(\frac{\pi(vt)a}{2} \right) \frac{1}{(vt)} + 8 \left(\frac{\pi(vt)^2}{2} \right) \frac{2}{(vt)} \right] \quad (\text{A.7})$$

$$= 0 + 3a + 4(vt). \quad (\text{A.8})$$

At $t = 0$, the growing cube of Figure (A.2) reduces to a standard cube, and so:

$$\mathcal{L}(\text{cube}) = 3a. \quad (\text{A.9})$$

Using this value of \mathcal{L} for the cube in Steiner’s equation, Equation A.1, and comparison of the result with Equation (A.3) again shows that $C_2 = \pi$. Indeed, C_2 , like the rest of the C_n , is the same for every convex shape!

A.5 Integrating Euler Widths

While the above definition allows us to calculate \mathcal{L} for a number of simple shapes, it is not general enough to apply to one- and two- dimensional objects. A more general definition of the mean width can be obtained by integrating the Euler characteristic χ rather than the mean curvature. We present here a brief description of the Euler characteristic.

The Euler characteristic is a 0-dimensional measure which represents a particular type of counting. For polygons and for the surfaces of polyhedra, the Euler characteristic is defined as follows:

$$\chi = V - E + F, \tag{A.10}$$

where V , E , and F are the number of vertices, edges, and faces, respectively. As a simple example, a triangle in the plane has 3 vertices, 3 edges, and 1 face and an Euler characteristic $\chi = 3 - 3 + 1 = 1$; a rectangle has 4 vertices, 4 edges, and 1 face, and therefore an Euler characteristic $\chi = 4 - 4 + 1 = 1$. In fact, any solid polygon in the plane has $\chi = 1$. The surface of a cube has 8 vertices, 12 edges, and 6 faces and $\chi = 8 - 12 + 6 = 2$. Interestingly, any polyhedral surface homeomorphic³ to the sphere will have $\chi = 2$. In the plane, χ is equal to the number of connected objects minus the number of holes in those objects (note, χ need not be positive).

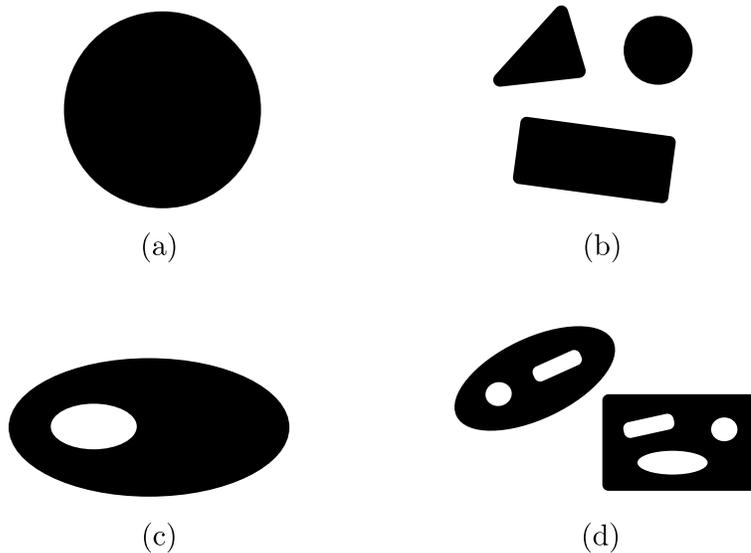


Figure A.4: Figures with different numbers of objects and holes.

A few illustrations (see Figure A.4) might help explain the Euler characteristic of objects em-

³Loosely speaking, two objects are homeomorphic if one can be deformed into the other without opening new holes or closing existing ones.

bedded in the plane. Figure (A.4a) shows only one object with no holes, so $\chi = 1$; Figure (A.4b) shows three objects, so $\chi = 3$; Figure (A.4c) shows a single object containing one hole, and so $\chi = 1 - 1 = 0$; and Figure (A.4d) has two objects and five holes, and so $\chi = 2 - 5 = -3$.

Using the Euler characteristic, we now define a one-dimensional measure of three-dimensional objects in the following manner. Take an object D and a line ℓ , and consider all cross-sections of D perpendicular to ℓ . Figure (A.5) shows an example with a few such cross-sections. Now, taking the Euler characteristic of each cross-section and integrating over *all* planes perpendicular to ℓ gives us the *Euler width* of D in the direction ℓ :

$$E_\ell(D) = \int_\ell \chi(D \cap P_\ell) d\ell, \tag{A.11}$$

where a cross-section of D is the intersection of the three-dimensional object D with a plane P_ℓ perpendicular to ℓ , i.e. $D \cap P_\ell$.

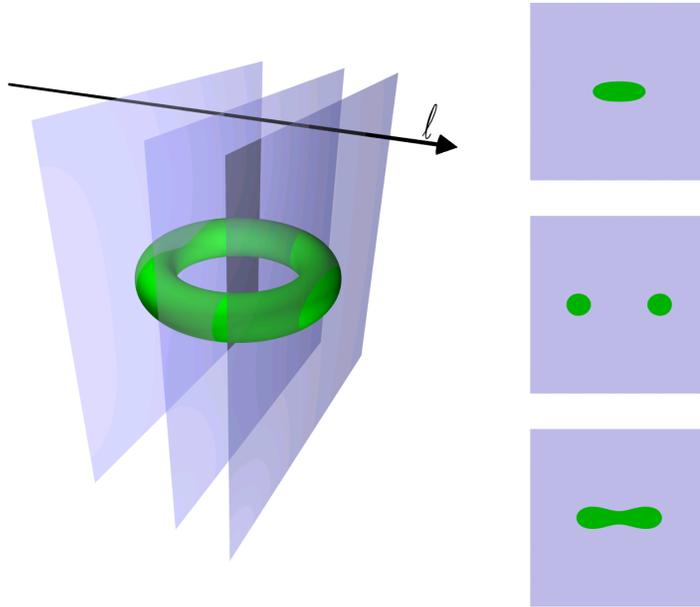


Figure A.5: Torus with cross-section cuts.

As a simple example, consider a sphere of radius r . It is easy to see that the Euler width in *any* direction is just $2r$. Slightly more complicated is the Euler width of the torus T of Figure (A.5) in the direction ℓ_1 . The radius of the tube is r and the inner radius of the torus is R . Every plane perpendicular to ℓ_1 and intersecting the torus has either $\chi = 1$ or $\chi = 2$; planes that do not intersect

the torus at all have $\chi = 0$. Figure (A.6) illustrates this for various regions of ℓ_1 . We then have:

$$E_\ell(T) = \int_\ell \chi(T \cap P_\ell) d\ell = (\chi = 1)2r + (\chi = 2)2R + (\chi = 1)2r = 4(R + r)$$

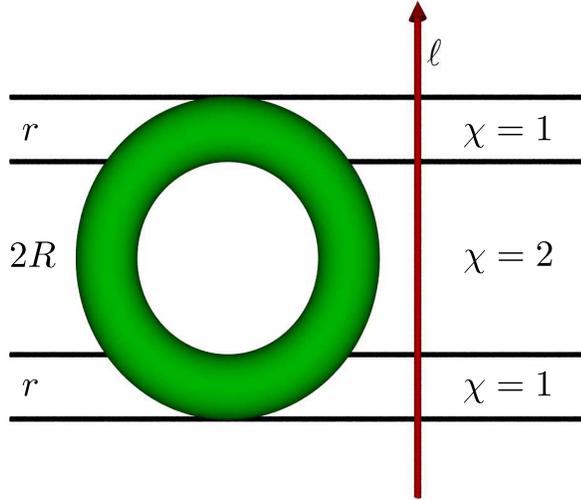


Figure A.6: Torus with values of χ .

Next, we consider the Euler width of the same torus but in the direction pointed directly through the hole of the ring. Every plane perpendicular to this line either fails to intersect the torus, in which case $\chi = 0$, or else intersects it to form an annulus, or a circle with a hole in its center, in which case again $\chi = 0$. Therefore, the Euler width in this direction is exactly 0. It should be clear that the value of the Euler width generally depends on the direction ℓ .

We now reintroduce the mean width in terms of the Euler width, without reference to the mean curvature. The mean width is twice the average Euler width over all lines ℓ passing through the origin; the particular choice of origin is immaterial as the mean width is invariant upon translation of the object or the origin. The mean width is then the mean *Euler* width.

In the case of a sphere of radius r , we saw above that the Euler width in every direction is $2r$. Considering all lines through the origin (which is most conveniently chosen to be the center of the sphere), it is obvious that the average is $2r$, and so $\mathcal{L} = 4r$, consistent with our earlier calculations.

One advantage of the definition based on the Euler characteristic over that based on the mean curvature is that it can be directly applied to objects that do not have smooth two-dimensional surfaces. As a simple example, consider a straight line segment of length L embedded in a three-dimensional space and choose the origin to lie along the line segment. The Euler width in the direction of the line segment will be L , the Euler width in a direction perpendicular to the line

segment will be 0, and the Euler width in all other directions ℓ will be $0 < E_\ell < L$. The average Euler width turns out to be $L/2$ and so the mean width \mathcal{L} , which is defined as twice the average Euler width, turns out to be exactly L . By this definition, it turns out that the mean width of any space curve with length L is always exactly L .

A.6 Numerical Evaluation of the Mean Width

The analysis above provides methods for the exact, analytic determination of the mean width. Unfortunately, the analytical determination of the mean width is rather difficult, except for a relatively small set of highly symmetric objects such as those considered here. Fortunately, there exists a relatively straightforward way to numerically approximate the mean width of an arbitrary shape provided that a piecewise linear discretization of the shape is available.

Consider the torus of Figure A.7. Calculating the Euler width in arbitrary directions is rather tricky even for such a relatively simple shape. However, numerically calculating the mean width of a discretized torus, as shown in Figure A.8, is relatively straightforward. Using analysis similar to what we used to calculate the mean width of the growing cube (Figure A.2), we can easily divide the integral into sums over the faces, edges, and vertices of this object. Because the triangles are flat, their mean curvature is 0 everywhere. All curvature of the object is concentrated in the edges. The integral of the mean curvature along each edge is $l_i\alpha_i$, where l_i is the length of edge i and α is the turning angle there between the two adjacent edges. This leaves us with:

$$\mathcal{L}(A) = \sum_i^n e_i\alpha_i \tag{A.12}$$

for a discretized surface. As the discretization triangles become smaller, the approximating surface converges to the object, and the calculated mean width converges to exact mean width. In Chapter 4, we use Equation A.12 to calculate the mean width of grains with triangulated surfaces.

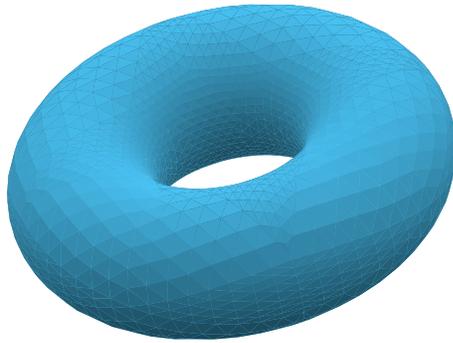


Figure A.7: Plain torus.

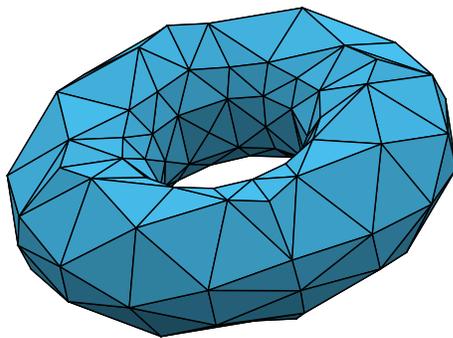


Figure A.8: Discretized torus.