FINITENESS OF FROBENIUS TRACES OF A SHEAF ON A 
FLAT ARITHMETIC SCHEME 

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ABSTRACT. For a lisse ℓ-adic sheaf on a scheme flat and of finite type over \( \mathbb{Z} \), we consider the field generated over \( \mathbb{Q} \) by Frobenius traces of the sheaf at closed points. Assuming conjectural properties of geometric Galois representations of number fields and the Generalized Riemann Hypothesis, we prove that the field is finite over \( \mathbb{Q} \) when the sheaf is de Rham at \( \ell \) pointwise. This is a number field analogue of Deligne’s finiteness result about Frobenius traces in the function field case.

1. Introduction

In [5], Fontaine and Mazur made the conjecture that an \( \ell \)-adic Galois representation of a number field that is geometric, i.e., unramified almost everywhere and de Rham above \( \ell \) comes from algebraic geometry via étale cohomology. Furthermore, based on standard conjectures on étale cohomology, they conjectured that such an \( \ell \)-adic Galois representation satisfies motivic properties (see Conjecture 1.3 for the precise form). One can also expect that such a statement is true in the relative setting: consider an irreducible lisse \( \mathbb{Q}_\ell \)-sheaf \( \mathcal{E} \) on an algebraic variety over \( \mathbb{Q} \). If the restriction of \( \mathcal{E} \) to each closed point is geometric as \( \ell \)-adic Galois representation of the residue field, then \( \mathcal{E} \) satisfies motivic properties (see also [9]).

One of the motivic properties that one can expect is the finiteness of Frobenius traces. To be more precise, we have the following conjecture.

Conjecture 1.1. Let \( \ell \) be a prime. Let \( X \) be a scheme flat and of finite type over \( \mathbb{Z}[\ell^{-1}] \) and \( \mathcal{E} \) a lisse \( \overline{\mathbb{Q}_\ell} \)-sheaf on \( X \). If \( \mathcal{E} \) is de Rham pointwise (see below for the definition), then there exists a number field \( E \) such that \( \text{tr}(\text{Frob}_x, \mathcal{E}_x) \in E \) for every closed point \( x \) of \( X \) (and a geometric point \( \overline{x} \) above \( x \)).

Here we say that \( \mathcal{E} \) is de Rham at \( \ell \) pointwise if for every finite extension \( F \) of \( \mathbb{Q}_\ell \) and every morphism \( \alpha: \text{Spec } F \to X \), \( \alpha^* \mathcal{E} \) regarded as an \( \ell \)-adic Galois representation of \( F \) is de Rham in the sense of Fontaine. Note that Ruochuan Liu and Xinwen Zhu proved that the de Rham assumption at a single point in each geometric connected component of \( X \) implies that the sheaf is de Rham at \( \ell \) pointwise ([9, Theorem 1.3]).

Since the Fontaine-Mazur conjecture on Galois representations (i.e., the case when \( \dim X = 1 \)) is still wide open, Conjecture 1.1 is far beyond our reach. However, it is still interesting to ask whether the Fontaine-Mazur conjecture implies Conjecture 1.1. In this note, we give a partial answer to this question. Here is our main theorem.

Theorem 1.2. Assume the Fontaine-Mazur conjecture (Conjecture 1.3) and the Generalized Riemann Hypothesis for Dedekind zeta functions. Then Conjecture 1.1 holds.
Let us now recall the precise form of consequences of the Fontaine-Mazur conjecture.

**Conjecture 1.3** ([5, §4(d)]). Let \( \ell \) be a prime. Let \( K \) be a number field and let \( G_K \) denote the absolute Galois group of \( K \). For a prime \( \ell' \) we denote by \( S_{\ell'} \) the set of primes of \( K \) above \( \ell' \). Fix a finite set \( S \) of primes of \( K \) and let \( \rho \) be an irreducible continuous representation \( G_K \to \text{GL}_r(\mathbb{Q}_\ell) \) which is unramified outside \( S \cup S_{\ell} \) and de Rham at primes of \( K \) above \( \ell \). Then the following assertions hold.

(i) For every finite prime \( v \) of \( K \) with \( v \notin S \cup S_{\ell} \), the roots of \( \det(1 - \rho(\text{Frob}_v) t) \) are algebraic numbers.

(ii) There exists a number field \( E \) in \( \mathbb{Q}_\ell \) such that \( \det(1 - \rho(\text{Frob}_v) t) \in E[t] \) for every prime \( v \) of \( K \) with \( v \notin S \cup S_{\ell} \).

(iii) There exists a sufficiently large number field \( E \) in \( \mathbb{Q}_\ell \) satisfying (ii) and the following condition: for every prime \( \ell' \) and every embedding \( \sigma : E \hookrightarrow \mathbb{Q}_{\ell'} \), there exists a continuous representation \( \rho' : G_K \to \text{GL}_r(\mathbb{Q}_{\ell'}) \) unramified outside \( S \cup S_{\ell'} \) such that

\[
\det(1 - \rho'(\text{Frob}_v) t) = \sigma \det(1 - \rho(\text{Frob}_v) t)
\]

for every prime \( v \) of \( K \) with \( v \notin S \cup S_{\ell} \cup S_{\ell'} \).

Note that Conjecture 1.1 is a natural generalization of Conjecture 1.3(ii).

We now turn to Theorem 1.2. It is an analogue of the following finiteness theorem of Frobenius traces in the function field case by Deligne.

**Theorem 1.4** ([2, Théorème 3.1]). Let \( \ell \) and \( p \) be distinct primes. Let \( X \) be a scheme of finite type over the finite field \( \mathbb{F}_p \) and \( E \) a lisse \( \mathbb{Q}_\ell \)-sheaf on \( X \) such that the Frobenius trace \( \text{tr}(\text{Frob}_x, E_x) \) is an algebraic number for every closed point \( x \in X \). Then there exists a finite extension \( E \) of \( \mathbb{Q} \) in \( \mathbb{Q}_\ell \) such that \( \text{tr}(\text{Frob}_x, E_x) \in E \) for every closed point \( x \in X \).

In the function field case, L. Lafforgue proved the Langlands correspondence for \( \text{GL}_r \) over function fields and obtained an analogue of Conjecture 1.3 ([6, Théorème VII.6]). Deligne used the Langlands correspondence as an input to prove Theorem 1.4. We remark that although Lafforgue’s result directly implies Deligne’s theorem when \( \dim X = 1 \), the general case is not simply reduced to the curve case; Deligne needed the Weil conjecture and all the statements of Lafforgue’s theorem including the existence of compatible systems. This may explain why we need to assume the Generalized Riemann Hypothesis and Conjecture 1.3(ii) in Theorem 1.2.

Let us briefly mention another consequence of the relative Fontaine-Mazur conjecture. Consider a relative version of Conjecture 1.3(iii), i.e., the existence of a compatible system attached to a given de Rham lisse \( \mathbb{Q}_\ell \)-sheaf on a flat scheme of finite type over \( \mathbb{Z} \). One can also ask whether Conjecture 1.3 implies such a motivic result in the relative setting. Results of Drinfeld in [3] answer this question affirmatively. Note that his main theorem ([3, Theorem 1.1]) is an unconditional result for the function field analogue, but his key theorem ([3, Theorem 2.5]) together with Conjecture 1.3 implies the existence of a compatible system in the number field case. One can also obtain unconditional results in some cases in the number field situation by using results on potential automorphy. See [11] in this direction.
Now we explain how to prove Theorem 1.2. In the sequel, a curve means an open subscheme of the spectrum of the ring of integers of a number field. Assume Conjecture 1.3. First we know that for each closed point $x \in X$ the Frobenius trace $\text{tr}(\text{Frob}_x, \mathcal{E}_x)$ is algebraic over $\mathbb{Q}$ by Conjecture 1.3(i) applied to some curve passing through $x$.

For a positive integer $N$ we denote by $\mathbb{Q}(\text{tr} \mathcal{E})_{\leq N}$ the field generated over $\mathbb{Q}$ by $\text{tr}(\text{Frob}_x, \mathcal{E}_x)$ with $\#k(x) \leq N$, where $k(x)$ denotes the residue field of $x$. Then $\mathbb{Q}(\text{tr} \mathcal{E})_{\leq N}$ is finite over $\mathbb{Q}$ since there are only finitely many closed points $x$ with $\#k(x) \leq N$.

By Conjecture 1.3(ii), for each curve $C$ and each morphism $\varphi: C \to X$, there exists a positive integer $N$ such that $\text{tr}(\text{Frob}_y, (\varphi^* \mathcal{E})_y) \in \mathbb{Q}(\text{tr} \varphi^* \mathcal{E})_{\leq N}$ for every closed point $y \in C$. Let $N(C, \varphi)$ be the smallest integer satisfying this property.

In the body of the note, we will prove the following:

(a) The integer $N(C, \varphi)$ is controlled by the ramification and boundary of $C$ (Proposition 3.9).

(b) For each closed point $x \in X$, there exists a curve $\varphi_x: C_x \to X$ passing through $x$ with small ramification and boundary (Proposition 2.2).

It follows from statements (a) and (b) (in the precise form) that for each closed point $x \in X$, there exists a curve $\varphi_x: C_x \to X$ passing through $x$ such that the associated integer $N(C_x, \varphi_x)$ does not grow as fast as $\#k(x)$.

Once this assertion is obtained, we can find a sufficiently large integer $N_0$ such that if $N_0 < \#k(x)$ then $N(C_x, \varphi_x^* \mathcal{E}) < \#k(x)$ and thus $\text{tr}(\text{Frob}_x, \mathcal{E}_x) \in \mathbb{Q}(\text{tr} \mathcal{E})_{\leq \#k(x)-1}$. From this we can show by induction on $\#k(x)$ that the Frobenius trace $\text{tr}(\text{Frob}_x, \mathcal{E}_x)$ lies in the number field $\mathbb{Q}(\text{tr} \mathcal{E})_{\leq N_0}$ for every closed point $x$ of $X$, which completes the proof.

Let us briefly mention the proof of part (a). It uses Faltings’ lemma appearing in his proof of the Shafarevich conjecture and the effective Chebotarev theorem. Our argument needs the Generalized Riemann Hypothesis to have a stronger version of the effective Chebotarev theorem. We also need Conjecture 1.3(iii) (the existence of a compatible system) when we apply Faltings’ lemma to estimate $N(C, \varphi)$. Note that the proof of part (a) is the only place where we use the Generalized Riemann Hypothesis and Conjecture 1.3(iii).

Finally, we explain the organization of the note. Section 2 presents part (b) (Proposition 2.2). In Section 3, we first review Faltings’ lemma and the effective Chebotarev theorem, and then we establish part (a) (Proposition 3.9). In Section 4, we combine the discussions in the previous sections and prove the main theorem.

**Notation.** A number field means a field that is finite over $\mathbb{Q}$. For a number field $E$ and a finite place $\lambda$ of $E$, we denote by $\mathcal{O}_E$ the ring of integers of $E$ and by $\overline{E}_\lambda$ a fixed algebraic closure of the $\lambda$-adic completion $E_\lambda$ of $E$. We denote by $d_E$ the absolute value of the discriminant of $E$ over $\mathbb{Q}$.

For a scheme $X$, we denote by $|X|$ the set of closed points of $X$. In particular, we identify $|\text{Spec } \mathbb{Z}|$ with the set of rational primes. We denote the residue field of a point $x$ of a scheme by $k(x)$ and $\pi$ denotes a geometric point above $x$. We omit the base points of étale fundamental groups to simplify the notation.

2. Curves with small ramification and boundary

The purpose of this section is to prove the existence of a curve (i.e., an open subscheme of the ring of integers of a number field) with small ramification and
boundary on a smooth scheme over \( \mathbb{Z} \) (Proposition 2.2). This is statement (b) in the introduction.

**Definition 2.1.** Let \( K \) be a number field and \( C \) an open subscheme of \( \text{Spec} \mathcal{O}_K \). We define subsets \( S_{K}^{\text{ram}}, S_{C}^{\text{id}}, \) and \( S_{C} \) of \( |\text{Spec} \mathbb{Z}| \) as follows:

- \( S_{K}^{\text{ram}} \) denotes the set of rational primes that ramify in \( K \).
- \( S_{C}^{\text{id}} := u(\text{Spec} \mathcal{O}_K - C) \) where \( u : \text{Spec} \mathcal{O}_K \to \text{Spec} \mathbb{Z} \) is the structure morphism.
- \( S_{C} := S_{K}^{\text{ram}} \cup S_{C}^{\text{id}} \).

We consider the following situation. Let \( X \) be a connected smooth scheme over \( \text{Spec} \mathbb{Z} \) equipped with an étale morphism \( \pi : X \to \mathbb{A}^n_{\mathbb{Z}} \). Denote by \( d_{\pi} \) the degree of \( \pi \) over the generic point of \( \mathbb{A}^n_{\mathbb{Z}} \). Then there exists an open subscheme \( U \subset \mathbb{A}^n_{\mathbb{Z}} \) such that \( U \subset \text{Im} \pi \) and that \( \pi^{-1}(U) \to U \) is finite and étale of degree \( d_{\pi} \).

Write \( \mathbb{A}^n_{\mathbb{Z}} = \text{Spec} \mathbb{Z}[t_1, \ldots, t_m] \) and let \( I \) denote the definition ideal of the reduced closed subscheme \( \mathbb{A}^n_{\mathbb{Z}} \setminus U \). We fix a nonzero element \( f(t_1, \ldots, t_m) \in I \). Let \( d_f \) be the total degree of \( f \) and \( B_f \) the maximum of the absolute values of the coefficients of \( f \).

**Proposition 2.2.** With the notation as above, let \( p \) be a prime and \( n \) a positive integer. Then for each point \( x \in X(\mathbb{F}_p^n) \) there exist a number field \( K \), an open subscheme \( C \subset \text{Spec} \mathcal{O}_K \), and a morphism \( \varphi : C \to X \) satisfying the following conditions:

- (i) There exists a point \( \tilde{x} \in C(\mathbb{F}_p^n) \) such that \( \varphi(\tilde{x}) = x \).
- (ii) \( [K : \mathbb{Q}] \leq d_{\pi} n \) and the following inequality holds:

\[
\prod_{p' \in S_{C}} p' \leq 2^{n(d_{\pi}+m)}B_f^n(d_f+1)^{2n}n^{2n}p^{d_f+n^2+2n-2}.
\]

The rest of the section is devoted to the proof of Proposition 2.2. The outline of the proof is as follows: we first choose a curve of the form \( \text{Spec} \mathcal{O}_M \) where \( M \) is a number field, using a lift of a minimal polynomial of the extension \( \mathbb{F}_p \subset \mathbb{F}_p^n \). Then we construct a morphism \( \text{Spec} \mathcal{O}_M \to \mathbb{A}^n_{\mathbb{Z}} \) passing through \( \pi(x) \), using the polynomial \( f(t_1, \ldots, t_m) \in I \). We pull back the curve by \( \pi \) and take the connected component which passes through \( x \). This is the curve \( C \). The construction of \( \text{Spec} \mathcal{O}_M \to \mathbb{A}^n_{\mathbb{Z}} \) gives a control of \( S_{C} \) and yields the proposition.

**Proof of Proposition 2.2.** The point \( x \in X(\mathbb{F}_p^n) \) and the morphism \( \pi : X \to \mathbb{A}^n_{\mathbb{Z}} \) define a ring homomorphism \( \varphi : \mathbb{Z}[t_1, \ldots, t_m] \to \mathbb{F}_p^n \). Write \( \mathbb{F}_p^n = \mathbb{F}_p[t]/(\mathfrak{g}(t)) \) with a monic irreducible polynomial \( \mathfrak{g}(t) \in \mathbb{F}_p[t] \) of degree \( n \).

Choose a polynomial \( g(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0 \in \mathbb{Z}[t] \) such that \( g \) is a lift of \( \mathfrak{g} \) and \( 0 \leq a_i \leq p-1 \) for each \( i = 0, \ldots, n-1 \). Denote \( \mathbb{Q}[t]/(g(t)) \) by \( M \). Then the \( \mathbb{Q} \)-algebra \( M \) is a field extension of \( \mathbb{Q} \) of degree \( n \).

**Lemma 2.3.**

(i) If \( \alpha \in C \) is a root of \( g(t) \), then \( |\alpha| \leq np \).

(ii) Let \( d_M \) denote the absolute value of the discriminant of \( M \) over \( \mathbb{Q} \). Then \( d_M \leq (n!)^2 n^2 p^{2n-2} \). In particular, \( \prod_{p' \in S_{K}^{\text{ram}}} p' \leq (n!)^2 n^2 p^{2n-2} \).
\textbf{Proof.} Note that \(|a_i| \leq p\) for each \(i\). Then (i) follows from the following inequalities:
\[
|\alpha^n| = |a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0| \\
\leq |a_{n-1}||\alpha|^{n-1} + \cdots + |a_1||\alpha| + |a_0| \\
\leq np \max\{1, |\alpha|^{n-1}\}.
\]

As for (ii), it suffices to show the first inequality since the second follows from
the first and the obvious inequality \(\prod_{g \in S\setminus \{t\}} p' \leq d_M\). Let \(\text{disc} g \in \mathbb{Z}\) be the
discriminant of the polynomial \(g(t)\). Since \(g(t)\) is monic, \(\mathbb{Z}[t]/(g(t))\) is an order of \(M\)
contained in \(\mathcal{O}_M\). Comparing the discriminants of the orders yields \(d_M \leq |\text{disc} g|\).
Thus it suffices to prove \(|\text{disc} g| \leq (n!)^2 n^n p^{2n-2}\).

Note that the discriminant \(\text{disc} g\) is a polynomial in \(a_0, \ldots, a_{n-1}\) of degree \(2n - 2\)
with integer coefficients. Moreover, \(\text{disc} g\) consists of at most \((n!)^2\) terms, and the
absolute value of each coefficient is at most \(n^n\); this follows from the relation
between \(\text{disc} g\) and the resultant of \(g(t)\) and its derivative \(g'(t)\). Since \(|a_i| \leq p\) for
every \(i\),
\[
|\text{disc} g| \leq (n!)^2 n^n (\max_i |a_i|)^{2n-2} \leq (n!)^2 n^n p^{2n-2}.
\]

\(\square\)

We fix a root \(\alpha\) of \(g(t)\) in \(\mathbb{C}\). This gives an embedding \(M \hookrightarrow \mathbb{C}\) and we regard \(\alpha\)
as an element of \(\mathcal{O}_M\). By construction, there exists a unique prime \(p\) of \(M\) above \(p\)
and it satisfies \(\mathcal{O}_M/p \cong \mathbb{F}_p^n\). Recall that we have fixed a nonzero polynomial
\(f(t_1, \ldots, t_m)\) of degree \(d_f\) that vanishes on \(\mathbb{A}^m_{\mathbb{Z}} \setminus U\).

\textbf{Lemma 2.4.} \textit{There exist \(m\) elements \(\beta_1, \ldots, \beta_m \in \mathcal{O}_M\) satisfying the following
conditions:}

\begin{itemize}
  \item Each \(\beta_j\) is of the form \(\sum_{0 \leq i \leq n-1} b_i \alpha^i\) (\(b_i \in \mathbb{Z}\)) with \(0 \leq b_i \leq (d_f + 1)p - 1\).
  \item The ring homomorphism \(\mathbb{Z}[t_1, \ldots, t_m] \rightarrow \mathcal{O}_M\) sending \(t_j\) to \(\beta_j\) is a lift of
    \(\varphi: \mathbb{Z}[t_1, \ldots, t_m] \rightarrow \mathbb{F}_p^n = \mathcal{O}_M/p\).
  \item \(f(\beta_1, \ldots, \beta_m) \neq 0\).
\end{itemize}

\textbf{Proof.} The lemma follows from an elementary fact: if \(f(t_1, \ldots, t_m) \in \mathbb{Z}[t_1, \ldots, t_m]\)
is a nonzero polynomial of total degree at most \(d_f\), then for any subsets \(S_1, \ldots, S_m\)
of \(\mathbb{C}\) of cardinality \(d_f + 1\), there exists an \(m\)-tuple \((\beta_1, \ldots, \beta_m) \in S_1 \times \cdots \times S_m\) such
that \(f(\beta_1, \ldots, \beta_m) \neq 0\).

The fact is easily proved by induction on \(m\) and the verification is left to the
reader. Since \(\mathcal{O}_M/p = \mathbb{F}_p[t]/(\varphi(t))\) is generated over \(\mathbb{F}_p\) by \(\alpha\) mod \(p\), each \(\varphi(t_j)\) in
\(\mathcal{O}_M/p\) has \((d_f + 1)\) distinct lifts in the set
\[
\left\{ \sum_{0 \leq i \leq n-1} b_i \alpha^i \in \mathcal{O}_M \mid 0 \leq b_i \leq (d_f + 1)p - 1 \right\}.
\]
Thus the fact implies the lemma. \(\square\)

We take \(\beta_i\) as in Lemma 2.4 and denote by \(\varphi_M\) the induced morphism \(\text{Spec} \mathcal{O}_M \rightarrow \mathbb{A}^m_{\mathbb{Z}}\). By construction, the prime \(p\) defines a point \(x_M \in (\text{Spec} \mathcal{O}_M)(\mathbb{F}_p^n)\) and it satisfies
\(\varphi_M(x_M) = \pi(x)\). We also have \(\text{Im} \varphi_M \not\subset \mathbb{A}^m_{\mathbb{Z}} \setminus U\) since \(f(\beta_1, \ldots, \beta_m) \neq 0\). Let
\(C_M \subset \text{Spec} \mathcal{O}_M\) denote the open subscheme \(\varphi^{-1}_M(U) = \text{Spec} \mathcal{O}_M \times \mathbb{A}^m_{\mathbb{Z}} U\).

We give an estimate of the boundary of \(C_M\). Recall that \(B_f\) denotes the maximum
of the absolute values of the coefficients of \(f(t_1, \ldots, t_m)\).
Lemma 2.5.
\[
\prod_{p' \in S_{\text{C}}^\text{bd}} p' \leq 2^{n(d_f+m)}B_f^n(d_f+1)^d n^d + p' n^2.
\]

Proof. Note that the complement \(\text{Spec} \mathcal{O}_M \setminus C_M = \text{Spec} \mathcal{O}_M \times_{\mathbb{A}_F} (\mathbb{A}_F^m \setminus U)\) is contained in \(\text{Spec} \mathcal{O}_M / f(\beta_1, \ldots, \beta_m)\) as underlying topological spaces. Therefore
\[
\prod_{p' \in S_{\text{C}}^\text{bd}} p' \leq |N_{M/Q} f(\beta_1, \ldots, \beta_m)|,
\]
where \(N_{M/Q} : M \to \mathbb{Q}\) is the norm map. We will estimate the norm of \(f(\beta_1, \ldots, \beta_m)\).

By Lemma 2.3(i), \(|\sigma| \leq np\) for each \(\sigma \in \text{Hom}_{\mathbb{Q}}(M, \mathbb{C})\). Hence
\[
|\sigma | \leq n((d_f+1)p-1)(np)^{n-1} \leq (d_f+1)n^np^n
\]
for each \(i = 1, \ldots, m\) (see the first condition in Lemma 2.4). As \(f(t_1, \ldots, t_m)\) is a polynomial in \(m\) variables of total degree \(d_f\), the number of the terms of \(f(t_1, \ldots, t_m)\) is at most
\[
\sum_{0 \leq i \leq d_f} \binom{i + m - 1}{i} = \binom{d_f + m}{d_f} \leq 2^{d_f + m}.
\]
Hence \(|\sigma f(\beta_1, \ldots, \beta_m)| \leq 2^{d_f + m} B_f ((d_f + 1)n^p)^d_f\) and
\[
|N_{M/Q} f(\beta_1, \ldots, \beta_m)| = \prod_{\sigma \in \text{Hom}_{\mathbb{Q}}(M, \mathbb{C})} |\sigma f(\beta_1, \ldots, \beta_m)|
\leq (2^{d_f + m}) B_f (d_f + 1)^d n^d p^d n^2
\leq 2^{n(d_f + m)} B_f^n (d_f + 1)^d n^d p^d n^2.
\]

Finally we construct a curve \(\varphi : C \to X\) satisfying the conditions in Proposition 2.2. We have two morphisms \(\pi : X \to \mathbb{A}_F^m\) and \(\varphi_M : \text{Spec} \mathcal{O}_M \to \mathbb{A}_F^m\) with \(\pi(x_M) = \varphi_M(x_M)\). Therefore there exists a point \(\bar{x} \in (X \times_{\mathbb{A}_F^m} \text{Spec} \mathcal{O}_M)[\mathbb{F}_p]\) mapping to \(\bar{x}\) and \(x_M\) under the projections. Define \(C\) to be the connected component of \(X \times_{\mathbb{A}_F^m} \text{Spec} \mathcal{O}_M\) containing \(\bar{x}\). Since \(\pi\) is étale, \(C\) is an open subscheme of the ring of integers of a number field. Let \(K\) be the fraction field of \(C\) and let \(\varphi\) denote the morphism \(C \to X\).

It remains to check that \(\varphi : C \to X\) satisfies (i) and (ii) in Proposition 2.2. By construction, (i) holds. Since \(\pi\) is étale of generic degree \(d_x\), we have \([K : M] \leq d_x\) and thus \([K : \mathbb{Q}] \leq d_x n\).

We estimate \(\prod_{p' \in S_C} p'\). As \(\pi : X \to \mathbb{A}_F^m\) is finite and étale over \(U\), so is \(C \to \text{Spec} \mathcal{O}_M\) over \(C_M\). This implies that \(S_{\text{C}}^\text{bd} \subset S_{\text{C}}^\text{bd}\) and \(S_{\text{K}}^\text{ram} \subset S_{\text{K}}^\text{ram} \cup S_{\text{C}}^\text{bd}\). In particular, we have \(S_C \subset S_{\text{C}}^\text{bd}\) and Lemmas 2.3(ii) and 2.5 yield
\[
\prod_{p' \in S_C} p' \leq \prod_{p' \in S_{\text{C}}^\text{bd}} p' \prod_{p' \in S_{\text{C}}^\text{bd}} p'
\leq ((n!)^2 n^p 2^{n-2}) (2^{n(d_f + m)}) B_f^n (d_f + 1)^d n^d + p' n^2 + 2n-2
\leq 2^{n(d_f + m)} B_f^n (d_f + 1)^d n^d + n^2 + 2n-2.
\]
This completes the proof of Proposition 2.2. □
3. Faltings' trick and the effective Chebotarev theorem

In this section, we first review Faltings’ trick and the effective Chebotarev theorem. Then we will prove statement (a) in the introduction (Proposition 3.9).

Fix a rational prime \( \ell \) and a finite extension \( E_\lambda \) of \( \mathbb{Q}_\ell \). We denote by \( q \) the cardinality of the residue field of \( E_\lambda \). Let \( r \) be a positive integer.

Faltings used the following lemma in the proof of the Shafarevich conjecture ([4]).

Lemma 3.1. Let \( K \) be a number field, \( C \) an open subscheme of \( \text{Spec} \mathcal{O}_K \), and \( Y \subset |C| \) a subset. Assume that for every finite Galois extension \( L \) of \( K \) that is unramified over \( C \) and has degree at most \( q^{2r^2} \), every conjugacy class of \( \text{Gal}(L/K) \) is the image of the Frobenius \( \text{Frob}_y \) under \( \pi_1(C) \to \text{Gal}(L/K) \) for some \( y \in Y \). Then for any two semisimple continuous representations \( \rho_1, \rho_2 : \pi_1(C) \to \text{GL}_r(\mathcal{E}_\lambda) \), \( \rho_1 \cong \rho_2 \) if \( \text{tr} \rho_1(\text{Frob}_y) = \text{tr} \rho_2(\text{Frob}_y) \) for every \( y \in Y \).

Proof. This follows from the Chebotarev density theorem and the Brauer-Nesbitt theorem; see the proof of [4, Satz 5] or [1, Théorème 3.1]. \( \square \)

Theorem 3.2 (the effective Chebotarev theorem).

(i) ([7]) There exists a positive constant \( A_0 \) such that for every number field \( K \), every finite Galois extension \( L \) of \( K \), and every conjugacy class \( c \) of \( \text{Gal}(L/K) \), there exists a prime \( p \) of \( K \) which is unramified in \( L \) and satisfies \( \text{Frob}_p = c \) and

\[
N_{K/Q}^p \leq 2d_L^{A_0}.
\]

Here \( N_{K/Q}^p \) is the absolute norm of \( p \) and \( d_L \) is the absolute value of the discriminant of \( L \) over \( \mathbb{Q} \).

(ii) ([8]) There exists a positive constant \( A \) such that if the Generalized Riemann Hypothesis holds for the Dedekind zeta function for \( L \), then (i) is valid with a sharper estimate

\[
N_{K/Q}^p \leq A(\log d_L)^2.
\]

Remark 3.3. Theorem 3.2(i) is unconditional, but the inequality is not strong enough for our purpose. This is why we assume the GRH in Theorem 1.2.

Remark 3.4. The inequality given in [8, Corollary 1.2] is of the form

\[
N_{K/Q}^p \leq A(\log d_L)^2(\log \log d_L)^4,
\]

and Theorem 3.2(ii) is mentioned with a sketch of a proof at the end of the paper (see also the introduction of [7]). In our setting, the above weaker estimate suffices; we use the inequality in Theorem 3.2(ii) only to simplify the growth rate argument. For details, see Lemma 4.4 and Remark 4.5, where we compare the estimates in Proposition 2.2 and Theorem 3.2(ii).

The goal of the rest of this section is to prove statement (a) in the introduction (Proposition 3.9). We first recall basic facts in algebraic number theory.

Lemma 3.5. For a number field \( M \), we have

\[
d_M \leq \prod_{p \in S_{M/Q}^\infty} p^{[M:Q](1+\log_p[M:Q])}.
\]
3.2 Assume the GRH for Dedekind zeta functions and let \( A \).

Proposition 3.8.

Let \( A \).

Proof. This is standard. For example, it follows from Remark 13 in [10, III-6]. \(\square\)

Lemma 3.6. Let \( m \) be a positive integer and let \( a \) be a nonzero integer. Denote a

primitive \( m \)-th root of unity by \( \zeta_m \). Then \( \mathbb{Q}(\sqrt[2r]{a}, \zeta_m) \) is a finite Galois extension of \( \mathbb{Q} \) of degree at most \( m(m-1) \). Moreover, if a rational prime \( p \) ramifies in \( \mathbb{Q}(\sqrt[2r]{a}, \zeta_m) \), then \( p \) divides \( ma \).

Proof. The proof is easy and left to the reader. \(\square\)

Lemma 3.7. Let \( K \) be a number field and \( C \) an open subscheme of \( \text{Spec} \, \mathcal{O}_K \). Let \( L \) be a finite Galois extension of \( K \) that is unramified over \( C \) and has degree at most \( q^{2r^2} \). Set

\[
m := [K : \mathbb{Q}] + 1, \quad a := \prod_{p \in \mathcal{S}_C} p, \quad \text{and} \quad L' := L(\sqrt[2r]{a}, \zeta_m).
\]

Then \( L' \) is a finite Galois extension of \( K \) of degree at most \( [K : \mathbb{Q}][[K : \mathbb{Q}] + 1]q^{2r^2} \), and the primes in \( \text{Spec} \, \mathcal{O}_K \setminus C \) ramify in \( L' \). Moreover, the following inequality holds:

\[
d_{L'} \leq \left( [K : \mathbb{Q}] + 1 \right) \prod_{p \in \mathcal{S}_C} p \left( [K : \mathbb{Q}] + 1 \right)^3 q^{2r^2} (1 + 3 \log_2 ([K : \mathbb{Q}] + 1) + 2r^2 \log_2 q).
\]

Proof. Since \( L' \) is the composite field of \( L \) and \( \mathbb{Q}(\sqrt[2r]{a}, \zeta_m) \) over \( \mathbb{Q} \), Lemma 3.6 implies that \( L' \) is a finite Galois extension \( K \) of degree at most \( [K : \mathbb{Q}][[K : \mathbb{Q}] + 1]q^{2r^2} \). If we denote by \( V(m) \) the set of rational primes dividing \( m = [K : \mathbb{Q}] + 1 \), then we have \( \mathcal{S}_C^{\text{ram}}(\sqrt[2r]{a}, \zeta_m) \subseteq \mathcal{S}_C \cup V(m) \). This implies

\[
\mathcal{S}_{L'}^{\text{ram}} = \mathcal{S}_L^{\text{ram}} \cup \mathcal{S}_C^{\text{ram}}(\sqrt[2r]{a}, \zeta_m) \subseteq \mathcal{S}_C \cup V(m).
\]

Take \( p \in \text{Spec} \, \mathcal{O}_K \setminus C \) and let \( p \) be the rational prime below \( p \). Then the absolute ramification index of \( p \) is at most \( [K : \mathbb{Q}] \). On the other hand, the absolute ramification index of a prime of \( \mathbb{Q}(\sqrt[2r]{a}, \zeta_m) \) above \( p \) is at least \( m = [K : \mathbb{Q}] + 1 \) as \( p \) divides \( a \) exactly once. Therefore the absolute ramification index of a prime of \( L' \) above \( p \) is also at least \( m = [K : \mathbb{Q}] + 1 \), and thus \( p \) ramifies in \( L' \).

Finally, the estimate of \( d_{L'} \) follows from Lemma 3.5 and the above discussions on \( [L' : K] \) and \( \mathcal{S}_{L'}^{\text{ram}} \). Namely, we have

\[
d_{L'} \leq \prod_{p \in \mathcal{S}_{L'}^{\text{ram}}} p^{|L' : \mathbb{Q}|(1 + \log_2 |L' : \mathbb{Q}|)} \leq \left( \prod_{p \in \mathcal{S}_C^{\text{ram}}} p \right)^{|L' : \mathbb{Q}|(1 + \log_2 |L' : \mathbb{Q}|)}
\]

\[
\leq \left( [K : \mathbb{Q}] + 1 \right) \prod_{p \in \mathcal{S}_C} p \left( [K : \mathbb{Q}] + 1 \right)^3 q^{2r^2} (1 + 3 \log_2 ([K : \mathbb{Q}] + 1) + 2r^2 \log_2 q).
\]

\(\square\)

Proposition 3.8. Let \( K \) be a number field and \( C \) an open subscheme of \( \text{Spec} \, \mathcal{O}_K \). Assume the GRH for Dedekind zeta functions and let \( A \) be the constant in Theorem 3.2(ii). Set

\[
D(C) = \left( [K : \mathbb{Q}] + 1 \right) \prod_{p \in \mathcal{S}_C} p \left( [K : \mathbb{Q}] + 1 \right)^3 q^{2r^2} (1 + 3 \log_2 ([K : \mathbb{Q}] + 1) + 2r^2 \log_2 q)
\]

\[\text{8}\]
and
\[ Y = \{ y \in C \mid \# k(y) \leq A(\log D(C))^2 \}. \]

Then for any two semisimple continuous representations \( \rho_1, \rho_2 : \pi_1(C) \to \text{GL}_r(E_\lambda) \), \( \rho_1 \cong \rho_2 \) if \( \text{tr} \rho_1(\text{Frob}_y) = \text{tr} \rho_2(\text{Frob}_y) \) for every \( y \in Y \).

**Proof.** It suffices to prove that \( Y \) satisfies the assumption in Lemma 3.1. Let \( L \) be a finite Galois extension of \( K \) that is unramified over \( C \) and has degree at most \( q^{2r^2} \). Set \( m := [K : \mathbb{Q}] + 1 \) and \( a := \prod_{p \in S_{\text{red}}} p \), and define \( L' := L(\sqrt{a}, \zeta_m) \). By Lemma 3.7, \( L' \) is a finite Galois extension of \( K \) and \( d_{L'} \leq D(C) \).

By Theorem 3.2(ii), for every conjugacy class \( c \in \text{Gal}(L'/K) \) there exists a prime \( p \) of \( K \) which is unramified in \( L' \) and satisfies \( \text{Frob}_p = c \) and
\[ \# k(p) = N_{K/\mathbb{Q}} p \leq A(\log d_{L'})^2 \leq A(\log D(C))^2. \]
Moreover, we have \( p \in C \) as all the primes in \( \text{Spec} \mathcal{O}_K \setminus C \) ramify in \( L' \). Since \( \text{Gal}(L/K) \) is a quotient of \( \text{Gal}(L'/K) \), we see that \( Y \) satisfies the assumption in Lemma 3.1. \( \square \)

Let us state the key proposition of this section.

**Proposition 3.9.** Assume Conjecture 1.3 and the GRH for Dedekind zeta functions. Let \( A \) be the constant in Theorem 3.2(ii). Let \( K \) be a number field and \( C \) an open subscheme of \( \text{Spec} \mathcal{O}_K \), and let \( \mathcal{F} \) be a lisse \( E_\lambda \)-sheaf on \( C \) that is de Rham at \( \ell \) pointwise. Denote by \( \mathbb{Q}(\text{tr} \mathcal{F}) \leq A(\log D(C))^2 \) the field generated over \( \mathbb{Q} \) by \( \text{tr}(\text{Frob}_y, \mathcal{F}_C) \) for all \( y \in C \) with \( \# k(y) \leq A(\log D(C))^2 \). Then for each closed point \( y \) of \( C \)
\[ \text{tr}(\text{Frob}_y, \mathcal{F}_C) \in \mathbb{Q}(\text{tr} \mathcal{F}) \leq A(\log D(C))^2. \]

**Remark 3.10.**

(i) With the notation as in the last part of the introduction, applying Proposition 3.9 to \( \mathcal{F} = \varphi^* \mathcal{E} \) yields \( N(C, \varphi) \leq A(\log D(C))^2 \). This is the precise formulation of statement (a) in the introduction, and it is also an analogue of [2, Proposition 2.10].

(ii) In addition to the GRH, the proof uses the existence of a compatible system (Conjecture 1.3(iii)).

**Proof.** Taking semisimplification if necessary, we may assume that \( \mathcal{F} \) is semisimple. Set \( E_0 = \mathbb{Q}(\text{tr} \mathcal{F}) \leq A(\log D(C))^2 \). By Conjecture 1.3(ii) and (iii), \( E_0 \) is a number field and there exists a sufficiently large Galois extension \( E \) of \( E_0 \) containing \( E_0 \) such that the following conditions hold:

- The polynomial \( \det(1 - \text{Frob}_y t, \mathcal{F}_C) \) has coefficients in \( E \) for every closed point \( y \in C \).
- For every embedding \( \sigma : E \hookrightarrow E_\lambda \) there exists a lisse \( E_\lambda \)-sheaf \( \sigma \mathcal{F} \) on \( C \) satisfying
\[ \det(1 - \text{Frob}_y t, (\sigma \mathcal{F})_C) = \sigma \det(1 - \text{Frob}_y t, \mathcal{F}_C) \]
for every closed point \( y \in C \).

Take any element \( \sigma \in \text{Gal}(E/E_0) \). Then \( \sigma \) defines an embedding \( E \ni \sigma \rightrightarrows E \hookrightarrow E_\lambda \) and we get a lisse sheaf \( \sigma \mathcal{F} \) by the second property above. Taking semisimplification if necessary, we may further assume that \( \sigma \mathcal{F} \) is semisimple. We will prove that \( \mathcal{F} \cong \sigma \mathcal{F} \).

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Consider the set
\[ Y = \{ y \in C \mid \#k(y) \leq A(\log D(C))^2 \}. \]
By the definition of \( E_0 \), we know that for each \( y \in Y \), \( \text{tr}(\text{Frob}_y, \mathcal{F}_\varpi) \in E_0 \) and thus
\[ \text{tr}(\text{Frob}_y, \mathcal{F}_\varpi) = \sigma \text{tr}(\text{Frob}_y, \mathcal{F}_\varpi) = \text{tr}(\text{Frob}_y, (\sigma \mathcal{F})_\varpi). \]

Consider the semisimple continuous representations of \( \pi_1(C) \) corresponding to \( \mathcal{F} \) and \( \sigma \mathcal{F} \). Then they are isomorphic by Proposition 3.8 (assuming the GRH). Hence \( \mathcal{F} \cong \sigma \mathcal{F} \). Therefore for every closed point \( y \in C \),
\[ \text{tr}(\text{Frob}_y, \mathcal{F}_\varpi) \in E_{\text{Gal}(\overline{E}/E_0)} = E_0. \]

\[ \square \]

4. FINITENESS OF FROBENIUS TRACES

In this section, we prove Theorem 1.2. More precisely, we show the following theorem.

**Theorem 4.1.** Let \( \ell \) be a prime and \( E_\lambda \) a finite extension of \( \mathbb{Q}_\ell \). Let \( X \) be a flat \( \mathbb{Z}[(\ell^{-1})] \)-scheme of finite type and \( \mathcal{E} \) a lisse \( E_\lambda \)-sheaf on \( X \) that is de Rham at \( \ell \) pointwise (see the paragraph after Conjecture 1.1 for definition). Assume Conjecture 1.3 and the Generalized Riemann Hypothesis for Dedekind zeta functions. Then there exists a number field \( E \subset \overline{E}_\lambda \) such that \( \text{tr}(\text{Frob}_x, \mathcal{E}_\varpi) \in E \) for every closed point \( x \in X \).

Note that Conjecture 1.1 and Theorem 1.2 are stated for \( \overline{\mathbb{Q}}_\ell \)-sheaves. However, since any \( \overline{\mathbb{Q}}_\ell \)-sheaf comes from an \( E_\lambda \)-sheaf for some finite extension \( E_\lambda \) of \( \mathbb{Q}_\ell \), Theorem 1.2 follows from Theorem 4.1.

**Remark 4.2.** This theorem implies that there exists a number field \( E \subset E_\lambda \) such that \( \det(1 - \text{Frob}_x t, \mathcal{E}_\varpi) \in E[t] \) for every closed point \( x \in X \). For this apply the theorem to irreducible constituents of \( \bigwedge^k \mathcal{E} \).

In the sequel we prove Theorem 4.1. We fix \( \ell \) and \( E_\lambda \). By extending \( E_\lambda \) if necessary, we may assume that \( E_\lambda \) is Galois over \( \mathbb{Q}_\ell \). We denote by \( q \) the cardinality of the residue field of \( E_\lambda \). Let \( r \) denote the rank of \( \mathcal{E} \).

**Definition 4.3.** For a \( \mathbb{Z} \)-scheme \( Z \) of finite type and a lisse \( E_\lambda \)-sheaf \( \mathcal{F} \) on \( Z \), we define \( \mathbb{Q}(\text{tr} \mathcal{F}) \) to be the field generated over \( \mathbb{Q} \) by \( \text{tr}(\text{Frob}_z, \mathcal{F}_\varpi) \) for all \( z \in |Z| \). For a positive integer \( N \), we denote by \( \mathbb{Q}(\text{tr} \mathcal{F})_{\leq N} \) the field generated over \( \mathbb{Q} \) by \( \text{tr}(\text{Frob}_z, \mathcal{F}_\varpi) \) for all \( z \in |Z| \) with \( \#k(z) \leq N \).

Note that Conjecture 1.3(i) implies that \( \mathbb{Q}(\text{tr} \mathcal{E}) \) is algebraic over \( \mathbb{Q} \); in fact, for every closed point \( x \in X \) there exist a number field \( K \), an open subscheme \( C \subset \text{Spec} \mathcal{O}_K \), a point \( \bar{x} \in C(k(x)) \), and a morphism \( \varphi: C \to X \) with \( \varphi(\bar{x}) = x \). Since the category of de Rham representations of the Galois group of an \( \ell \)-adic field is stable under subquotients, each constituent of \( \varphi^* \mathcal{E} \) is de Rham above \( \ell \). Hence Conjecture 1.3(i) implies that \( \text{tr}(\text{Frob}_x, \mathcal{E}_\varpi) = \text{tr}(\text{Frob}_y, \varphi^* \mathcal{E}_\varpi) \) is an algebraic number.

Next we reduce to the case where \( X \) is connected and smooth over \( \mathbb{Z} \). Note that if \( X \) is covered by finitely many locally closed subschemes \( X_i \), then \( \mathbb{Q}(\mathcal{E}) = \bigcup_i \mathbb{Q}(\mathcal{E}|_{X_i}) \) and thus it suffices to prove Theorem 4.1 for each \( X_i \).
Since there are at most finitely many irreducible components of $X$, we may assume that $X$ is irreducible. We may further assume that $X$ is reduced, hence integral. Consider the generic fiber of $X \to \text{Spec} \mathbb{Z}$. It admits a stratification by finitely many locally closed smooth subschemes over $\mathbb{Q}$, and this defines a stratification of the whole $X$ by finitely many locally closed subschemes with smooth generic fibers. Thus we may assume that $X$ has smooth generic fiber.

Then there exists an open subscheme $U \subset X$ such that $U$ is smooth over $\mathbb{Z}$ and contains the generic fiber of $X$. Note that the complement $X \setminus U$ is fibered over finitely many closed points of $\text{Spec} \mathbb{Z}$. Applying Theorem 1.4 to each of these fibers, we conclude that $\mathbb{Q}(\text{tr} \mathcal{E}|_{X\setminus U})$ is finite over $\mathbb{Q}$. Thus we may replace $X$ by $U$ and assume that $X$ is connected and smooth over $\mathbb{Z}$.

Now that $X \to \text{Spec} \mathbb{Z}$ is smooth, we may shrink $X$ and assume that there is an étale morphism $\pi: X \to \mathbb{A}_\mathbb{Z}^m$. This is the situation considered in Section 2. We use the same notation as in Section 2, which we now recall.

Denote by $d_\pi$ the degree of $\pi$ over the generic point of $\mathbb{A}_\mathbb{Z}^m$. Let $U$ be an open subscheme of $\mathbb{A}_\mathbb{Z}^m$ such that $U \subset \text{Im} \pi$ and that $\pi^{-1}(U) \to U$ is finite and étale of degree $d_\pi$. Write $\mathbb{A}_\mathbb{Z}^m = \text{Spec} \mathbb{Z}[t_1, \ldots, t_m]$ and let $I$ denote the definition ideal of the reduced closed subscheme $\mathbb{A}_\mathbb{Z}^m \setminus U$. We fix a nonzero element $f(t_1, \ldots, t_m) \in I$. Let $d_f$ be the total degree of $f$ and $B_f$ the maximum of the absolute values of the coefficients of $f$.

We also recall the notation from Section 3. Let $A$ be the constant in Theorem 3.2(ii). For a number field $K$ and an open subscheme $C$ of $\text{Spec} \mathcal{O}_K$, we set

$$D(C) = (|K: \mathbb{Q}| + 1) \prod_{p \in S_C} p \left(\left(|K: \mathbb{Q}| + 1\right)^3 q^{r^2} (1 + 3 \log_q (|K: \mathbb{Q}| + 1) + 2 r^2 \log_2 q)\right).$$

**Lemma 4.4.** There exists a positive integer $N_0$ satisfying the following: let $p$ be a prime, $n$ a positive integer, and $x \in X(\mathbb{F}_{p^n})$. If $p^n > N_0$, then for every number field $K$ and every curve $\varphi: C \to X$ with fraction field $K$ satisfying condition (ii) in Proposition 2.2, the following inequality holds:

$$A(\log D(C))^2 < p^n.$$

**Proof.** Let $p$ be a prime, $n$ a positive integer, and $x \in X(\mathbb{F}_{p^n})$. Take any number field $K$ and any curve $\varphi: C \to X$ satisfying the conditions in Proposition 2.2. Condition (ii) in Proposition 2.2 gives $|K: \mathbb{Q}| \leq d_\pi n$ and thus

$$\log D(C) \leq (d_\pi n + 1)^3 q^{2 r^2} (1 + 3 \log_2 (d_\pi n + 1) + 2 r^2 \log_2 q) \times \log((d_\pi n + 1)^{d_f(n+m)} B^n_p (d_f + 1)^{d_f(n+1)} 2^n (d_f n^2 + n^2) p^{d_f n^2 + 2n - 2}).$$

Note that $q, r, d_\pi, m, d_f$, and $B_f$ are constants that do not depend on $p, n$, or $x$. We denote by $M(p, n)$ the latter quantity in the above inequality. Then the lemma reduces to the following claim.

**Claim.** There exists a positive integer $N_0$ such that for every prime $p$ and every positive integer $n$ with $p^n > N_0$, the inequality

$$A(M(p, n))^2 \leq p^n$$

holds.

We prove the claim. Ignoring the constants $q, r, d_\pi, m, d_f$, and $B_f$, we have

$$M(p, n) = O\left(\log n (\log n + \log p) n^5\right) \quad \text{as} \quad p^n \to \infty,$$
and thus
\[ A(M(p, n))^2 = O\left((\log n)^2 (\log n + \log p)^2 n^{10}\right) \text{ as } p^n \to \infty. \]

If we set \( N = p^n \), then \( p \leq N \) and \( n \leq \log_2 N \). Hence we have
\[ (\log n)^2 (\log n + \log p)^2 n^{10} \leq (\log \log_2 N)^2 (\log \log_2 N + \log N)^2 (\log_2 N)^{10}. \]
Since the growth rate of the right hand side is \( o(N) \) as \( N \to \infty \), the claim follows. \( \square \)

**Remark 4.5.** As the proof shows, one can replace the inequality in Lemma 4.4 by
\[ A(\log D(C))^2 (\log \log D(C))^4 < p^n. \]
However, it cannot be replaced by an inequality of the form
\[ 2(D(C))^{A_0} < p^n. \]
So the unconditional estimate in Theorem 3.2(i) does not work.

We complete the proof of Theorem 4.1. Let \( N_0 \) be the positive integer in Lemma 4.4. We will prove that
\[ \text{tr}(\text{Frob}_x, \mathcal{E}_x) \in \mathbb{Q}(\text{tr} \mathcal{E})_{\leq N_0} \]
for every closed point \( x \in X \). For this, by induction, it suffices to show that for every prime \( p \), a positive integer \( n \) with \( p^n > N_0 \), and a point \( x \in X(F_{p^n}) \), the trace \( \text{tr}(\text{Frob}_x, \mathcal{E}_x) \) lies in \( \mathbb{Q}(\text{tr} \mathcal{E})_{\leq p^n-1} \). We fix such \( p, n, \) and \( x \).

Take \( \varphi : C \to X \) and \( \bar{x} \in C(F_{p^n}) \) as in Proposition 2.2. Then \( \text{tr}(\text{Frob}_x, \mathcal{E}_x) = \text{tr}(\text{Frob}_{\bar{x}}, (\varphi^* \mathcal{E})_{\bar{x}}) \). Since we assume Conjecture 1.3 and the GRH for Dedekind zeta functions, we can apply Proposition 3.9. This implies
\[ \text{tr}(\text{Frob}_x, \mathcal{E}_x) \in \mathbb{Q}(\text{tr} \varphi^* \mathcal{E})_{\leq A(\log D(C))^2}. \]
On the other hand, Lemma 4.4 implies \( A(\log D(C))^2 < p^n \) and thus \( \text{tr}(\text{Frob}_x, \mathcal{E}_x) \in \mathbb{Q}(\text{tr} \mathcal{E})_{\leq p^n-1} \). This finishes the proof of Theorem 4.1.

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**References**


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