Review

Čech cohomology of a sheaf
of a complex of sheaves (double complex of Čech cochains, form single complex, take $H^k$)

Cohomology via injective resolution of a sheaf
of a complex of sheaves $S \to I_\ast$ via Q.I.

Q.I. $\iff$ use on stalk cohomology $V_x$
$\iff$ use on cohomology sheaves

$SS$ of a double complex
$SS$ of a complex of sheaves $E_{ij} = H^i(X, H^j(A)) \Rightarrow H^{i+j}(X, A)$

Homotopy category of complexes
Derived category morphisms $\simeq$ equi classes of $A \to B$:

module $C$

commute up to the homotopy

Fact if $A \to B$ is a Q.I. and if $B$ is injective
$\Rightarrow$ there exists morphism $A \to B$ whose composites
is chain homotopic to $I$.

Then $D^b(C) = \text{equivalent to the homotopy category of complexes of injectives}$. 
1. Abelian groups has enough injectives. Recall: $G \rightarrow$ any injective, and this is enough to define injective resolutions.

An ab gp $G$ is injective $\iff$ divisible

$$\iff \quad \forall n \in \mathbb{Z} \quad \forall g \in G \quad \exists g' \text{ s.t. } ng' = g$$

$$\iff$$ $G$ is injective, $K < G$ any subgp $\Rightarrow G/K$ injective

Given $G$ exists a free ab gp and a surjection $\mathbb{F} \twoheadrightarrow G$

$$F = \bigoplus_{g \in G} \mathbb{Z} \rightarrow G \quad m \cdot [g] \mapsto mg$$

Let $0 \rightarrow K \rightarrow F \rightarrow G \rightarrow 0$ $K = \ker \text{nlcl so } G \cong F/K$

Then $F \leftarrow F \otimes \mathbb{Q}$

$\downarrow \quad \downarrow$

$$\frac{F}{K} \leftarrow (F \otimes \mathbb{Q})/K = \text{mncx}$$

Case of a general comm ring $R$. Given an $R$-module $M$, put $M \xrightarrow{\alpha} I$ = en injective

Then $M \hookrightarrow \text{Hom}_R (R, M) \xrightarrow{\alpha} \text{Hom}_R (R, I)$

$$m \mapsto [r \mapsto \alpha(m) \cdot r]$$

This is an injective $R$-module

$$(r \cdot m) = \alpha(r \cdot m)$$

Case of a sheaf of $R$-modules $\mathcal{A}$

$I(U) = \bigoplus_{x \in U} \mathcal{A}_x$ (sum of sections over $U$)

This is a sheaf of injective $R$-modules of $\mathcal{A}$.
2. Simplicial sheaves

\[ \sigma \quad \text{map} \quad \Rightarrow \quad \text{const} \quad \circ \quad \text{maps} \quad \Rightarrow \quad \text{const} \quad \circ \quad \text{maps} \]

So presheaves go from big simplices to small simplices.

Exercise: \[ \text{is simplex} \]

Ex: these are maps

but not back

\[ \text{so, an internal resolution of } \sigma \text{ is } \Rightarrow \]

\[ \text{is a} \]
Properties of derived category

Mapping

Let $C$ be an ab. cat. (e.g. sheaves of $R$-modules)

Let $K(C)$ be the category of complexes, bounded from below

Then $D(C)$ derived category, functor

$$K(C) \xrightarrow{F} D(C)$$

so that if $A \to B$ then $F(A) \to F(B)$ is an isomorphism.

(Recall: an isomorphism in a category is a morphism that has an
inverse \rightarrow the inverse is unique.)

AND

For any category $D'$, $K(C) \xrightarrow{F} D'$ s.t. $F \circ i$ is iso,

exist a functor $G$

$D(C) \xrightarrow{G} D(G)$ so that $F' = G \circ F$

in $D(G)$ is the smallest such category.

Mapping Cone, Triangle

Given $A' \rightarrow B'$

complex $\text{defin } C(x)$

with 3 maps

$$A'[x] \leftarrow C(x) \leftarrow B'$$

or

$$A \rightarrow B$$

Complex $\text{defin } C(x)$

with 3 maps

$$A'[x] \leftarrow C(x) \leftarrow B'$$

$$A \rightarrow B$$
This is the replacement for exact sequence.

**Exercise:** If \( 0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0 \) is a short exact sequence, then the natural map \( C(x) \rightarrow C \) is a chain map and \( A^{i+1} \oplus B^i \rightarrow C^i \) is an isomorphism, \( (a, b) \rightarrow \beta(b) \).

**Exercise:** If \( 0 \rightarrow C \xrightarrow{\gamma} A \xrightarrow{\alpha} B \rightarrow 0 \) is a short exact sequence, then the natural map \( C^i \rightarrow C^i \) is a chain map and it is an A.I.

So the core is basically \( \ker(x) \oplus \operatorname{coker}(x) \) if \( \Delta \) of \( \alpha \) groups is a long exact sequence.

**Def:** A triangle of morphisms \( A \xrightarrow{\alpha} B \xrightarrow{\beta} C \) is a distinguished triangle if \( \alpha \) is a weak equivalence and the third.
octahedral axiom: when you have \( A \to B \to C \)
you get 3 mapping cones & a huge diagram

**Derived functors**

let \( f: C \to B \) be an abelian category
for simplicity, assume \( C \) has enough injectives
and each \( A \in C \) has a canonical, functorial
injective resolution

if \( f \) is left exact:

\[
RF: D_+(C) \to D_+(B)
\]

\[
RF(A^\cdot): = F(I^\cdot(A)) \quad A^\cdot \to I^\cdot
\]

caution: this is a complex & \( f^\bullet \) cohomology \( H^i(f(I^\cdot(A))) \)
called \( R^iF(A) \in D
\]

**example**

then if \( A^\cdot \to B^\cdot \) is a q.t. and

\( B^\cdot \) is \( F \)-acyclic

**meaning:**

\[
\Rightarrow RF(A^\cdot) = F(B^\cdot)
\]

\[
H^i(RF(B^\cdot)) = 0 \quad \forall i \geq 0
\]

**don't need to resolve \( B^\cdot \)**

**Theory memo:**

\( H^i(f(I^\cdot)) \) when \( B^\cdot \to fI^\cdot \)

is a problem

**Fact**

\( RF \) takes \( A^\cdot \) to \( A^\cdot \)

**Example**

\( RF \)

\[
f: X \to Y
\]

take a sheaf \( A \) on \( X \)

\[
RF(A)(U) := A(f^{-1}(U))
\]

so \( RF \) on constant sheaf

\[
RF(\mathbb{R}) = \mathbb{R}
\]

so \( H^x(f^{-1}(\mathbb{R})) = H^x(Y) \)

**But:** Take on my resolution \( RF_{f^\bullet}(\mathbb{R}) = f^\bullet(\mathbb{R}^0 \to \mathbb{R}^1 \to \mathbb{R}^2 \to \cdots) \)

so \( U \to \mathbb{R}(f^{-1}(U)) \to \mathbb{R}(f^{-1}(U)) \)

so \( H^x = \text{coker} \) \( U = X \Rightarrow H^x(Y, RF_{f^\bullet}(\mathbb{R})) = H^x(X) \).
Careful $R^f_x \mathbb{R}$ is a complex of sheaves

$$R^f_x \mathbb{R} := H^i(R_f x \mathbb{R}) = \text{cohomology sheaf}$$

thin stalk: $H^i(R_f x \mathbb{R})_y$ is $H^i(f^{-1}(y), \mathbb{R})$

more generally $H^i(R_f x A')_y = H^i(f^{-1}(y), A')$

cohomology sheaf = a sheaf whose stalks are the $H^i$ of fiber.

The SS of $R_f x \mathbb{R}$ is Leray-Serre SS

$$E_2^{pq} = H^p(Y; H^q(R_f x \mathbb{R})) = H^p(Y; H^q(\text{fiber})) \Rightarrow H^{p+q}(X; \mathbb{R})$$

How to compute $H^i(X)$ using local data on $Y$ over $R_f x A'$ has a SS

$$E_2^{pq} = H^p(Y; H^q(\text{fiber}; A')) \Rightarrow H^{p+q}(X; A')$$

Special case: $Y = pt \Rightarrow f = T$

so $R^f(A')$ = complex of groups $I^0 \to I^1 \to I^2 \to \cdots$

and $H^i(R^f(A')) = H^i(X; A')$

Kashiwara & Schapira say $R^f A'$ simply specializes from $H^i(X, A')$

But this is a complex $H^i(X; A')$
Recall $\text{Hom}(A, B)$ of sheaves

$$\text{Hom}(A, B) = \text{the prechek}$$
$$U \mapsto \text{Hom}(A|_U, B|_U)$$

Hom of complexes $A \to B$ (cochain or complex of sheaves)

$$\text{Hom}^n_{\text{C}}(A, B) = \prod_{k \geq 0} \text{Hom}(A^k, B^{k+n})$$
$$d^n : \text{Hom}^n_{\text{C}}(A, B) \to \text{Hom}^{n+1}_{\text{C}}(A, B)$$
$$d^0 : \text{Hom}^0_{\text{C}}(A, B) \to \text{Hom}^1_{\text{C}}(A, B)$$

So $d^0 : \text{Hom}^0_{\text{C}}(A, B) \to \text{Hom}^1_{\text{C}}(A, B)$

$\text{ker} d^0 = \text{Hom}_{\text{C}}(A, B)$

$$\text{Im} d^0 = [A, B] \text{ homotopy classes of maps}$$

$$\text{Thm} \quad \text{Im} d^0 = H^0(\text{Hom}_{\text{C}}(A, B))$$

$$\text{Thm} \quad H^0(\text{RHom}_{\text{C}}(A, B)) = [A, I] \text{ where } B \to I \text{ by } 0$$

So $\text{Hom}$ descends to $d^0$.
Triangle again: $A \to B \to C$ is a triangle of sheaves.

\[
\begin{align*}
RT(A) \to RT(B) &\xrightarrow{\delta} (RP(A)) \\
\to \text{long exact sequence} \\
RT(C) &\text{take cohomology}
\end{align*}
\]

\[
\to H^i(X,A) \to H^i(X,B) \to H^i(X,C) \to H^{i+1}(X,A) \to \ldots
\]

OR

\[
\Hom(M,A) \to \Hom(M,B) \quad \text{long exact sequence on } H^*
\]

\[
\Hom(M,C)
\]

\[
\to \Hom_{D^b(G)}(M,A) \to \Hom_{D^b(G)}(M,B) \to \Hom_{D^b(G)}(M,C) \to \Ext^1_{D^b(G)}(A,A') \to \ldots
\]

A different way:

There is a natural isomorphism $f: X \to Y$ such that

\[
\Hom_{D^b(G)}(f^{-1}Y,F) \xrightarrow{\sim} \Hom_{D^b(G)}(X,f^*F)
\]

\[
f_* \Hom_{D^b(G)}(f^{-1}Y,F) \xrightarrow{\sim} \Hom_{D^b(G)}(X,f^*F)
\]

Consistently

\[
Rf^* \Hom_{D^b(G)}(f^{-1}Y,F) \xrightarrow{\sim} \Hom_{D^b(G)}(X,F) \xrightarrow{\sim} Rf^* \Hom_{D^b(G)}(f^{-1}Y,F)
\]

\[
Rf^* \Hom_{D^b(G)}(f^{-1}Y,F) \xrightarrow{\sim} \Hom_{D^b(G)}(X,f^*F) \xrightarrow{\sim} \Hom_{D^b(G)}(X,F) \xrightarrow{\sim} Rf^* \Hom_{D^b(G)}(f^{-1}Y,F)
\]
compact supports

$$\Gamma_c^e (X,A) = \{ x \text{ s.t. } s(x) \neq 0 \text{ } \text{compact} \}$$

Associated $$\Gamma_c^e (X,A)$$ as a complex

$$H_c^i (X,A)$$ is its cohomology = $$H_c^i (X, I^* )$$ where $$I^*$$ is an

indef. rep. of $$A$$

$$\mathcal{R}^i \Gamma_c^e (X,A)$$ is we have complex $$\mathcal{R} \Gamma_c^e (X,A)$$

Note we also have "opt. stalk" $$\lim_{U \downarrow x} \Gamma_c^e (U,A)$$

its cohomology is $$H^i_\text{opt} (A)$$ or $$H^i_{(x)} (A)$$

Similarly $$f : X \to Y$$

$$(f! A^e) (U) = \{ \text{sections of } f^{-1} U \text{ of } A \text{ over } f^{-1} (U) \}$$

where $$\text{spt}$$ is proper over $$Y$$

if $$f$$ is proper $$\Rightarrow f_! = f^*$$

$$\Rightarrow$$ we have $$\mathcal{R} f_! : D^+ (X) \to D^+ (Y)$$

Then $$\exists$$ an adjoint $$f^\sim$$ to $$\mathcal{R} f_!$$ in

$$\mathcal{R} f_* \mathcal{R} \text{Hom}^e (A^e, f^\sim B^e) \to \mathcal{R} \text{Hom}^e (\mathcal{R} f_! A^e, B^e)$$

mor about this, later. when we do duality
Example: \( \text{Cone}(\Delta) \)

\[ X = \text{Cone}(\Delta) \]

\[ L = \text{a space or manifold} \]

\[ U = X - \{pt\} \quad \xrightarrow{i} \quad X \]

Let \( \mathcal{I} \) be the sheaf on \( X \). What is the stalk at \( x_0 \)?

\[ \Gamma(U, i^\ast \mathcal{I}) = \Gamma(i^{-1}(U), i^\ast \mathcal{I}) \]

...my resolution of the coset sheaf

So \( H^i(U, i^\ast \mathcal{I}) = H^i(i^{-1}(U), i^\ast \mathcal{I}) = \}

...my resolution of the coset sheaf

So we have a sheaf \( \mathcal{O} \) in degree 0 (systematic \( x_0 \), \( i^{-1}(U) \) is connected)

and \( H^1(U), H^2(U), \ldots, H^m(U) \) at the sing part.

This is \( H^\ast(\mathcal{R}i^\ast \mathcal{O}) \).

Now consider \( \mathcal{R}i^\ast \mathcal{O} \)

\[ H^i(\text{sections in } U \xrightarrow{pr} X; \mathcal{O}) \]

...sections w. sp. pt.

So they vanish near \( x_0 \).

A section you can make \( U \) so small that the section = 0

\[ \Rightarrow H^\ast_x(\mathcal{R}i^\ast \mathcal{O}) = 0 \]

We have a sheaf that is \( \mathcal{O} \) on \( U \) and 0 at \( x_0 \).

**EXACT SEQ OF A PAIR**

\[ \xymatrix{ \mathcal{R}i^\ast A' \ar[r] & A' \ar[r] & \mathcal{H}^i(X, Y; A') \ar[r] & \mathcal{H}^i(X, A) } \]

\[ \xymatrix{ \mathcal{R}j^\ast A' \ar[r] & \mathcal{H}^i(Y, A') \ar[r] & \mathcal{H}^i(Y, A; Y) } \]

Recall \( \xymatrix{ 0 \ar[r] & c_\ast(X, Y) \ar[r] & c_\ast(X) \ar[r] & c_\ast(Y) \ar[r] & 0 } \)

Is cochain that vanished on \( Y \).
Homology as a homology theory

If the sheaf of chains \( g \) of chains or \( \omega \) algebra chain

A chain is \( \in \Gamma \mathcal{M} \circ \) Finite sum of oriented algebra module refined

To make ashy: \( C_i(U; \mathbb{Z}) \)

But if \( V \subset U \) wrong cut the chain

When it allow chains that are not compact

By Brui—Nevre chain = (geometric)

Chain locally finite, allow arbitrary

Subdomain OR \( C_i(U; \omega) \) if \( U \) is

all vary \( \Gamma(U, C^k) \) in regular chain

Thm: \( H^\bullet_X \) stalk chain

\[ H_i(U; \omega) = H_i(X(x-x)) \]

\[ \lim_{U \subset X} \]

local homology sheaf

\[ H^i(X; \mathbb{Z}; H_j(X(x-x))) \to H^i_{X-j}(X) \]

Lemma SS: If \( X = \text{manifold} \) the kernel

\[ H^i(X; \mathbb{Z}) \to H^n_{n-i}(X) \text{ P.D.} \]
1. It is possible to construct a sheaf of chains but they are not compact (locally finite). Use P.L. or singular chains.

2. The cohomology sheaf is

\[ H^m_j(U) = H^m_j(\overline{U}, \partial \overline{U}) \]

so the stalk cohomology is

\[ H^j(U, \partial U) = H^j(X, X - x) \]

provided the space is reasonable.

3. Simplicially compactifiable space \( X = Y - Y_\varepsilon \),

\[ H^{-i}(X, \mathbb{C}) = H^m_{X} (X) = H^i(Y, Y_\varepsilon) \]

4. The diagram:

\[
\begin{array}{ccc}
U \subseteq X \subseteq Y & \xrightarrow{i} & C \xrightarrow{\alpha} \mathbb{C} \\
\downarrow & & \downarrow \\
Rj_\ast j_\ast & & Rj_\ast j_\ast \mathbb{C} \\
\end{array}
\]

\[ H^i(U) \rightarrow H^i(X) \]