Geometric Rank of Tensors

Jeroen Zuiddam (IAS)

Joint work with Swastik Kopparty and Guy Moshkovitz
Tensors are 3-d arrays
Tensors play a central role in computer science, mathematics and physics

Algebraic Complexity Theory:
- Complexity of Matrix Multiplication

Quantum Information Theory:
- Understanding Entanglement

Extremal Combinatorics:
- Cap set problem
Motivated by these problems we introduce a new tensor parameter

Geometric Rank
Geometric Rank of tensors extends the classical rank of matrices

Matrix Rank

Geometric Rank

Slice Rank
Subrank
Analytic Rank
Tensor Rank
Border Rank
Main results on Geometric Rank

- basic properties and invariances
- develop tools to reason about, and sometimes exactly compute it
- intimate connections to the other important notions for tensors
- answer an old question of Strassen on the (Border) Subrank of matrix multiplication, the “dual” of the more famous Tensor Rank.
Geometric Rank provides new interesting route to upper bound

- Subrank of tensors
  important in complexity theory for *matrix multiplication and barriers*

- Independence number of Hypergraphs
  important in combinatorics in the context of specific natural hypergraphs,
  as in *cap set problem and Erdős–Szemerédi sunflower problem*
Geometric Rank

\[ T = M_1 \cdots M_n \]
Geometric Rank

system of equations

\[(x_1, \ldots, x_n) \begin{pmatrix} M_1 \\ \vdots \\ M_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = 0\]
Geometric Rank

\[
\text{GR}(T) = 2n - \text{dimension of } V(T)
\]

\[
(x_1, \ldots, x_n) M_1 \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = 0 \implies V(T) \text{ is the set of all solutions}
\]
Geometric Rank

\[ \text{GR}(T) = 2n - \text{dimension of set of solutions } V(T) \]

\[ (x_1, \ldots, x_n) M_1 \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = 0 \]

\[ (x_1, \ldots, x_n) M_2 \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = 0 \]

\[ \vdots \]

\[ (x_1, \ldots, x_n) M_n \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = 0 \]
Dimension measures continuous degrees of freedom

“length of maximal chain of irreducible subvarieties”

Computational intuition:

• If $V$ is a linear space then the dimension equals the one from linear algebra

• If $V = \bigcup_i W_i$ then $\dim V = \max_i \dim W_i$

• If $V \subseteq W$ then $\dim V \leq \dim W$
Example of Geometric Rank (W-tensor)

\[ T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \]

\[ M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \]

\[ (x_1, x_2) M_2 \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x_2 y_1 + x_1 y_2 = 0 \]

\[ (x_1, x_2) M_1 \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x_1 y_1 = 0 \]

\[ V(T) = \{ x_1 y_1 = 0, x_2 y_1 + x_1 y_2 = 0 \} \]

\[ = \{ x_1 = 0, x_2 = 0 \} \cup \{ y_1 = 0, y_2 = 0 \} \cup \{ x_1 = 0, y_1 = 0 \} \]

\[ \text{GR}(T) = 4 - \dim V(T) = 4 - 2 = 2 \]
Geometric Rank takes values between 0 and $n$ because the system is bilinear

\[ (x_1, \ldots, x_n) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = 0 \]

Always:
\[ \{ x_1 = \cdots = x_n = 0 \} \subseteq V(T) \]
\[ n \leq \dim V(T) \]

\[ (x_1, \ldots, x_n) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = 0 \]

\[ \vdots \]

\[ (x_1, \ldots, x_n) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = 0 \]

\[ \text{GR}(T) = 2n - \dim V(T) \leq n \]
Computing Geometric Rank is easy in practice for small tensors

system of equations:
\[ x_2 y_1 + x_1 y_2 = 0 \]
\[ x_1 y_1 = 0 \]

\[ \Rightarrow \]
\[ \dim = 2 \]

Macaulay2

```plaintext
R = CC[x1,x2,y1,y2];
dim ideal(x1*y1, x2*y1 + x1*y2)
```

Sage

```plaintext
A.<x1,x2,y1,y2> = AffineSpace(4, CC);
Ideal([x1*y1, x2*y1 + x1*y2]).dimension()
```
Computational complexity of Geometric Rank is not known

Computing dimension of variety that is:

- **linear**: easy
- **bilinear**: not known to be easy or hard (at least we are not aware)
- **general**: hard

**Koiran:**

NP-hard $\leq$ dimension of **general** variety $\leq$ PSPACE
The outline of this talk:

I. Tensors and Applications
II. Fundamental Properties of Geometric rank
III. As upper bound on Subrank
I. Tensors and Applications
Guassian elimination

\[
\begin{bmatrix}
1 & -2 \\
0 & 1 \\
\end{bmatrix} \cdot \begin{bmatrix}
1 & 2 \\
2 & 1 \\
\end{bmatrix} \cdot \begin{bmatrix}
1 & \frac{2}{3} \\
0 & -\frac{2}{3} \\
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}
\]
“Guassian order” on matrices

Example:

\[
\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \geq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]
Matrix Rank completely determines the Gaussian order

\[
M \geq N \quad \text{if and only if} \quad \text{rank}(M) \geq \text{rank}(N)
\]
Recall once more:

\[ M \geq M \]

not necess. invertible column operations

not necess. invertible row operations
Gaussian order on Tensors generalizes the one on matrices

\[ T = \begin{array}{c} \end{array} = \begin{array}{c} \end{array} = \begin{array}{c} \end{array} \]

not necess. invertible slice operations in any of the three directions
Examples of slice operations:
Gaussian order in Mathematics, Physics and Computer Science

Complexity of **Matrix Multiplication**

Classifying **Quantum Entanglement**

Hypergraph **Independence Number**
Matrix Rank completely determines the Gaussian order on matrices

\[
\begin{align*}
M & \geq N \\ \iff \quad R(M) & \geq R(N)
\end{align*}
\]

For tensors that level of complete understanding is out of reach

\[
\begin{align*}
S & \geq T \\ \iff \quad ?
\end{align*}
\]

(NP-hard problem)
Our aim is to find monotones for the Gaussian order:

\[ S \geq T \implies F(S) \geq F(T) \]

Monotones serve as obstructions:

\[ S \not\geq T \iff F(S) < F(T) \]
II. Fundamental Properties of Geometric Rank
Theorem 1

(Geometric Rank is monotone under the Gaussian order on tensors)

\[ S \geq T \implies \text{GR}(S) \geq \text{GR}(T) \]
Theorem 2
(“Fundamental Theorem of Multilinear Algebra”, by analogy)

\[ \text{GR}(T) = \text{codim} \{ (u, v) : \forall i \ u^\top M_i v = 0 \} \]

(definition)

\[ = \text{codim} \{ (u, v) : \forall i \ u^\top N_i v = 0 \} \]

\[ = \text{codim} \{ (u, v) : \forall i \ u^\top L_i v = 0 \} \]
Theorem 1 (Monotonicity) \( T \geq T' \Rightarrow \text{GR}(T) \geq \text{GR}(T') \)

Proof:

By Fundamental Theorem we may focus on the first step.
Focus on one step:

\[ T = \{ (u, v) : \forall i \ u^\top M_i v = 0 \} \]
\[ V(T') = \{ (u, v) : \forall i \ u^\top M'_i v = 0 \} \]

- By assumption: \( M'_i \) are in the span of the \( M_i \)
- \( V(T) \subseteq V(T') \)
- \( \dim V(T) \leq \dim V(T') \).
- \( \text{GR}(T) = \text{codim} V(T) \geq \text{codim} V(T') = \text{GR}(T') \). 

\[ \Box \]
Theorem 3
(Method for computing Geometric Rank)

\[
GR(T) = \min \text{codim} \{ u : \text{rank } T(u) = j \} + j
\]

Proof: relies on a fiber dimension theorem applied to the projection \((u, v) \mapsto u\)
Theorem 1
\[ S \geq T \implies \text{GR}(S) \geq \text{GR}(T) \]

Theorem 2
\[ T = M_i = N_i = L_i \implies \text{same notion of GR} \]

Theorem 3
\[ T(u) := u_1M_1 + \cdots + u_nM_n \]
\[ \text{GR}(T) = \min_{j} \text{codim} \{u : \text{rank} T(u) = j\} + j \]
III. As upper bound on Subrank
The Subrank of $T$ is the size of the largest diagonal tensor smaller than $T$

Strassen 1987

$Q(T)$ is the largest possible $q$
Subrank of tensors

Complexity theory
*matrix multiplication and barriers*

Combinatorics
*Hypergraph independence number, cap set problem,*
*and Erdős–Szemerédi sunflower problem*

Quantum Information
*distilling GHZ states by SLOCC*
Subrank upper bounds hypergraph independence number

**Hypergraph:** symmetric subset $E \subseteq V \times V \times V$

**Independent set:** $A \subseteq V$ such that $E \cap (A \times A \times A) = \emptyset$

**Tensor $T$** supported on $E \cup \{(i, i, i) : i \in V\}$.

$$Q(T) \geq |A|$$
Upper bounds on Subrank

Slice Rank
Analytic Rank
Geometric Rank
Slice Rank is the smallest number of slice rank one tensors summing to $T$.

Slice rank one tensor has slices that are multiples of one slice.
Slice Rank upper bounds Subrank

Proof: Monotone \((T \geq S \Rightarrow \text{SR}(T) \geq \text{SR}(S))\)

+ Normalized \((\text{SR}(I_n) = n)\)
Analytic Rank for tensors over finite fields $\mathbb{F}_p$ (say $\mathbb{F}_2$)

Gowers and Wolf

\[ T(u) := u_1 M_1 + \cdots + u_n M_n \]

\[ \text{bias}(T) := \mathbb{E}_{u,v,w} \left[ (-1)^{v^T T(u) w} \right] \in (0, \infty) \]

\[ \text{AR}(T) := -\log_2 \text{bias}(T) \]
Analytic Rank upper bounds Subrank

AR/AR(I₁)

Lovett

Briët: application in combinatorics

Q

Proof: Monotone + Normalized
Geometric Rank “extends” Analytic Rank to characteristic 0

Theorem
\[ \liminf_{p \to \infty} \text{AR}(T \mod p) = \text{GR}(T) \]

Proof ingredients:

- \( \text{AR}(T \mod p) = 2n - \log_p |V(T \mod p)(\mathbb{F}_p)| \)
- Generalized Schwartz–Zippel lemma (Dvir–Kollár–Lovett)
- Lang–Weil Theorem
- Bertini–Noether Theorem: \( V(T) \leftrightarrow V(T \mod p) \)
Geometric rank upper bounds Subrank and is at most Slice Rank

![Diagram]

Proof: Monotone + Normalized
Example (matrix multiplication)

Matrix multiplication tensor

$$T = M_{(i,j)}$$

As quantum state: triangle of level-$n$ EPR pairs
Example (matrix multiplication)

Previously (Christandl, Lucia, Vrana and Werner)

\[ Q(T') \leq n^2 - n + 1 \]
Example (matrix multiplication)

\[
\text{SR}(T) = n^2 \\
\text{GR}(T) = \left\lceil \frac{3}{4}n^2 \right\rceil \\
Q(T) \geq \left\lceil \frac{3}{4}n^2 \right\rceil \\
Q(T) \geq n^{2-o(1)}
\]

Improves:
\[Q(T) \leq n^2 - n + 1\]

Proof uses Theorem 3:
\[
\dim V(T) = \max_r \dim \{ M \in \mathbb{F}^{n \times n} : \text{rank } M = r \} + (n - r)n
\]

Strassen 1987
Question 1
Computational complexity of GR?
(SR is NP-hard.)

Question 2
How much smaller than SR can GR be?
(Big open problem for SR and AR.)

Question 3
Is GR($T$) the limit of analytic ranks?