



## Local Intersection Cohomology of Baily–Borel Compactifications

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**Abstract.** The local intersection cohomology of a point in the Baily–Borel compactification (of a Hermitian locally symmetric space) is shown to be canonically isomorphic to the weighted cohomology of a certain linear locally symmetric space (an arithmetic quotient of the associated self-adjoint homogeneous cone). Explicit computations are given for the symplectic group in four variables.

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### 1. Introduction

**1.1.** Suppose  $D$  is a Hermitian symmetric domain,  $\Gamma$  is a neat arithmetic group of automorphisms of  $D$ , and  $X = \Gamma \backslash D$  is the corresponding locally symmetric space. The Baily–Borel Satake compactification  $\hat{X}$  of  $X$  is a projective algebraic variety. The Zucker conjecture (proven by Looijenga [Lo], Saper and Stern [SS]) states that the complex of sheaves of  $L^2$  differential forms on  $\hat{X}$  is canonically isomorphic to the complex of sheaves  $\mathbf{IC}^\bullet(\hat{X})$  of intersection chains on  $\hat{X}$ . Both proofs proceed by showing that, in some sense, these two complexes of sheaves have the same stalk cohomology at any point  $x \in \hat{X}$ . These stalk cohomology groups  $IH_x^i(\hat{X})$  are usually viewed as being extremely complicated objects. In this paper (Theorem 6.3) we give an explicit interpretation for the stalk cohomology, together with its weight filtration which arises from mixed Hodge theory. In Theorem 7.8, we evaluate the formula of Theorem 6.3 for the case  $X = \Gamma(p) \backslash \mathbf{Sp}_4(\mathbb{R})/\mathbf{U}_2$ .

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The identity mapping  $X \rightarrow X$  has a unique continuous extension  $\Phi: \bar{X} \rightarrow \hat{X}$  to the ‘reductive Borel–Serre compactification’  $\bar{X}$ , which may be thought of as a (nonalgebraic) partial resolution of singularities of  $\hat{X}$ . The projection  $\Phi$  is stratified by the natural stratifications of  $\bar{X}$  and of  $\hat{X}$ , but the singularities of  $\bar{X}$  are particularly explicit and easy to understand. The fiber  $\Phi^{-1}(x)$  over a boundary point  $x \in \hat{X}$  is itself the reductive Borel–Serre compactification  $\bar{X}_\ell$  of a certain ‘linear’ locally symmetric space (6.1.2),

$$X_\ell = \Gamma_\ell \backslash Q_\ell / A_Q K_\ell. \tag{1.1.1}$$

Our formula expresses the stalk cohomology of the intersection cohomology of  $\hat{X}$  as a direct sum of *weighted cohomology* groups of this auxiliary space  $\Phi^{-1}(x)$ .

**1.2.** Weighted cohomology groups for locally symmetric spaces were introduced in [GHM] and the present paper relies heavily on the results and notations of [GHM] and [GM2]. Like intersection cohomology, the weighted cohomology  $W^p H^*(\bar{X})$  is the hypercohomology of a complex of sheaves  $\mathbf{W}^p \mathbf{C}^\bullet(\bar{X})$  which is obtained by ‘truncation’ of the direct image sheaf  $Ri_*(\mathbb{C})$  (where  $i: X \rightarrow \bar{X}$  denotes the inclusion). However the weighted cohomology complex is obtained by truncating with respect to weights of a certain torus action, rather than with respect to dimension.

A central result in [GHM] states that the pushforward  $\Phi_* \mathbf{W}^v \mathbf{C}^\bullet(\bar{X})$  of the weighted cohomology complex (with ‘middle’ weight  $v$ ) on  $\bar{X}$  is canonically isomorphic to the (‘middle’) intersection complex  $\mathbf{IC}^\bullet(\hat{X})$  of  $\hat{X}$ . It follows that the stalk intersection cohomology at a point  $x \in \hat{X}$  is isomorphic to the hypercohomology of the restriction

$$\mathbf{W}^v \mathbf{C}^\bullet |_{\Phi^{-1}(x)} \tag{1.2.1}$$

of the weighted cohomology complex to the fiber  $\Phi^{-1}(x) \cong \bar{X}_\ell$ .

**1.3.** The key technical achievement in this paper is the identification of the restriction (1.2.1) as a sum of shifted weighted cohomology complexes of the reductive Borel–Serre compactification  $\bar{X}_\ell$ . In Theorem 4.3 and Corollary 4.6 we show, more generally, that for any weight profile  $p$ , the restriction  $\mathbf{W} \mathbf{C}^\bullet(\bar{X}) |_{\bar{X}_Q}$  of the weighted cohomology complex to the closure  $\bar{X}_Q$  of any boundary stratum  $X_Q \subset \bar{X}$  breaks into a direct sum of weighted cohomology complexes of  $\bar{X}_Q$  with shifts. (This result even holds when we drop the assumption that  $D$  is Hermitian.)

Combining this with (1.2.1) gives a formula (6.5.1) (notation explained in Section 6) for the local intersection cohomology at a point  $x$  in the Baily–Borel compactification,

$$IH_x^i(\hat{X}, \mathbf{E}) \cong \bigoplus_{\beta \geq -\rho_Q} \bigoplus_i W^{\nu \boxplus \beta} H^{k-i}(\bar{X}_\ell, \mathbf{H}^i(\mathcal{R}_Q, \mathbf{E})_\beta). \tag{1.3.1}$$

Theorem 4.3 is one of several properties which make weighted cohomology a somewhat simpler object to study than intersection cohomology. The analogous statements which may be formulated for the intersection cohomology of  $\hat{X}$  or of  $\bar{X}$  are false. (The restriction of the intersection complex  $\mathbf{IC}^\bullet(\hat{X})$  to the closure  $\hat{Y} \subset \hat{X}$  of a boundary stratum  $Y \subset \hat{X}$  may fail to be isomorphic to a sum of intersection cohomology complexes of  $\hat{Y}$ .)

The results of [LR] show that the weight filtration from mixed Hodge theory on the stalk cohomology  $IH_x^i(\hat{X})$  is given by the torus weights which define the truncation for weighted cohomology. This gives a precise formula (6.3.1) for the associated graded of the weight filtration of the stalk cohomology.

In [F], J. Franke introduced an important family of invariants, the weighted  $L^2$  cohomology groups of  $X$ , which are the Lie algebra cohomology groups of a certain space of functions on  $X$ . In [N] it was shown that Franke’s weighted  $L^2$  cohomology groups coincide with the weighted cohomology groups. Consequently, the above formula (1.3.1) for the stalk cohomology  $IH_x^*(\hat{X})$  may be translated into Lie algebra cohomology. (In the rank two case this stalk cohomology was previously described in terms of automorphic forms [BC].)

Finally, in Section 7 we evaluate the local cohomology and intersection cohomology Euler characteristic for the Siegel modular threefolds given by principal congruence subgroups of level  $\geq 3$ .

**2. Some Linear Algebra**

**2.1.** Suppose  $X$  is a  $\mathbb{Q}$ -vector space. Denote by  $X^* = \text{Hom}(X, \mathbb{Q})$  the dual vector space. Let  $\Delta \subset X^*$  be a finite collection of linearly independent functionals. Define

$$\ker(\Delta) = \bigcap_{\alpha \in \Delta} \ker(\alpha) \text{ and } C(\Delta) = \left\{ \sum_{\alpha \in \Delta} m_\alpha \alpha \mid m_\alpha \geq 0, m_\alpha \in \mathbb{Q} \right\}.$$

The set  $C(\Delta)$  is called the positive cone spanned by the elements of  $\Delta$ . If  $v \in X^*$  define

$$(X^*)_{\geq v(\Delta)} = \{ \gamma \in X^* \mid \gamma - v \in C(\Delta) \}. \tag{2.1.1}$$

Every element  $\gamma \in (X^*)_{\geq v(\Delta)}$  satisfies  $\gamma|_{\ker(\Delta)} = v|_{\ker(\Delta)}$ .

Let  $W = \ker(\Delta)$  and let  $\{t_\alpha \mid \alpha \in \Delta\} \subset X/W$  be the basis of  $X/W$  which is dual to the basis determined by  $\Delta$ . If  $\gamma \in X^*$  satisfies  $\gamma|_W = v|_W$  then  $\gamma - v$  passes to a linear functional on  $X/W$ . Hence,

$$(X^*)_{\geq v(\Delta)} = \{ \gamma \in X^* \mid \gamma|_W = v|_W \text{ and } \langle \gamma - v, t_\alpha \rangle \geq 0, \forall \alpha \in \Delta \}, \tag{2.1.2}$$

where  $\langle \cdot, \cdot \rangle: (X/W)^* \times (X/W) \rightarrow \mathbb{Q}$  is the canonical pairing.

Fix a subset  $J \subset \Delta$  and set  $Y = \ker(J)$  and  $Z = \ker(\Delta - J)$ . The sequence

$$0 \rightarrow W \rightarrow Y \oplus Z \rightarrow X \rightarrow 0$$

is exact, where the maps are given by  $w \mapsto (w, -w)$  and  $y \oplus z \mapsto y + z$ . It follows by duality that if  $v \in X^*$ , if  $\beta \in Y^*$ , and if  $v|_W = \beta|_W$  then there is a unique element  $q = v \boxplus \beta \in X^*$  so that  $q(y + z) = \beta(y) + v(z)$  for all  $y \in Y$  and  $z \in Z$ . Moreover,  $q|_W = v|_W = \beta|_W$ .

If  $\alpha \in \Delta - J$  then  $t_\alpha \in Y/W$ . The elements of  $\Delta - J$  restrict to a basis of  $(Y/W)^*$  whose dual basis is  $\{t_\alpha \mid \alpha \in \Delta - J\}$ . The composition  $Z/W \subset X/W \rightarrow X/Y$  is an isomorphism. If  $\alpha \in J$  then  $t_\alpha \in Z/W$  projects to a nonzero element  $\bar{t}_\alpha \in X/Y$ . The elements of  $J$  determine a basis of  $(X/Y)^*$  whose dual basis is  $\{\bar{t}_\alpha \mid \alpha \in J\}$ .

2.2. PROPOSITION. Let  $X$  be a rational vector space, let  $\Delta \subset X^*$  be a finite set of linearly independent elements, and set  $W = \ker(\Delta)$ . Let  $J \subset \Delta$  and let  $Y = \ker(J)$ . Fix  $v, \gamma \in X^*$  with  $v|_W = \gamma|_W$ . Let  $\beta = \gamma|_Y \in Y^*$ . Then

$$\gamma \in (X^*)_{\geq v(\Delta)} \iff \beta \in (Y^*)_{\geq v(\Delta-J)} \quad \text{and} \quad \gamma \in (X^*)_{\geq v \boxplus \beta(J)}$$

(where we have also written  $v$  for its restriction to  $Y$ ).

*Proof.* If  $\alpha \in \Delta - J$  then  $t_\alpha \in Y/W$  and the first condition says that  $\langle \beta - v, t_\alpha \rangle \geq 0$ . But  $\beta = \gamma|_Y$  so  $\langle \gamma - v, t_\alpha \rangle \geq 0$  for all  $\alpha \in \Delta - J$ .

If  $\alpha \in J$  then  $t_\alpha \in Z/W$ . Let  $\hat{t}_\alpha \in Z \subset X$  be any lift of  $t_\alpha$ ; then  $v \boxplus \beta(\hat{t}_\alpha) = v(\hat{t}_\alpha)$ . Therefore the second condition says

$$0 \leq \langle \gamma - v \boxplus \beta, \bar{t}_\alpha \rangle = \langle \gamma - v \boxplus \beta, \hat{t}_\alpha \rangle = \langle \gamma - v, \hat{t}_\alpha \rangle = \langle \gamma - v, t_\alpha \rangle$$

for all  $\alpha \in J$ . The reverse implication is similar. □

### 3. Weighted Cohomology

#### 3.1. LOCALLY SYMMETRIC SPACES

Algebraic groups will be designated by bold face type ( $\mathbf{G}, \mathbf{P}$ , etc.). If an algebraic group is defined over the real numbers  $\mathbb{R}$ , then its group of real points will be in Roman ( $G = \mathbf{G}(\mathbb{R})$ ). Throughout this paper we fix a connected reductive group  $\mathbf{G}$  which is defined over  $\mathbb{Q}$ . Let  $\mathbf{S}_{\mathbf{G}}$  be the maximal  $\mathbb{Q}$ -split torus in the center of  $\mathbf{G}$  and let  $A_G = \mathbf{S}_{\mathbf{G}}(\mathbb{R})^0$  be the identity component of its group of real points. Let  $K$  be a maximal compact subgroup of  $G$ , set  $K' = KA_G$  and  $D = G/K'$ . This is a homogeneous space on which  $G$  acts transitively. The group  $K'$  corresponds to a choice of basepoint  $x_0 \in D$ . We also fix a neat arithmetic subgroup  $\Gamma \subset \mathbf{G}(\mathbb{Q})$  and set  $X = \Gamma \backslash D$ . By abuse of terminology we will refer to  $X$  as a locally symmetric space.

In this paper,  $\tilde{X}$  denotes the Borel–Serre compactification of  $X$  ([BS]) and  $\bar{X}$  denotes the reductive Borel–Serre compactification ([Z3], Section 4.2, [GHM] Section 8). If  $X$  is Hermitian then  $\hat{X}$  will denote the Baily–Borel Satake compactification.

#### 3.2. PARABOLIC SUBGROUPS

Let  $\mathbf{P}$  be a rationally defined parabolic subgroup of  $\mathbf{G}$ . Then we have the following groups:

- $\mathcal{U}_{\mathbf{P}}$  = unipotent radical of  $\mathbf{P}$
- $\mathfrak{L}_{\mathbf{P}} = \text{Lie}(\mathcal{U}_{\mathbf{P}})$
- $\mathbf{R}_{\mathbf{d}}\mathbf{P}$  = the  $\mathbb{Q}$ -split radical of  $\mathbf{P}$  ([BS] Section 0.3)
- $\mathbf{L}_{\mathbf{P}}$  = the Levi quotient,  $v_{\mathbf{P}} : \mathbf{P} \rightarrow \mathbf{L}_{\mathbf{P}}$  the projection
- $\mathbf{S}_{\mathbf{P}} = \mathbf{R}_{\mathbf{d}}\mathbf{P}/\mathcal{U}_{\mathbf{P}}$
- $\chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}}) = \chi^*(\mathbf{S}_{\mathbf{P}}) \otimes_{\mathbb{Z}} \mathbb{Q}$
- $\chi_{*}^{\mathbb{Q}}(\mathbf{S}_{\mathbf{P}}) = \text{Hom}(\chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}}), \mathbb{Q})$

- $A_P = \mathbf{S}_P(\mathbb{R})^0$
- $\Gamma_P = \Gamma \cap P$  and  $\Gamma_L = \nu_P(\Gamma_P)$
- $K_P = K_P(x_0) = K \cap P$  and  $K'_P = K' \cap P$ .

The torus  $\mathbf{S}_P$  may also be identified as the greatest  $\mathbb{Q}$ -split torus in the center of  $\mathbf{L}_P$ . It contains  $\mathbf{S}_G$  and we denote the quotient by  $\mathbf{S}'_P = \mathbf{S}_P/\mathbf{S}_G$ .

The choice of basepoint  $x_0 \in D$  determines a Cartan involution  $\theta: G \rightarrow G$  with fixed point set  $K(x_0)$ . There is a unique lift ([BS], Sections 1.6, 1.9)  $i_{x_0}: L_P \rightarrow P$  of the Levi quotient such that the image  $L_P(x_0) = i_{x_0}(L_P)$  is  $\theta$ -stable. This determines a lift of  $\mathbf{S}_P(\mathbb{R})$ . Note that  $K_P \subset L_P(x_0)$ .

Fix once and for all a minimal rational parabolic subgroup  $\mathbf{P}_0 \subset \mathbf{G}$ . The (rational ‘relative’) root system  $\Phi(\mathbf{S}_{\mathbf{P}_0}, \mathbf{G})$  admits a linear order such that the positive roots are those in  $\mathfrak{N}_0 = \text{Lie}(\mathcal{U}_{\mathbf{P}_0})$ . Let  $\Delta$  denote the set of simple (rational) roots. The rational parabolic subgroups containing  $\mathbf{P}_0$  are called *standard*. They form a unique set of representatives of the  $\mathbf{G}(\mathbb{Q})$  conjugacy classes of rational parabolic subgroups of  $\mathbf{G}$ . They are in one to one correspondence with subsets  $J \subset \Delta$ . The parabolic subgroup  $\mathbf{P}(J)$  corresponding to  $J$  is determined by the condition that  $\mathbf{S}_{\mathbf{P}(J)} \subset \ker(\alpha)$  for all  $\alpha \in J$ . The (restrictions of the) roots  $\alpha \in \Delta - J$  form a basis of the rational (quasi-) character module  $\chi_{\mathbb{Q}}^*(\mathbf{S}'_{\mathbf{P}(J)})$ . Denote by  $\Delta_{\mathbf{P}(J)}$  the collection of restrictions  $\{\alpha|_{\mathbf{S}_{\mathbf{P}(J)}}\}$  (for  $\alpha \in \Delta - J$ ), which we refer to as the set of simple roots of  $\mathbf{P}(J)$  occurring in  $\mathfrak{N}_J = \text{Lie}(\mathcal{U}_{\mathbf{P}(J)})$ . These notions extend to arbitrary rational parabolic subgroups  $\mathbf{P} \subset \mathbf{G}$  by conjugation.

### 3.3. TWO PARABOLIC SUBGROUPS

If  $\mathbf{P} \subset \mathbf{Q} \subseteq \mathbf{G}$  are rational parabolic subgroups of  $\mathbf{G}$ , then  $\mathcal{U}_{\mathbf{Q}} \subset \mathcal{U}_{\mathbf{P}}$  and  $\mathbf{R}_d\mathbf{Q} \subset \mathbf{R}_d\mathbf{P}$ . This gives an embedding  $\mathbf{S}_{\mathbf{Q}} \hookrightarrow \mathbf{S}_{\mathbf{P}}$ . Let  $\nu_{\mathbf{Q}}: \mathbf{Q} \rightarrow \mathbf{L}_{\mathbf{Q}}$  denote the projection to the Levi quotient and set  $\bar{\mathbf{P}} = \nu_{\mathbf{Q}}(\mathbf{P}) = \mathbf{P}/\mathcal{U}_{\mathbf{Q}}$ . Then  $\mathcal{U}_{\bar{\mathbf{P}}} = \mathcal{U}_{\mathbf{P}}/\mathcal{U}_{\mathbf{Q}}$  and the resulting isomorphism

$$\mathbf{L}_{\bar{\mathbf{P}}} = \bar{\mathbf{P}}/\mathcal{U}_{\bar{\mathbf{P}}} = (\mathbf{P}/\mathcal{U}_{\mathbf{Q}})/(\mathcal{U}_{\mathbf{P}}/\mathcal{U}_{\mathbf{Q}}) \cong \mathbf{L}_{\mathbf{P}}$$

induces an isomorphism  $\mathbf{S}_{\mathbf{P}} \cong \mathbf{S}_{\bar{\mathbf{P}}}$ . However,  $\mathbf{P}$  is regarded as a parabolic subgroup of  $\mathbf{G}$ , so  $\mathbf{S}'_{\mathbf{P}} = \mathbf{S}_{\mathbf{P}}/\mathbf{S}_G$ , while  $\bar{\mathbf{P}}$  is regarded as a parabolic subgroup of  $\mathbf{L}_{\mathbf{Q}}$  so  $\mathbf{S}'_{\bar{\mathbf{P}}} = \mathbf{S}_{\bar{\mathbf{P}}}/\mathbf{S}_{\mathbf{Q}}$ .

### 3.4. BOUNDARY STRATA

Let  $\tilde{X}$  denote the Borel–Serre compactification of  $X$  ([BS]) and let  $\bar{X}$  denote the reductive Borel–Serre compactification of  $X$  ([Z3], Section 4.2, [GHM], Section 8). These spaces are Whitney stratified by their boundary strata  $Y_P \subset \tilde{X}$  and  $X_P \subset \bar{X}$  which are in one to one correspondence with  $\Gamma$ -conjugacy classes of proper rational parabolic subgroups of  $\mathbf{G}$ . A choice of representative  $\mathbf{P}$  in this  $\Gamma$ -conjugacy class determines an identification

$$Y_P = \Gamma_P \backslash P / K_P A_P(x_0) \tag{3.4.1}$$

and an identification

$$X_P = \Gamma_P \backslash P / \mathcal{U}_P K_P A_P(x_0). \tag{3.4.2}$$

The identity map  $X \rightarrow X$  has a unique continuous extension to a mapping  $\pi : \tilde{X} \rightarrow \bar{X}$ . Then  $\pi$  is surjective, takes strata to strata, and its restriction to each boundary stratum  $Y_P$  is the smooth proper fiber bundle  $\pi_P : Y_P \rightarrow X_P$  which is given by collapsing the ‘orbits’ of  $\mathcal{U}_P$  ([GHM] Section 7.4).

### 3.5. WEIGHTED COHOMOLOGY

Fix a minimal rational parabolic subgroup  $\mathbf{P}_0 \subset \mathbf{G}$ . A *weight profile* is an element of  $\chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}_0})$ . (This is slightly more general than the definition in [GHM], but it agrees with the definition in [GKM] and [GM2].) For any standard rational parabolic subgroup  $\mathbf{P} \supset \mathbf{P}_0$  the weight profile together with the set  $\Delta_P \subset \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}})$  determines a ‘high’ subset of weights as in (2.1.1) and (2.1.2),

$$\chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}})_{\geq p(\Delta_P)} = \{ \gamma \in \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}}) \mid \gamma|_{\mathbf{S}_{\mathbf{G}}} = p|_{\mathbf{S}_{\mathbf{G}}} \text{ and } \langle \gamma - p, t_{\alpha} \rangle \geq 0, \forall \alpha \in \Delta_P \}, \tag{3.5.1}$$

where  $\{t_{\alpha} \mid \alpha \in \Delta_P\}$  is the basis of the rational co-character module  $\chi_{*}^{\mathbb{Q}}(\mathbf{S}'_{\mathbf{P}})$  which is dual to the basis of  $\chi_{\mathbb{Q}}^*(\mathbf{S}'_{\mathbf{P}})$  determined by  $\Delta_P$ . When there is no possibility of confusion we will abbreviate the notation to  $\chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}})_{\geq p}$  (which agrees with the notation of [GKM] and [GM2], and which was denoted  $\chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}})_{+}$  in [GHM]). If a rational vector space  $V$  is a module over  $\mathbf{S}_{\mathbf{P}}$  let  $V_{\alpha}$  be the subspace of weight  $\alpha \in \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}})$  and set

$$V_{\geq p} = V_{\geq p(\Delta_P)} = \bigoplus_{\alpha \in \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}})_{\geq p}} V_{\alpha}. \tag{3.5.2}$$

Let  $\mathbf{G} \rightarrow \mathbf{GL}(E)$  be an irreducible representation of  $\mathbf{G}$  on some finite dimensional complex vector space  $E$  and let  $\mathbf{E} = E \times_{\Gamma} D$  be the resulting local coefficient system on  $X$ . Since  $E$  is irreducible, the torus  $\mathbf{S}_{\mathbf{G}}$  acts by a single character  $\lambda_E \in \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{G}})$ . Let  $p$  be a weight profile such that  $p|_{\mathbf{S}_{\mathbf{G}}} = \lambda_E$ . The construction of [GHM] defines a weighted complex of sheaves  $\mathbf{W}^p \mathbf{C}^{\bullet}(\bar{X}, \mathbf{E})$  on the reductive Borel–Serre compactification  $\bar{X}$  of  $X$ . This is a (cohomologically) constructible complex of sheaves on  $\bar{X}$  which is obtained by truncating the sheaf of smooth differential forms  $i_* \Omega^{\bullet}(X, \mathbf{E})$  along the boundary strata  $X_P$  so as to keep only the differential forms with ‘weights’ in  $\chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}})_{\geq p}$ . (Here,  $i : X \rightarrow \bar{X}$  denotes the inclusion of  $X$  into its reductive Borel–Serre compactification.)

Let  $\mathfrak{N}_P = \text{Lie}(\mathcal{U}_P)$  be the Lie algebra of the unipotent radical of  $P$ . The Lie algebra cohomology  $H^i(\mathfrak{N}_P, E)$  is a module over  $\mathbf{L}_{\mathbf{P}}$  and hence also over  $\mathbf{S}_{\mathbf{P}}$ . The torus  $\mathbf{S}_{\mathbf{G}}$  acts on  $H^*(\mathfrak{N}_P, E)$  via the weight  $\lambda_E$ . Then ([GHM], Section 17) the stalk cohomology at a (singular) point  $x \in X_P$  of the weighted cohomology complex  $\mathbf{W}^p \mathbf{C}^{\bullet}(\bar{X}, \mathbf{E})$  is the (finite) sum,

$$W^p H_x^i \cong H^i(\mathfrak{N}_p, E)_{\geq p} = \bigoplus_{\beta \in \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}})_{\geq p}} H^i(\mathfrak{N}_p, E)_{\beta}. \quad (3.5.3)$$

**3.6.** For some purposes it is necessary to consider weight truncations of the form

$$\chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}})_{>p} = \{\gamma \in \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}}) \mid \gamma|_{\mathbf{S}_{\mathbf{G}}} = p|_{\mathbf{S}_{\mathbf{G}}} \text{ and } \langle \gamma - p, t_{\alpha} \rangle > 0, \forall \alpha \in \Delta_P\}.$$

Since only finitely many weights occur in  $H^*(\mathfrak{N}_p, E)$ , for any  $\epsilon > 0$  sufficiently small we have

$$\bigoplus_{\beta \in \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}})_{\geq p+\epsilon}} H^i(\mathfrak{N}_p, E)_{\beta} = \bigoplus_{\beta \in \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}})_{>p}} H^i(\mathfrak{N}_p, E)_{\beta}.$$

The weighted cohomology sheaf constructed with respect to this weight truncation will be denoted  $\mathbf{W}^{p+\epsilon} \mathbf{C}^{\bullet}(\bar{X}, \mathbf{E})$ .

Choose a Cartan subgroup  $\mathbf{H}$  and a Borel subgroup  $\mathbf{B}$  of  $\mathbf{G}$  so that

$$\mathbf{S}_{\mathbf{P}_0} \subset \mathbf{H} \subset \mathbf{B} \subset \mathbf{P}_0 \subset \mathbf{G}.$$

Let  $\Phi^+ = \Phi^+(\mathbf{H}(\mathbb{C}), \mathbf{G}(\mathbb{C}))$  be the resulting set of positive roots and let  $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ . The ‘lower middle’ weight profile is  $\nu = -\rho|_{\mathbf{S}_{\mathbf{P}_0}}$ , the ‘upper middle’ weight profile is  $\nu = -\rho|_{\mathbf{S}_{\mathbf{P}_0}} + \epsilon$ . (The modification by  $\epsilon$  corresponds exactly to the  $\pm \log$  modification which occurs in the weighted  $L^2$  theory.) The ‘dualizing’ weight profile is  $d = -2\rho|_{\mathbf{S}_{\mathbf{P}_0}} + \epsilon$ . The weighted cohomology sheaf  $\mathbf{W}^d \mathbf{C}^{\bullet}(\bar{X}, \mathbb{C})$  is canonically (quasi-) isomorphic ([GHM], Section 19.4) to the dualizing complex  $\mathbb{D}_{\bar{X}}$  on  $\bar{X}$ .

### 3.7. DUALITY

A morphism  $E_1 \otimes E_2 \rightarrow E$  of irreducible representations of  $\mathbf{G}$  induces a morphism of (complexes of) sheaves,

$$\mathbf{W}^p \mathbf{C}^{\bullet}(\bar{X}, \mathbf{E}_1) \otimes \mathbf{W}^q \mathbf{C}^{\bullet}(\bar{X}, \mathbf{E}_2) \rightarrow \mathbf{W}^{p+q} \mathbf{C}^{\bullet}(\bar{X}, \mathbf{E}).$$

Let  $E$  be an irreducible representation of  $\mathbf{G}$  and let  $p \in \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}_0})$  be a weight profile such that  $p|_{\mathbf{S}_{\mathbf{G}}}$  coincides with the character  $\lambda_E$  by which  $\mathbf{S}_{\mathbf{G}}$  acts on  $E$ . Let  $E^*$  be the dual representation and let  $q = d - p$  be the ‘dual’ weight profile. Then the resulting morphism

$$\mathbf{W}^p \mathbf{C}^{\bullet}(\bar{X}, \mathbf{E}) \otimes \mathbf{W}^q \mathbf{C}^{\bullet}(\bar{X}, \mathbf{E}^*) \rightarrow \mathbf{W}^d \mathbf{C}^{\bullet}(\bar{X}, \mathbb{C}) = \mathbb{D}_{\bar{X}}$$

is a (Borel–Moore-) Verdier dual pairing. In particular, the upper and lower middle weight profiles are dual.

**3.8.** One may drop the assumption that the representation  $E$  is irreducible, requiring instead that in the  $\mathbf{S}_{\mathbf{G}}$ -isotypical decomposition  $E = \bigoplus_{\lambda} E_{\lambda}$ , each weight  $\lambda \in \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{G}})$  appears at most once and each summand  $E_{\lambda}$  is irreducible. Sums of this form (with shifts) appear in the restriction to the boundary (Theorem 4.3).

**3.9.** The singularities of  $\bar{X}$  are relatively easy to understand and the weighted cohomology complex on  $\bar{X}$  is relatively simple. If  $X$  is Hermitian then the identity mapping  $X \rightarrow X$  has a unique continuous extension  $\Phi : \bar{X} \rightarrow \hat{X}$  [Z3]. The pushforward  $R\Phi_*(\mathbf{W}^v\mathbf{C}^\bullet(\bar{X}, \mathbf{E}))$  of the middle weighted cohomology is canonically isomorphic to the middle intersection complex  $\mathbf{IC}^\bullet(\hat{X}, \mathbf{E})$  which, by the Zucker conjecture ([Lo], [SS]) is canonically isomorphic to the sheaf of  $L^2$  differential forms with coefficients in  $\mathbf{E}$ . So the weighted cohomology complex on  $\bar{X}$  may be thought of as a sort of (non algebraic) partial resolution of singularities of  $\hat{X}$  together with its sheaf of  $L^2$  differential forms. (In fact, the relation with  $L^2$  cohomology may be described ([N]) completely in terms of  $\bar{X}$ .)

3.10. WEIGHTED  $L^2$  COHOMOLOGY

For completeness we sketch how the results in this paper may be translated into the language of Lie algebra cohomology using Franke’s theory [F] of weighted  $L^2$  cohomology groups.

An element  $\lambda \in \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}_0})$  determines (via reduction theory) a certain real-valued weight function ([N], Section 1.6)  $w_\lambda$  on  $\Gamma A_G \backslash G$ . When  $\lambda = 0$  we have  $w_\lambda = 1$ . For any element  $D$  in the universal enveloping algebra of  $\mathfrak{g} = \text{Lie}(G)$  there is a left-invariant differential operator  $R_D$  which acts on the smooth functions on  $\Gamma \backslash G$ . (For  $D \in \mathfrak{g}$  this is simply the derivative of the right regular representation). Let  $\xi \in \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{G}})$ . Let  $S_{\lambda, \xi}(\Gamma)$  be the space of smooth  $\mathbb{C}$ -valued functions  $f$  on  $\Gamma \backslash G$  satisfying

- (1)  $f(ag) = \xi(a)f(g)$  for all  $a \in A_G$  and
- (2)  $w_\lambda \xi^{-1} R_D f$  is square-integrable on  $\Gamma A_G \backslash G$  for every  $D$ .

This space is a  $(\mathfrak{g}, K)$ -module ([V]). When  $\lambda = 0$  it is the familiar space of smooth uniformly  $L^2$  functions. Its  $(\mathfrak{g}, K)$ -cohomology is often infinite-dimensional.

Let  $\epsilon \in \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}_0})$  be any dominant weight. Let  $S_{\lambda+\log, \xi}(\Gamma) \supset S_{\lambda, \xi}(\Gamma)$  be the module of smooth functions for which

$$w_\lambda \log(w_\epsilon)^m \xi^{-1} R_D f \in L^2(\Gamma A_G \backslash G) \tag{3.10.1}$$

for every positive integer  $m$  and for every  $D$ . This is also a  $(\mathfrak{g}, K)$ -module and it always has finite-dimensional cohomology. The main result of [N], extended to reductive groups, gives an isomorphism

$$W^p H^i(\bar{X}, \mathbf{E}) \cong H^i(\mathfrak{g}, K; S_{\lambda+\log, \xi}(\Gamma) \otimes E) \tag{3.10.2}$$

where  $\lambda = (-\rho - p)|\mathbf{S}_{\mathbf{P}_0}$  and  $\xi = -(p|\mathbf{S}_{\mathbf{G}}) = -\lambda_E$ . Here  $\rho$  is the half-sum of positive roots (see 4.5). In fact, there is a cohomologically constructible complex of sheaves on  $\bar{X}$  with (hyper)cohomology equal to the right-hand side of (3.10.2). The isomorphism (3.10.2) is induced from an (explicit and natural) quasi-isomorphism of this object with  $\mathbf{W}^p\mathbf{C}^\bullet(\bar{X}, \mathbf{E})$ .



## 4. Computations Along Boundary Strata

### 4.1. THE BOUNDARY STRATUM

In this section we restrict the weighted cohomology sheaf to the *closure* of a boundary stratum in the reductive Borel–Serre compactification, and state (in Theorem 4.3) that this restriction decomposes as a direct sum of weighted cohomology sheaves of the closure of the boundary stratum. Throughout this section we fix a weight profile  $p \in \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}_0})$  and we fix a local system  $\mathbf{E}$  on  $X$  which corresponds to an irreducible complex representation of  $\mathbf{G}$  for which  $\mathbf{S}_{\mathbf{G}}$  acts through the character  $\lambda_E = p|_{\mathbf{S}_{\mathbf{G}}}$ . Fix a boundary stratum  $X_Q$  (corresponding to a proper rational parabolic subgroup  $\mathbf{Q} \subset \mathbf{G}$ ). The closure  $\bar{X}_Q$  coincides with the reductive Borel–Serre compactification of  $X_Q$  which we consider to be the locally symmetric space

$$X_Q = \Gamma_L \backslash L_Q / K_Q A_Q$$

associated to the reductive group  $\mathbf{L}_Q$  (notation as in Section 3.2). The restriction  $\mathbf{W}^p \mathbf{C}^\bullet(\bar{X}, \mathbf{E})|_{X_Q}$  to the interior of the boundary stratum is quasi-isomorphic to the sheaf of differential forms  $\Omega^\bullet(X_Q, \mathbf{H}^*(\mathfrak{Y}_Q, E)_{\geq p})$  (see [GHM], Section 14.1.2). Here,  $\mathbf{H}^*(\mathfrak{Y}_Q, E)_{\geq p}$  is the local system on  $X_Q$  associated to the  $\mathbf{L}_Q$  module  $H^*(\mathfrak{Y}_Q, E)_{\geq p}$ .

### 4.2. WEIGHT PROFILE ON THE BOUNDARY STRATUM

The boundary stratum  $X_Q$  corresponds to a rational parabolic subgroup  $\mathbf{Q} \subset \mathbf{G}$  which we may assume to be standard ( $\mathbf{Q} \supset \mathbf{P}_0$ ) and which therefore corresponds to a subset  $J \subset \Delta$  of the simple roots, so that

$$\chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{G}}) = \ker(\Delta) \subset \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{Q}}) = \ker(J) \subset \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}_0})$$

in the notation of Section 2. The elements of  $\Delta - J$  restrict to a linearly independent set  $\Delta_Q \subset \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{Q}})$  and determine a basis of  $\chi_{\mathbb{Q}}^*(\mathbf{S}'_{\mathbf{Q}})$  (where  $\mathbf{S}'_{\mathbf{Q}} = \mathbf{S}_{\mathbf{Q}}/\mathbf{S}_{\mathbf{G}}$ ). Let  $\{t_\alpha \mid \alpha \in \Delta_Q\}$  denote the dual basis of  $\chi_{\mathbb{Q}}^*(\mathbf{S}'_{\mathbf{Q}})$ . Let  $\beta \in \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{Q}})_{\geq p}$ , that is,  $\beta|_{\mathbf{S}_{\mathbf{G}}} = p|_{\mathbf{S}_{\mathbf{G}}}$  and  $\langle \beta - p, t_\alpha \rangle \geq 0$  for all  $\alpha \in \Delta_Q$ . The projection  $v_Q: \mathbf{Q} \rightarrow \mathbf{L}_Q$  determines an identification  $\mathbf{S}_{\mathbf{P}_0} \cong \mathbf{S}_{\bar{\mathbf{P}}_0}$  where

$$\bar{\mathbf{P}}_0 = v_Q(\mathbf{P}_0) \subset \mathbf{L}_Q$$

is the corresponding minimal rational parabolic subgroup of  $\mathbf{L}_Q$ . The elements of  $J \subset \Delta$  may be identified with the set  $\Delta_{\bar{\mathbf{P}}_0}$  of simple rational roots of  $\mathbf{S}_{\bar{\mathbf{P}}_0}$  occurring in the unipotent radical of  $\bar{\mathbf{P}}_0$ .

As in Section 2 define

$$p \boxplus \beta \in \chi_{\mathbb{Q}}^*(\mathbf{S}_{\bar{\mathbf{P}}_0}) \tag{4.2.1}$$

to be the unique rational (quasi-) character such that  $(p \boxplus \beta)(y + z) = \beta(y) + p(z)$  for all  $y \in \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{Q}})$  and all  $z \in \ker(\Delta - J)$ . We may consider  $p \boxplus \beta$  to be a weight profile for the reductive Borel–Serre compactification  $\bar{X}_Q$  of the locally symmetric space corresponding to the reductive group  $\mathbf{L}_Q$ . It satisfies  $(p \boxplus \beta)|_{\mathbf{S}_{\mathbf{Q}}} = \beta$ . The proof of the following theorem will appear in Section 5.

**THEOREM 4.3.** *Let  $p \in \chi_{\mathbb{Q}}^*(\mathbf{S}_0)$  be a weight profile. The restriction of the weighted cohomology complex to the closure of the boundary stratum  $X_Q$  is quasi-isomorphic to the (finite) direct sum of weighted cohomology sheaves:*

$$\mathbf{W}^p \mathbf{C}^\bullet(\bar{X}, \mathbf{E})|_{\bar{X}_Q} \cong \bigoplus_{\beta \in \chi_{\mathbb{Q}}^*(\mathbf{S}_Q)_{\geq p}} \bigoplus_i \mathbf{W}^{p \boxplus \beta} \mathbf{C}^\bullet(\bar{X}_Q, \mathbf{H}^i(\mathfrak{N}_Q, E)_\beta)[-i], \tag{4.3.1}$$

where  $[-i]$  denotes a shift in degree:  $\mathbf{C}^k[-i] = \mathbf{C}^{k-i}$ .

**4.4.** The following remarks will be needed for the proof of Theorem 4.3. Suppose  $\mathbf{P} \subset \mathbf{Q}$  is another standard rational parabolic subgroup. Then  $\mathbf{S}_G \subset \mathbf{S}_Q \subset \mathbf{S}_P \subset \mathbf{S}_{P_0}$ . Let  $p \in \chi_{\mathbb{Q}}^*(\mathbf{S}_{P_0})$ ,  $\beta \in \chi_{\mathbb{Q}}^*(\mathbf{S}_Q)$ , and suppose  $\beta|_{\mathbf{S}_G} = p|_{\mathbf{S}_G}$  as above. We claim that

$$(p \boxplus \beta)|_{\mathbf{S}_P} = (p|_{\mathbf{S}_P}) \boxplus \beta \tag{4.4.1}$$

where, for the sake of notational simplicity, we confuse the restriction to  $\mathbf{S}_P$  with the restriction to  $\chi_{\mathbb{Q}}^*(\mathbf{S}_P)$ . The right-hand side of this equation needs some explanation.

Let  $\Delta_P$  be the simple roots for  $\mathbf{P}$ . They determine a basis of  $\chi_{\mathbb{Q}}^*(\mathbf{S}_P/\mathbf{S}_G)$ . Let  $J' \subset \Delta_P$  be the subset corresponding to  $\mathbf{Q}$ , that is,

$$\chi_{\mathbb{Q}}^*(\mathbf{S}_Q) = \ker(J'). \tag{4.4.2}$$

Here, we consider the elements of  $J'$  to be linear functionals on the vector space

$$X = \chi_{\mathbb{Q}}^*(\mathbf{S}_P). \tag{4.4.3}$$

Then we may apply the construction of Section 2 with

$$\begin{aligned} Y &= \chi_{\mathbb{Q}}^*(\mathbf{S}_Q) = \ker(J') \\ Z &= \ker(\Delta_P - J') \subset \chi_{\mathbb{Q}}^*(\mathbf{S}_P) \end{aligned}$$

to obtain an element  $(p|_{\mathbf{S}_P}) \boxplus \beta \in \chi_{\mathbb{Q}}^*(\mathbf{S}_P)$  such that

$$(p|_{\mathbf{S}_P}) \boxplus \beta(y + z) = \beta(y) + p(z)$$

for all  $y \in Y$  and all  $z \in Z$ . This defines the right-hand side of (4.4.1).

The proof of the claim is a straightforward matter of bookkeeping. The sets  $J' \subset \Delta_P$  may be considered as subsets of  $\Delta$ , that is, as characters of the larger torus  $\mathbf{S}_{P_0}$ . Set

$$\begin{aligned} \Delta_P &= \Delta - I \text{ so } \chi_{\mathbb{Q}}^*(\mathbf{S}_P) = \ker(I) \\ \Delta_Q &= \Delta - J \text{ so } \chi_{\mathbb{Q}}^*(\mathbf{S}_Q) = \ker(J). \end{aligned}$$

It follows that  $J = J' \cup I$  (disjoint union) and that

$$Y = \ker(J') \cap \chi_{\mathbb{Q}}^*(\mathbf{S}_P) \text{ and } Z = \ker(\Delta_P - J') \cap \chi_{\mathbb{Q}}^*(\mathbf{S}_P).$$

To verify the claim we have to check that the subspaces  $Y$  and  $Z$  used to define the right-hand side of (4.4.1) agree with the subspaces (say,  $Y'$  and  $Z'$ ) used to define  $(p \boxplus \beta)|_{\mathbf{S}_P}$  on the left hand side. Clearly,  $Y = Y' = \chi_{\mathbb{Q}}^*(\mathbf{S}_Q)$ . Moreover,

$$Z' = \ker(\Delta - J) \cap \chi_{\mathbb{Q}}^*(\mathbf{S}_P) = \ker(\Delta - I - J') \cap \chi_{\mathbb{Q}}^*(\mathbf{S}_P) = \ker(\Delta_P - J') \cap \chi_{\mathbb{Q}}^*(\mathbf{S}_P)$$

which completes the proof of (4.4.1).

4.5. KOSTANT’S THEOREM

Fix a Cartan subgroup and a Borel subgroup,  $\mathbf{H}(\mathbb{C}) \subset \mathbf{B}(\mathbb{C}) \subset \mathbf{L}_{\mathbf{Q}}(\mathbf{x}_0)(\mathbb{C})$ . Let  $\Phi^+ = \Phi^+(\mathbf{H}(\mathbb{C}), \mathbf{G}(\mathbb{C}))$  be the resulting system of positive roots for  $\mathbf{G}$  and let  $W_Q = W(\mathbf{H}(\mathbb{C}), \mathbf{L}_{\mathbf{Q}}(\mathbb{C}))$  be the Weyl group for  $\mathbf{L}_{\mathbf{Q}}(\mathbb{C})$ . For each  $w \in W_Q$  set

$$\Phi^+(w) = \{ \alpha \in \Phi^+ \mid w^{-1}\alpha \in \Phi^- \}.$$

Then  $|\Phi^+(w)| = \ell(w)$  is the length of  $w$ . The set

$$W_Q^1 = \{ w \in W_Q \mid \Phi^+(w) \subset \Phi(\mathbf{H}, \mathfrak{N}_{\mathbf{Q}}(\mathbb{C})) \} \tag{4.5.1}$$

consists of the unique element of minimal length from each of the cosets  $W_Q x \in W_Q \setminus W$  ([Sp], Section 10.2, [V], Section 3.2.1). (Here,  $\Phi(\mathbf{H}, \mathfrak{N}_{\mathbf{Q}}(\mathbb{C}))$  denotes the set of (positive) roots which occur in the nilradial  $\mathfrak{N}_{\mathbf{Q}}(\mathbb{C})$ .) Let  $\Lambda$  be the highest weight of the irreducible representation  $\mathbf{G} \rightarrow \mathbf{GL}(E)$  (cf. Section 3.5) and let  $V_{\mu}$  be the irreducible  $\mathbf{L}_{\mathbf{Q}}(\mathbb{C})$  module with highest weight  $\mu$ . Then Kostant’s theorem ([K], Section 5.14, [V], Section 3.2.16) states that, as a representation of  $\mathbf{L}_{\mathbf{Q}}$ , the Lie algebra cohomology  $H^*(\mathfrak{N}_{\mathbf{Q}}, E)$  is given by

$$H^*(\mathfrak{N}_{\mathbf{Q}}, E) \cong \bigoplus_{w \in W_Q^1} V_{w(\Lambda+\rho)-\rho}[-\ell(w)],$$

where  $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$  is one-half the sum of the positive roots of  $\mathbf{G}$ . (In this formula, we view  $V_{w(\Lambda+\rho)-\rho}$  as a trivial complex concentrated in degree 0, so that  $V_{w(\Lambda+\rho)-\rho}[-\ell(w)]$  is concentrated in degree  $\ell(w)$ . We use  $H^*$  rather than  $H^\bullet$  to indicate that the cohomology is viewed as a complex with trivial differential.)

It follows that the expression (4.3.1) may be rewritten as

$$\bigoplus_{w \in W_Q^1} \mathbf{W}^{p \boxplus \beta(w)} \mathbf{C}^\bullet(\bar{X}_Q, \mathbf{V}_{w(\Lambda+\rho)-\rho})[\ell(w)], \tag{4.5.2}$$

where the symbol  $\mathbf{V}_{\alpha}$  denotes the local system  $V_{\alpha} \times_{\Gamma_L} (Q/K_Q A_Q) \rightarrow X_Q$  which arises from an irreducible representation  $V_{\alpha}$  of  $\mathbf{L}_{\mathbf{Q}}$  with highest weight  $\alpha$ , and where

$$\beta(w) = (w(\Lambda + \rho) - \rho) | \mathbf{S}_{\mathbf{Q}} \in \chi_{\mathbf{Q}}^*(\mathbf{S}_{\mathbf{Q}}), \tag{4.5.3}$$

$$W_Q^p(E) = \left\{ w \in W_Q^1 \mid \beta(w) \in \chi_{\mathbf{Q}}^*(\mathbf{S}_{\mathbf{Q}})_{\geq p} \right\}. \tag{4.5.4}$$

See [GHM] Section 11.5, Section 11.7. This gives the following corollary:

4.6. COROLLARY. *Suppose  $E$  is an irreducible representation of  $\mathbf{G}$  with highest weight  $\Lambda$ . Then the restriction of the weighted cohomology sheaf to the closure  $\bar{X}_Q$  of the boundary stratum  $X_Q$  decomposes as the sum*

$$\mathbf{W}^p \mathbf{C}^\bullet(\bar{X}, E) | \bar{X}_Q \cong \bigoplus_{w \in W_Q^p(E)} \mathbf{W}^{p \boxplus \beta(w)} \mathbf{C}^{\bullet-\ell(w)}(\bar{X}_Q, \mathbf{V}_{w(\Lambda+\rho)-\rho}).$$

**5. Proof of Theorem 4.3**

5.1. SPECIAL DIFFERENTIAL FORMS

Let  $\mathbf{Q} \subset \mathbf{P}$  be a rational parabolic subgroup, let  $e_Q = Q/K_Q A_Q$  be the Borel–Serre boundary component and let  $Y_Q = \Gamma_Q \backslash e_Q$  be the corresponding stratum in the Borel–Serre compactification  $\tilde{X}$  of  $X$ , with  $\tau: e_Q \rightarrow Y_Q$  the projection. Let  $\mathbf{E} = E \times_{\Gamma_Q} e_Q$  be the local system on  $Y_Q$  arising from some irreducible representation  $\mathbf{G} \rightarrow \mathbf{GL}(E)$ . Recall ([GHM], Section 12.3) that a differential form  $\omega_Q \in \Omega^i(Y_Q, \mathbf{E})$  is ‘invariant’ if its pullback  $\tau^*(\omega_Q) \in \Omega^i(e_Q, \mathbf{E})$  is invariant under  $\mathcal{U}_Q$ . The invariant differential forms give rise to a complex of sheaves  $\Omega_{\text{inv}}^\bullet(Y_Q, \mathbf{E})$  on  $Y_Q$ .

Recall ([GHM], Section 13) that a differential  $i$ -form  $\omega$  on  $X = \Gamma \backslash D$  with values in  $\mathbf{E}$  is called *special* if for each stratum  $Y_Q$  of the Borel–Serre compactification  $\tilde{X}$ , there exists a neighborhood of  $Y_Q$  in  $\tilde{X}$  (which depends on the differential form  $\omega$ ), such that in this neighborhood, the following two conditions hold:

- (1) the differential form  $\omega$  is the pull-up of a differential form  $\omega_Q \in \Omega^i(Y_Q, \mathbf{E})$  from the boundary stratum, via the geodesic retraction, and
- (2) the form  $\omega_Q$  is  $\mathcal{U}_Q$ -invariant, i.e.  $\omega_Q \in \Omega_{\text{inv}}^i(Y_Q, \mathbf{E})$ .

We denote by  $\Omega_{\text{sp}}^\bullet$  the complex of pre-sheaves of special differential forms on  $X$ , whose sections over an open set  $U \subset X$  are

$$\Gamma(U; \Omega_{\text{sp}}^\bullet) = \left\{ \omega \in \Omega^\bullet(U, \mathbf{E}) \mid \begin{array}{l} \omega \text{ is the restriction to } U \\ \text{of a special differential form} \end{array} \right\}$$

Let  $Sh$  denote the sheafification functor, let  $j: X \hookrightarrow \tilde{X}$  be the inclusion of  $X$  into its Borel–Serre compactification, and let  $\pi: \tilde{X} \rightarrow \bar{X}$  denote the projection from the Borel–Serre compactification to the reductive Borel–Serre compactification. Then

$$\tilde{\Omega}_{\text{sp}}^\bullet(\tilde{X}, \mathbf{E}) = Sh(j_* \Omega_{\text{sp}}^\bullet) \tag{5.1.1}$$

is the complex of sheaves of special differential forms on  $\tilde{X}$ , and

$$\bar{\Omega}_{\text{sp}}^\bullet(\bar{X}, \mathbf{E}) = \pi_*(\tilde{\Omega}_{\text{sp}}^\bullet(\tilde{X}, \mathbf{E})) \tag{5.1.2}$$

is the complex of sheaves of special differential forms on  $\bar{X}$  ([GHM], Section 13.8).

For any boundary stratum  $Y_Q \subset \tilde{X}$  the restriction  $\pi|_{Y_Q}: Y_Q \rightarrow X_Q$  is a smooth fiber bundle with nilmanifold fiber  $\pi^{-1}(x) \cong N_Q = (\Gamma \cap \mathcal{U}_Q) \backslash \mathcal{U}_Q$ . The complex  $\mathbf{C}^\bullet(N_Q, \mathbf{E})$  of  $\mathcal{U}_Q$ -invariant differential forms along the fibers of  $\pi$  constitute a complex of flat vector bundles over the stratum  $X_Q$  which is associated (see [GHM], Section 12.5) to the adjoint representation of  $\mathbf{L}_Q$  on the (Koszul) complex

$$\mathbf{C}^\bullet(\mathfrak{N}_Q, E) = \text{Hom}_{\mathbf{R}}(\wedge^\bullet \mathfrak{N}_Q, E) = \text{Hom}_{\mathbf{C}}(\wedge^\bullet \mathfrak{N}_Q(\mathbf{C}), E). \tag{5.1.3}$$

In fact, the choice of basepoint  $x_0 \in D$  determines an isomorphism ([GHM], Section 12.13)

$$\mathbf{C}^\bullet(N_Q, \mathbf{E}) \cong \mathbf{C}^\bullet(\mathfrak{N}_Q, E) \times_{\Gamma_L} (L_Q/K_Q A_Q). \tag{5.1.4}$$

The theorem of Nomizu and van Est identifies the cohomology of this complex with the flat vector bundle on  $X_Q$  which is determined by the representation of  $L_Q$  on  $H^*(N_Q, \mathbf{E})$ .

5.2. LEMMA. *The restriction  $\bar{\Omega}_{\text{sp}}^\bullet(\bar{X}, \mathbf{E})|_{\bar{X}_Q}$  of this sheaf to the closure of the boundary stratum  $X_Q$  decomposes as a direct sum*

$$\bar{\Omega}_{\text{sp}}^\bullet(\bar{X}, \mathbf{E})|_{\bar{X}_Q} \cong \bigoplus_{q+r=\bullet} \bar{\Omega}_{\text{sp}}^q(\bar{X}_Q, \mathbf{C}^r(N_Q, \mathbf{E})),$$

where the differential is given by the differential of the double complex.

5.3. Remark. In [GHM], Section 12.6, it is shown that the integrable connection on the vector bundle  $\mathbf{C}^q(N_Q, \mathbf{E})$  determines an isomorphism

$$\bar{\Omega}_{\text{sp}}^\bullet(\bar{X}, \mathbf{E})|_{X_Q} \cong \bigoplus_{q+r=\bullet} \Omega^q(X_Q, \mathbf{C}^r(N_Q, \mathbf{E}))$$

over the interior of the stratum  $X_Q$ . Denote this sheaf by  $\mathbf{T}_Q^\bullet$ . The content of Lemma 5.2 is that this decomposition extends over the closure of the stratum  $X_Q$ .

5.4. Proof of Lemma 5.2. The closure  $\bar{X}_Q$  of the ‘reductive Borel–Serre stratum’  $X_Q$  may be canonically identified with the reductive Borel–Serre compactification of  $X_Q$ . Restricting (5.1.2) to  $\bar{X}_Q$  we have

$$\bar{\Omega}_{\text{sp}}^\bullet(\bar{X}, \mathbf{E})|_{\bar{X}_Q} = \pi_*(\tilde{\Omega}_{\text{sp}}^\bullet(\tilde{X}, \mathbf{E})|_{\tilde{Y}_Q}), \tag{5.4.1}$$

where  $\tilde{Y}_Q$  is the closure in  $\tilde{X}$  of the Borel–Serre stratum  $Y_Q$ ; it may be canonically identified with the Borel–Serre compactification of  $Y_Q$ . The sheaf  $\tilde{\Omega}_{\text{sp}}^\bullet(\tilde{X}, \mathbf{E})|_{\tilde{Y}_Q}$  may be canonically identified with the sheaf  $\tilde{\Omega}_{\text{sp,inv}}^\bullet(\tilde{Y}_Q, \mathbf{E})$  of  $\mathcal{U}_Q$ -invariant differential forms  $\omega_Q$  on  $Y_Q$  (with values in  $\mathbf{E}$ ) which are ‘special’ near each boundary stratum  $Y_P \subset \tilde{Y}_Q$  and for which the resulting differential form  $\omega_P \in \Omega^\bullet(Y_P, \mathbf{E})$  is  $\mathcal{U}_P$ -invariant. The map  $\pi: \tilde{Y}_Q \rightarrow \bar{X}_Q$  (from the closure of the Borel–Serre boundary stratum to the closure of the reductive Borel–Serre boundary stratum) factors through the ‘hybrid’ Borel–Serre compactification  $\tilde{X}_Q$  of the reductive Borel–Serre stratum  $X_Q$  as the composition,

$$\tilde{Y}_Q \xrightarrow{\alpha} \tilde{X}_Q \xrightarrow{\beta} \bar{X}_Q. \tag{5.4.2}$$

The map  $\alpha$  is a fibration with fiber  $N_Q$  and the integrable connection on  $\alpha|_{Y_Q} = \pi|_{Y_Q}: Y_Q \rightarrow X_Q$  extends to an integrable connection on the closure,  $\tilde{Y}_Q \rightarrow \tilde{X}_Q$ . It may be verified that for any boundary stratum  $Y_P \subset \tilde{Y}_Q$ , the geodesic retraction to  $Y_P$  preserves the flat connection on  $Y_Q$ . It follows that the isomorphism [GHM], Section (12.6),

$$\mathbf{T}_Q^\bullet = \pi_*\tilde{\Omega}_{\text{sp,inv}}^\bullet(Y_Q, \mathbf{E}) \cong \bigoplus_{q+r=\bullet} \Omega^q(X_Q, \mathbf{C}^r(N_Q, \mathbf{E})) \tag{5.4.3}$$

extends uniquely to an isomorphism

$$\alpha_* \tilde{\Omega}_{\text{sp,inv}}^\bullet(\tilde{Y}_Q, \mathbf{E}) \cong \bigoplus_{q+r=\bullet} \tilde{\Omega}_{\text{sp}}^q(\tilde{X}_Q, \mathbf{C}^r(N_Q, \mathbf{E})). \tag{5.4.4}$$

Now apply  $\beta_*$  to obtain an isomorphism,

$$\begin{aligned} \tilde{\Omega}_{\text{sp}}^\bullet(\tilde{X}, \mathbf{E})|_{\tilde{X}_Q} &\cong \beta_* \alpha_* (\tilde{\Omega}_{\text{sp,inv}}^\bullet(\tilde{Y}_Q, \mathbf{E})) \\ &\cong \bigoplus_{q+r=\bullet} \beta_* \tilde{\Omega}_{\text{sp}}^q(\tilde{X}_Q, \mathbf{C}^r(N_Q, \mathbf{E})) \quad \text{by (5.4.4)} \\ &\cong \bigoplus_{q+r=\bullet} \tilde{\Omega}_{\text{sp}}^q(\tilde{X}_Q, \mathbf{C}^r(N_Q, \mathbf{E})) \quad \text{by (5.1.2)}. \end{aligned}$$

This completes the proof of Lemma 5.2. □

**5.5.** The flat bundle  $\mathbf{C}^\bullet(N_Q, \mathbf{E})$  decomposes ([GHM], Section 12.8.1) as a sum of flat subbundles,

$$\mathbf{C}^\bullet(N_Q, \mathbf{E}) \cong \bigoplus_{\beta \in \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbb{Q}})} \mathbf{C}^\bullet(N_Q, \mathbf{E})_\beta$$

according to the weights of  $\mathbf{S}_{\mathbb{Q}}$ . The weight subbundle ([GHM], Section 12.9) is defined to be

$$\mathbf{C}^\bullet(N_Q, \mathbf{E})_{\geq p} = \bigoplus_{\beta \in \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbb{Q}})_{\geq p}} \mathbf{C}^\bullet(N_Q, \mathbf{E})_\beta, \tag{5.5.2}$$

where  $p$  denotes the weight profile chosen in Theorem 4.3.

Recall ([GHM], Section 14) that the weighted cohomology sheaf is defined to be the subsheaf of  $\tilde{\Omega}_{\text{sp}}^\bullet(\tilde{X}, \mathbf{E})$  (5.1.2) which is obtained by truncating with respect to the weight profile  $p$  on  $\mathbf{G}$ . In other words, it is the unique subsheaf such that for every boundary stratum  $X_P$ ,

$$\mathbf{W}^p \mathbf{C}^\bullet(\tilde{X}, \mathbf{E})|_{X_P} = (\mathbf{T}_P^\bullet)_{\geq p} = \bigoplus_{q+r=\bullet} \Omega^q(X_P, \mathbf{C}^r(\mathcal{Y}_P, \mathbf{E})_{\geq p}), \tag{5.5.3}$$

the identification being determined by the choice of basepoint  $x_0 \in D$ . If  $\beta \in \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbb{Q}})$  then the weight profile  $p \boxplus \beta$  (4.2.1) is defined on  $\mathbf{L}_{\mathbb{Q}}$ .

**5.6. DEFINITION.** Using Lemma 5.2, define the subsheaf  $\Theta^\bullet \subset \tilde{\Omega}_{\text{sp}}^\bullet(\tilde{X}, \mathbf{E})|_{\tilde{X}_Q}$  as follows:

$$\Theta^\bullet = \bigoplus_{\beta \in \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbb{Q}})_{\geq p}} \bigoplus_{q+r=\bullet} \mathbf{W}^{p \boxplus \beta} \mathbf{C}^q(\tilde{X}_Q, \mathbf{C}^r(N_Q, \mathbf{E})_\beta) \subset \bigoplus_{q+r=\bullet} \tilde{\Omega}_{\text{sp}}^q(\tilde{X}_Q, \mathbf{C}^r(N_Q, \mathbf{E})). \tag{5.6.1}$$

**5.7. LEMMA.** For any boundary stratum  $X_P \subseteq \tilde{X}_Q$ , the restriction

$$\Theta^\bullet|_{X_P} \subset \tilde{\Omega}_{\text{sp}}^\bullet(\tilde{X}, \mathbf{E})|_{X_P}$$

coincides with the subsheaf

$$(\mathbf{T}_P^\bullet)_{\geq p} = \bigoplus_{q+r=\bullet} \Omega^q(X_P; \mathbf{C}^r(N_P, \mathbf{E})_{\geq p}).$$

Consequently,

$$\Theta^\bullet = \mathbf{W}^p \mathbf{C}^\bullet(\bar{X}, \mathbf{E})|_{\bar{X}_Q}. \quad (5.7.1)$$

5.8. *Proof.* First consider the case  $\mathbf{P} = \mathbf{Q}$ . The restriction  $\Theta^\bullet|_{X_Q}$  is given by

$$\Theta^\bullet|_{X_Q} = \bigoplus_{\beta \in \chi_Q^*(\mathbf{S}_Q)_{\geq p}} \bigoplus_{q+r=\bullet} \Omega^q(X_Q, \mathbf{C}^r(N_Q, \mathbf{E})_\beta) \quad (5.8.1)$$

since the weight conditions at the edge of  $X_Q$  do not affect the sections of this sheaf over the interior of  $X_Q$ . But this is precisely the weight subcomplex  $(T_Q^\bullet)_{\geq p}$ .

Now suppose that  $\mathbf{P} \subset \mathbf{Q}$  is a proper parabolic subgroup. As in Section 3.3, set  $\bar{\mathbf{P}} = v_Q(\mathbf{P})$ . Consider the restriction to  $X_P$  of a single weighted cohomology sheaf  $\mathbf{W}^{p \boxplus \beta} \mathbf{C}^q(\bar{X}_Q, \mathbf{C}^r(N_Q, \mathbf{E})_\beta)$  which occurs in (5.6.1). The choice of basepoint  $x_0 \in D$  determines an isomorphism (5.5.3) between this restriction and

$$\bigoplus_{a+b=r} \bigoplus_{\gamma \in \chi_Q^*(\mathbf{S}_{\bar{\mathbf{P}}})_{\geq p \boxplus \beta}} \Omega^q(X_P, \mathbf{C}^a(N_{\bar{\mathbf{P}}}, \mathbf{C}^b(N_Q, \mathbf{E})_\beta)_\gamma).$$

Thus we obtain an identification

$$\Theta^\bullet|_{X_P} \cong \bigoplus_{\beta \in \chi_Q^*(\mathbf{S}_Q)_{\geq p}} \bigoplus_{q+r=\bullet} \bigoplus_{a+b=r} \bigoplus_{\gamma \in \chi_Q^*(\mathbf{S}_{\bar{\mathbf{P}}})_{\geq p \boxplus \beta}} \Omega^q(X_P; \mathbf{C}^a(N_{\bar{\mathbf{P}}}; \mathbf{C}^b(N_Q, \mathbf{E})_\beta)_\gamma). \quad (5.8.2)$$

Let us compare this with  $(T_P^\bullet)_{\geq p}$ . It suffices to show that the coefficient subbundles coincide. By (5.1.4) this amounts to comparing the following two representations of  $\mathbf{L}_{\bar{\mathbf{P}}}$ :

$$\Theta = \bigoplus_{a+b=r} \bigoplus_{\beta \in \chi_Q^*(\mathbf{S}_Q)_{\geq p}} \bigoplus_{\gamma \in \chi_Q^*(\mathbf{S}_{\bar{\mathbf{P}}})_{\geq p \boxplus \beta}} \mathbf{C}^a(\mathfrak{N}_{\bar{\mathbf{P}}}, \mathbf{C}^b(\mathfrak{N}_Q, \mathbf{E})_\beta)_\gamma. \quad (5.8.3)$$

and

$$\mathbf{C}^r(\mathfrak{N}_P, E)_{\geq p}. \quad (5.8.4)$$

The basepoint also determines a splitting  $i: \mathcal{U}_{\bar{\mathbf{P}}} \rightarrow \mathcal{U}_P$  of the sequence

$$1 \rightarrow \mathcal{U}_Q \rightarrow \mathcal{U}_P \rightarrow \mathcal{U}_{\bar{\mathbf{P}}} \rightarrow 1$$

and, hence, a decomposition  $\mathfrak{N}_P \cong \mathfrak{N}_{\bar{\mathbf{P}}} \oplus \mathfrak{N}_Q$ . Therefore, we obtain an isomorphism

$$\mathbf{C}^\bullet(\mathfrak{N}_P, E) = \text{Hom}(\wedge^\bullet \mathfrak{N}_P, E) \cong \bigoplus_{a+b=\bullet} \text{Hom}(\wedge^a \mathfrak{N}_{\bar{\mathbf{P}}}, \text{Hom}(\wedge^b \mathfrak{N}_Q, E)) \quad (5.8.5)$$

and hence an isomorphism

$$\mathbf{C}^r(\mathfrak{N}_P, E)_{\geq p} \cong \bigoplus_{a+b=r} \mathbf{C}^a(\mathfrak{N}_{\bar{\mathbf{P}}}; \mathbf{C}^b(\mathfrak{N}_Q, E))_{\geq p}, \quad (5.8.6)$$

where ‘ $\geq p$ ’ indicates the direct sum of those  $\mathbf{S}_{\mathbf{P}}$ -isotypical components with weights  $\alpha \in \chi_Q^*(\mathbf{S}_{\mathbf{P}})_{\geq p}$ .

Now it is important to keep track of which simple roots are involved in the weight truncations. The parabolic subgroup  $\mathbf{Q}$  corresponds to a subset  $J \subset \Delta_P$  of the simple roots with  $\chi_*^{\mathbf{Q}}(\mathbf{S}_{\mathbf{Q}}) = \ker(J)$ . The elements of  $\Delta_P - J$  restrict to a linearly independent set  $\Delta_Q \subset \chi_{\mathbf{Q}}^*(\mathbf{S}_{\mathbf{Q}})$  and determine a basis of  $\chi_{\mathbf{Q}}^*(\mathbf{S}'_{\mathbf{Q}})$ . The elements of  $J$  may be identified with the set  $\Delta_{\bar{P}}$  of simple (rational) roots of  $\mathbf{S}_{\bar{P}}$  occurring in the unipotent radical of  $\bar{P}$ .

Let  $\alpha \in \chi_{\mathbf{Q}}^*(\mathbf{S}_{\mathbf{Q}})$  and let  $\beta = \alpha|_{\mathbf{S}_{\mathbf{Q}}}$ . By taking  $X = \chi_*^{\mathbf{Q}}(\mathbf{S}_{\mathbf{P}})$ ,  $Y = \ker(J) = \chi_*^{\mathbf{Q}}(\mathbf{S}_{\mathbf{Q}})$  and  $Z = \ker(\Delta_P - J)$ , and by forming the restriction  $p|_{\mathbf{S}_{\mathbf{P}}} \in \chi_{\mathbf{Q}}^*(\mathbf{S}_{\mathbf{P}})$ , the construction of Section 2 defines an element  $(p|_{\mathbf{S}_{\mathbf{P}}}) \boxplus \beta \in \chi_{\mathbf{Q}}^*(\mathbf{S}_{\bar{P}})$  such that  $(p|_{\mathbf{S}_{\mathbf{P}}}) \boxplus \beta(y + z) = \beta(y) + p(z)$  for all  $y \in Y$  and  $z \in Z$ . By (4.4.1), this agrees with the restriction  $(p \boxplus \beta)|_{\mathbf{S}_{\mathbf{P}}}$  of the weight profile  $p \boxplus \beta \in \chi_{\mathbf{Q}}^*(\mathbf{S}_{\mathbf{P}_0})$  defined in Section 4.2, so we may refer to it simply as  $p \boxplus \beta$ . Let  $\gamma \in \chi_{\mathbf{Q}}^*(\mathbf{S}_{\bar{P}})$  be the element corresponding to  $\alpha \in \chi_{\mathbf{Q}}^*(\mathbf{S}_{\mathbf{P}})$  under the canonical isomorphism  $\mathbf{S}_{\mathbf{P}} \cong \mathbf{S}_{\bar{P}}$ . Then Proposition 2.2 says

$$\alpha \in \chi_{\mathbf{Q}}^*(\mathbf{S}_{\mathbf{P}})_{\geq p(\Delta_P)} \text{ iff } \beta \in \chi_{\mathbf{Q}}^*(\mathbf{S}_{\mathbf{Q}})_{\geq p(\Delta_Q)} \text{ and } \gamma \in \chi_{\mathbf{Q}}^*(\mathbf{S}_{\bar{P}})_{\geq p \boxplus \beta(\Delta_{\bar{P}})}. \quad (5.8.7)$$

Let us regard the double complex (5.8.6) as a module over  $L_{\bar{P}}$  and decompose it into  $\mathbf{S}_{\bar{P}}$ -isotypical components. We obtain

$$C^r(\mathfrak{N}_P, E)_{\geq p(\Delta_P)} \cong \bigoplus_{a+b=r} \bigoplus_{\beta} \bigoplus_{\gamma} C^a(\mathfrak{N}_{\bar{P}}, C^b(\mathfrak{N}_Q, E)_{\beta})_{\gamma}$$

where the second sum is over those  $\beta \in \chi_{\mathbf{Q}}^*(\mathbf{S}_{\mathbf{Q}})_{\geq p(\Delta_Q)}$  and the third sum is over those  $\gamma \in \chi_{\mathbf{Q}}^*(\mathbf{S}_{\bar{P}})_{\geq p \boxplus \beta(\Delta_{\bar{P}})}$ . This is exactly (5.8.3).

It is easy to see that the isomorphisms and splittings are compatible, so that the representations (5.8.3) and (5.8.4) are actually the same subspace of  $C^r(\mathfrak{N}_P, E)$ . In fact, as in [GHM], Section 10.4, these splittings do not depend on the choice of basepoint. This completes the proof of Lemma 5.7.  $\square$

5.9. *Proof of Theorem 4.3.* By [GHM], Section 12.15 the complex of local systems  $\mathbf{C}^{\bullet}(N_Q, \mathbf{E})_{\beta}$  is quasi-isomorphic to its cohomology sheaves,  $\mathbf{H}^*(N_Q, \mathbf{E})_{\beta}$  (where we use  $*$  rather than  $\bullet$  to indicate that this is to be considered a complex of sheaves with trivial differentials). From (5.6.1) this determines a quasi-isomorphism

$$\Theta^{\bullet} \cong \bigoplus_{\beta \in \chi_{\mathbf{Q}}^*(\mathbf{S}_{\mathbf{Q}})_{\geq p}} \bigoplus_{q+r=\bullet} \mathbf{W}^{p \boxplus \beta} \mathbf{C}^q(\bar{X}_Q, \mathbf{H}^r(N_Q, \mathbf{E})_{\beta}). \quad (5.9.1)$$

On the other hand, (5.7.1) identifies  $\Theta^{\bullet}$  with the desired weighted cohomology complex.  $\square$

## 6. Local Intersection Cohomology

### 6.1. BAILY–BOREL COMPACTIFICATION

In this section we suppose (until Section 6.11) that  $\mathbf{G}$  is a semi-simple  $\mathbb{Q}$ -algebraic group, and that  $D = G/K$  is a Hermitian symmetric space (where  $K \subset G$  is a maximal compact subgroup corresponding to a choice of basepoint  $x_0 \in D$ ). Let  $\Phi: \bar{X} \rightarrow \hat{X}$  be the projection from the reductive Borel–Serre compactification of  $X = \Gamma \backslash G/K$  to the



Baily–Borel Satake compactification ([Z3], [GHM], Section 22). Fix a boundary stratum  $F \subset \hat{X}$ . Then  $F$  corresponds to a  $\Gamma$ -conjugacy class of proper maximal rational parabolic subgroups of  $\mathbf{G}$ , from which we choose one and denote it by  $\mathbf{Q}$ . Let us consider the reductive Borel–Serre stratum  $X_Q$  to be a locally symmetric space associated to the connected reductive algebraic group  $\mathbf{L}_Q$  (Section 3.2). By [AMRT], III Section 4 (see also [LR], Section 6.1) the group  $\mathbf{L}_Q$  decomposes as an almost direct product,  $\mathbf{L}_Q = \mathbf{Q}_h \mathbf{Q}_\ell$  of commuting algebraic subgroups with finite intersection. The compact factors (if any) of  $\mathbf{L}_Q$  may be distributed among  $\mathbf{Q}_h$  and  $\mathbf{Q}_\ell$  so that both  $\mathbf{Q}_h$  and  $\mathbf{Q}_\ell$  are defined over  $\mathbb{Q}$ . The group  $\mathbf{Q}_h$  acts by holomorphic automorphisms of the boundary component  $F$ . It contains no rational anisotropic subgroup of positive dimension. The group  $\mathbf{Q}_\ell$  is reductive with split center  $\mathbf{S}_Q$ , and it acts by linear automorphisms on a certain self-adjoint homogeneous symmetric cone. Set

$$\begin{aligned} K_Q &= K \cap Q \subset K_\ell \subset L_Q(x_0), & \Gamma_L &= v_Q(\Gamma \cap Q), \\ K_\ell &= K_Q \cap Q_\ell, & K_h &= \mu(K_Q), \\ \Gamma_\ell &= \Gamma_L \cap Q_\ell, & \Gamma_h &= \mu(\Gamma_L), \end{aligned}$$

where  $v_Q: \mathbf{Q} \rightarrow \mathbf{L}_Q$  and  $\mu: \mathbf{L}_Q \rightarrow \mathbf{Q}_h$  are the projections. Then the boundary stratum  $F$  is diffeomorphic to  $\Gamma_h \backslash Q_h / K_h$ . For any  $x \in F$ , a choice of  $q \in Q_h$  which projects to  $x$  determines a stratum preserving homeomorphism ([GHM], Section 22.6),

$$\tilde{f}_q: \Phi^{-1}(x) \rightarrow \tilde{X}_\ell \tag{6.1.1}$$

(which is smooth on each stratum) between the fiber  $\Phi^{-1}(x)$  and the reductive Borel–Serre compactification of the locally symmetric space

$$X_\ell = \Gamma_\ell \backslash Q_\ell / A_Q K_\ell. \tag{6.1.2}$$

The restriction  $(\Phi|_{X_Q}): X_Q \rightarrow F$  agrees with the projection determined by  $\mu$ .

6.2. WEIGHT PROFILES

A weight profile for  $\mathbf{L}_Q = \mathbf{Q}_h \mathbf{Q}_\ell$  determines weight profiles for  $\mathbf{Q}_h$  and  $\mathbf{Q}_\ell$ . In fact, if  $\mathbf{S}_{0h} \subset \mathbf{Q}_h$  and  $\mathbf{S}_{0\ell} \subset \mathbf{Q}_\ell$  are maximal  $\mathbb{Q}$ -split tori then their product defines a maximal  $\mathbb{Q}$ -split torus  $\mathbf{S}_{0L}$  in  $\mathbf{L}_Q$  and a canonical isomorphism  $\chi_{\mathbb{Q}}^*(\mathbf{S}_{0L}) \cong \chi_{\mathbb{Q}}^*(\mathbf{S}_{0h}) \oplus \chi_{\mathbb{Q}}^*(\mathbf{S}_{0\ell})$ . So for any weight profile  $p$  for  $\mathbf{L}_Q$  we may speak of the weight profile  $p$  which is obtained by restriction to the linear factor  $\mathbf{Q}_\ell$ .

Now, fix an algebraic irreducible representation  $G \rightarrow GL(E)$  with highest weight  $\Lambda$ . It determines a local system  $\mathbf{E}$  on  $X$  and by restriction, a local system (which we also denote by  $\mathbf{E}$ ) on  $X_\ell$ . An irreducible  $\mathbf{L}_Q$ -module  $V_\alpha$  of highest weight  $\alpha$  may be considered, by restriction, as a module over  $\mathbf{Q}_\ell$  and hence determines a local system  $\mathbf{V}_\alpha$  over the space  $X_\ell$ . The following is the second main result in this paper.

6.3. THEOREM. *Let  $p$  be a weight profile for the reductive Borel–Serre compactification,  $\tilde{X}$  of  $X = \Gamma \backslash G / K$ . Let  $R\Phi_* \mathbf{W}^p \mathbf{C}^\bullet(\tilde{X}, \mathbf{E})$  denote the pushforward of the weighted*

cohomology sheaf to the Baily-Borel compactification  $\hat{X}$ . Let  $F \subset \hat{X}$  be a boundary component, corresponding to a maximal rational parabolic subgroup  $\mathbf{Q} \subset \mathbf{G}$ . Then the stalk cohomology  $H_x^k(R\Phi_* \mathbf{W}^p \mathbf{C}^\bullet)$  of this sheaf at a point  $x \in F \subset \hat{X}$  is given by

$$\bigoplus_{\beta \in \chi_{\mathbf{Q}}^*(\mathbf{S}_{\mathbf{Q}})_{\geq p}} \bigoplus_i W^{p \boxplus \beta} H^{k-i}(\bar{X}_\ell; \mathbf{H}^i(\mathfrak{N}_{\mathbf{Q}}, E)_\beta) \tag{6.3.1}$$

$$\cong \bigoplus_{w \in W_{\mathbf{Q}}^p(E)} W^{p \boxplus \beta(w)} H^{k-\ell(w)}(\bar{X}_\ell; \mathbf{V}_{w(\Lambda+\rho)-\rho}) \tag{6.3.2}$$

where  $\chi_{\mathbf{Q}}^*(\mathbf{S}_{\mathbf{Q}})_{\geq p}$  is given by (3.5.1),  $\beta(w)$  is given by (4.5.3), and  $W_{\mathbf{Q}}^p(E)$  is given by (4.5.4).

6.4. *Proof.* Since  $\Phi: \bar{X} \rightarrow \hat{X}$  is proper, the stalk cohomology of  $R\Phi_*(\mathbf{W}^p \mathbf{C}^\bullet)$  at a point  $x \in \hat{X}$  is canonically isomorphic to the cohomology of the fiber,  $H^*(\Phi^{-1}(x); \mathbf{W}^p \mathbf{C}^\bullet(\mathbf{E}))$ . If  $F$  denotes the stratum of  $\hat{X}$  which contains  $x$  and if  $\mathbf{Q}$  denotes a corresponding maximal rational parabolic subgroup of  $\mathbf{G}$ , then  $\Phi^{-1}(F)$  is a union of strata in  $\bar{X}_{\mathbf{Q}}$  and the restriction  $\Phi: \Phi^{-1}(F) \rightarrow F$  is a fiber bundle. For any weight profile  $q$  it follows that the homeomorphism (6.1.1) determines a (quasi-) isomorphism

$$\mathbf{W}^q \mathbf{C}^\bullet(\bar{X}_{\mathbf{Q}}, \mathbf{E})|_{\Phi^{-1}(x)} \cong \mathbf{W}^q \mathbf{C}^\bullet(\bar{X}_\ell, \mathbf{E}) \tag{4.6.4}$$

where  $q$  also denotes the restriction of the weight profile to the linear factor. (This follows from the Poincaré lemma for  $F$  and the fact that no weight truncation occurs relative to the Hermitian factor.) The restriction  $\mathbf{W}^q \mathbf{C}^\bullet|_{\Phi^{-1}(x)}$  may be obtained by first restricting to  $\bar{X}_{\mathbf{Q}}$  then further restricting to  $\Phi^{-1}(x)$ . So (6.3.1) follows from (4.3.1) and (6.4.1), while (6.3.2) follows from (4.5.2) and (6.4.1).  $\square$

6.5. COROLLARY. Let  $p = \mu$  or  $p = \nu$  denote the upper middle or lower middle weight profile, respectively (3.6). Let  $\rho_{\mathbf{Q}} = \rho|_{\mathbf{S}_{\mathbf{Q}}}$  be the restriction of  $\rho$ . Then the stalk cohomology of the intersection cohomology, at a point  $x \in F \subset \hat{X}$  is given by Theorem 6.3. In other words,

$$IH_x^k(\hat{X}, \mathbf{E}) \cong \bigoplus_{\beta \geq -\rho_{\mathbf{Q}}} \bigoplus_i \mathbf{W}^{p \boxplus \beta} H^{k-i}(\bar{X}_\ell; \mathbf{H}^i(\mathfrak{N}_{\mathbf{Q}}, E)_\beta). \tag{6.5.1}$$

The first sum may be replaced by  $\bigoplus_{\beta > -\rho_{\mathbf{Q}}}$ .

This expression may also be evaluated as in (6.3.2) using Kostant’s theorem and it may be translated as in Section 3.10 into Lie algebra cohomology.

6.6. *Proof.* The proof follows by combining Theorem 6.3 above and Theorem 23.2 of [GHM] which constructs a canonical isomorphism

$$R\Phi_* \mathbf{W}^\mu \mathbf{C}^\bullet(\mathbf{E}) \cong R\Phi_* \mathbf{W}^\nu \mathbf{C}^\bullet(\mathbf{E}) \cong \mathbf{IC}^\bullet(\hat{X}, \mathbf{E}) \tag{6.6.1}$$

between the pushforward of the middle weighted cohomology on  $\bar{X}$  with the intersection cohomology of  $\hat{X}$ .  $\square$

Taking  $p = -\infty$  gives the following well-known result [LR], the étale version of which is proven in [P1], [P2]:

**6.7. COROLLARY.** *Let  $\hat{\iota}: X \hookrightarrow \hat{X}$  be the inclusion of the locally symmetric space into the Baily–Borel compactification. Then the stalk cohomology of the sheaf  $R\hat{\iota}_*(\mathbf{E})$  is given by*

$$H_X^k(R\hat{\iota}_*(\mathbf{E})) \cong \bigoplus_{w \in W_Q^1} H^{k-\ell(w)}(\Gamma_\ell; V_{w(\Lambda+\rho)-\rho}).$$

Here,  $W_Q^1$  is given by (4.5.1), and  $V_\alpha$  is the irreducible  $\mathbf{L}_Q$ -module with highest weight  $\alpha$ . It is considered as a module over  $\Gamma_\ell$  by way of the inclusion  $\Gamma_\ell \subset \Gamma_L = v_Q(\Gamma_Q) \subset L_Q$ .

**6.8. Proof.** In Theorem 6.3, take  $p = -\infty$  to be the weight profile which involves no truncation. Then  $W_Q^p(E) = W_Q^1$ . The weighted cohomology  $W^p H^i(\bar{X}_\ell; V_\alpha)$  is equal to the ordinary cohomology of  $X_\ell$ , (with coefficients in  $V_\alpha$ ) which (since  $\Gamma$  is neat) in turn coincides with the group cohomology  $H^i(\Gamma_\ell, V_\alpha)$ .  $\square$

**6.9. Remarks.** The vanishing of the stalk cohomology of the intersection cohomology, and more generally, the purity theorem of Looijenga (i.e. that the stalk cohomology in degree  $i$  of the intersection cohomology has weight  $\leq i$ ) may be translated into vanishing theorems for certain weighted cohomology groups of the the ‘linear’ locally symmetric space  $X_\ell$  using Theorem 6.3. A general framework for such vanishing theorems has been developed in [S]. See [B2] for a related vanishing theorem for the  $L^2$  cohomology of linear locally symmetric spaces.  $\square$

#### 6.10. MIXED HODGE WEIGHTS

By [LR], the direct sum (6.5.1) over  $\beta \in \chi_Q^*(\mathbf{S}_Q)$  is a splitting of the weight filtration (on the stalk cohomology) which comes from Saito’s theory of mixed Hodge modules.

#### 6.11. THE $L^2$ EULER CHARACTERISTIC

When  $G/K$  is Hermitian, the Zucker conjecture ([Lo],[SS]) implies that the  $L^2$  cohomology Euler characteristic  $L^2\chi(\Gamma, E)$  is equal to the intersection cohomology Euler characteristic of the Baily–Borel compactification, i.e.

$$L^2\chi(\Gamma, E) = I\chi(\hat{X}, \mathbf{E}) = \sum_i (-1)^i \dim IH^i(\hat{X}, \mathbf{E}).$$

By (6.6.1) this is equal to the Euler characteristic of the weighted cohomology complex of the reductive Borel–Serre compactification,

$$L^2\chi(\Gamma, E) = \sum_i (-1)^i \dim W^u H^i(\bar{X}, \mathbf{E}). \tag{6.11.1}$$

In fact this relation holds more generally. Suppose only that  $\mathbf{G}$  is a reductive  $\mathbb{Q}$ -algebraic group such that the derived group of  $G$  possesses a compact Cartan subgroup. Then  $W^u H^*(\bar{X}, \mathbf{E})$  is isomorphic to the  $L^2$  cohomology of  $\Gamma$  by [N], and therefore (6.11.1) holds.

As in [GM1], (6.11.1) is equal to a sum over strata of  $\bar{X}$ , with the contribution from a single stratum  $X_P$  being given by the compactly supported Euler characteristic  $\chi_c(X_P)$  of the stratum  $X_P$  times the stalk Euler characteristic of the weighted cohomology at any point  $x \in X_P$ . Moreover,  $\chi_c(X_P) = \chi(X_P)$ . Let  $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_l$  be a collection of representatives, one from each  $\Gamma$ -conjugacy class of proper rational parabolic subgroups of  $\mathbf{G}$ . For each representative  $\mathbf{P}_j$ , let  $\Gamma_{L_j} = v(\Gamma \cap P_j)$  be the projection of  $\Gamma \cap P_j$  to the Levi quotient. Since  $\Gamma$  is neat, the Euler characteristic  $\chi(X_{P_j}) = \chi(\Gamma_{L_j})$  is equal to the Euler characteristic of the discrete group  $\Gamma_{L_j}$ . Let  $V_\alpha^{(j)}$  be the irreducible  $L_j$ -module with highest weight  $\alpha$ . Let us suppose that  $\mathbf{E}$  is the local system associated to an irreducible representation of  $\mathbf{G}$  with highest weight  $\Lambda$ . Evaluating (3.5.3) using Kostant’s theorem gives the following formula (which is in [GHM], Section 17.9, [S] for Hermitian  $X$ ):

**6.12. THEOREM.** *Let  $\mathbf{G}$  be a reductive  $\mathbb{Q}$ -algebraic group and suppose the derived group of  $G$  has a compact Cartan subgroup. Then the Euler characteristic of the  $L^2$  cohomology of the locally symmetric space  $X = \Gamma \backslash G / A_G K$  is given by*

$$L^2\chi(\Gamma, E) = \chi(\Gamma) \dim(E) + \sum_{j=1}^l \chi(\Gamma_{L_j}) \cdot \sum_{w \in W_{P_j}^u} (-1)^{\ell(w)} \dim(V_{w(\Lambda+\rho)-\rho}^{(j)}).$$

## 7. Computations for $\mathbf{Sp}_4$

### 7.1. THE SYMMETRIC SPACE

Throughout this section we fix an integer  $N \geq 3$  and we take  $\Gamma = \Gamma(N)$  to be the principal congruence subgroup of  $\mathbf{Sp}_4(\mathbb{Z})$  consisting of matrices which are congruent to the identity modulo  $N$ . Define  $X = \Gamma \backslash \mathbf{Sp}_4(\mathbb{R}) / \mathbf{U}_2$  to be the associated (real) 6-dimensional Hermitian locally symmetric space. The Baily–Borel compactification  $\hat{X}$  has boundary strata of real dimension 2 and of real dimension 0. In this section we will compute the intersection Euler characteristic of  $\hat{X}$  and also the local  $L^2$  (or intersection) cohomology (with constant coefficients) at a most singular point (i.e. at a 0-dimensional stratum) of  $\hat{X}$ . Denote by  $\Phi: \bar{X} \rightarrow \hat{X}$  the projection from the reductive Borel–Serre compactification to the Baily–Borel compactification. Let  $\mathfrak{h}$  denote the upper half plane.

7.2. PARABOLIC SUBGROUPS

There are three types of rational proper parabolic subgroups  $\mathbf{P}$  of  $\mathbf{G}$ .

- (A)  $P$  is the stabilizer of a rational one-dimensional (and hence isotropic) subspace  $F^1 \subset \mathbb{Q}^4$ . Such a parabolic subgroup is maximal with Hermitian Levi factor  $L = P_h \cong \mathbf{SL}_2(\mathbb{R})$ . The associated reductive Borel–Serre stratum  $X_P$  is (real) two-dimensional, in fact it is a modular curve.
- (B)  $P$  is the stabilizer of a rational Lagrangian subspace  $F^2 \subset \mathbb{Q}^4$ . Such a parabolic subgroup is maximal with linear Levi factor  $L = P_\ell \cong \mathbf{GL}_2(\mathbb{R})$ . The associated reductive Borel–Serre stratum  $X_P$  is (real) two-dimensional and is diffeomorphic to a modular curve.
- (C)  $P$  is a rational Borel subgroup: it is the stabilizer of a rational isotropic flag  $F^1 \subset F^2 \subset \mathbb{Q}^4$ . The associated reductive Borel–Serre stratum  $X_P$  is a point. Such a point is simultaneously a cusp for a single type A stratum and a single type B stratum in the reductive Borel–Serre compactification.

The projection  $\Phi$  takes each type A stratum  $X_P \subset \bar{X}$  isomorphically to a (real) two dimensional stratum  $Y_P \subset \hat{X}$ . The projection  $\Phi$  collapses each type B stratum  $X_P$  to a single point in  $\hat{X}$ . The preimage  $\Phi^{-1}(x)$  of such a point is the reductive Borel–Serre compactification of the type B stratum  $X_P$ , and it is obtained by adding type C strata as cusps of  $X_P$ .

7.3. Consider the case of the trivial local system  $\mathbf{E} = \mathbb{C}$ . By Corollary 6.5, the  $L^2$  cohomology, or the intersection cohomology of  $\hat{X}$  is isomorphic to either of the middle weighted cohomology groups of  $\bar{X}$ . In fact, the upper and lower middle weight profiles  $\mu, \nu$  give rise to the same weighted cohomology sheaf on  $\bar{X}$  since in this case, the middle weight does not appear in the  $\mathfrak{N}_P$  cohomology for any parabolic subgroup  $P$ .

7.4. First we consider the weighted Euler characteristic. Let  $n_i$  be the number of  $\Gamma(N)$ -conjugacy classes of rational parabolic subgroups of type  $i$  (for  $i = A, B, C$ ). Let  $\Gamma_i$  denote the projection of  $\Gamma$  into the Levi quotient  $L = P/U_P$  for each parabolic subgroup  $P$  of type  $i = A, B, C$ . Let

$$\chi_i = \sum_{w \in W_P^\mu} (-1)^{\ell(w)} \dim(V_{w\rho - \rho})$$

be the factor which appears in Theorem (6.12) and which arises from a parabolic subgroup  $P$  of type  $i = A, B, C$ . Then the Euler characteristic of the intersection cohomology of  $\hat{X}$  is given by

$$I\chi(\hat{X}) = \chi(\Gamma) + n_A \chi(\Gamma_A) \chi_A + n_B \chi(\Gamma_B) \chi_B + n_C \chi(\Gamma_C) \chi_C. \tag{7.4.1}$$

7.5. THEOREM. For  $\Gamma = \Gamma(N)$  and  $N \geq 3$  the intersection cohomology Euler characteristic of  $\hat{X}$  is

$$I\chi(\hat{X}) = -\left[\frac{N^{10}}{2^5 3^2 5} - \frac{N^7}{8} + \frac{N^6}{4}\right] \prod_{p|N} \left(1 - \frac{1}{p^2}\right) \left(1 - \frac{1}{p^4}\right). \tag{7.5.1}$$

For  $N = 4$  this gives  $I\chi(\hat{X}) = 208$ , in agreement with the computation  $1 - 0 + 118 - 30 + 118 - 0 + 1 = 208$  on the final page of [HW]. If  $N \geq 5$  then  $I\chi(\hat{X})$  is negative.

7.6. Proof. From the root system for  $\mathbf{Sp}_4$  we find

$$\chi_A = -1, \quad \chi_B = -2, \quad \chi_C = -1. \tag{7.6.1}$$

The groups  $\Gamma_A$  and  $\Gamma_B$  both turn out to be the principal congruence subgroup of  $\mathbf{SL}_2(\mathbb{Z})$  with level  $N$ . The modular curve  $\Gamma_B \backslash \mathfrak{h}$  has

$$v_\infty(N) = \frac{N^2}{2} \prod_{p|N} \left(1 - \frac{1}{p^2}\right)$$

cusps ([Sh] Lemma 1.42, [Mi] Theorem 4.2.10) and Euler characteristic

$$\chi(\Gamma_A) = \chi(\Gamma_B) = -\frac{1}{12} |\mathbf{SL}_2(\mathbb{Z}/(N))| \tag{7.6.2}$$

by the Gauss–Bonnet theorem [H], or by [Sh] (1.6.4) and [Mi], Section 4.2, where

$$|\mathbf{SL}_2(\mathbb{Z}/(N))| = N^3 \prod_{p|N} \left(1 - \frac{1}{p^2}\right).$$

Each type C boundary component is simultaneously a cusp of a type A boundary component and of a type B boundary component of the reductive Borel–Serre compactification so  $n_A = n_B$  and  $n_C = n_A v_\infty(N)$ . The number of lines in  $\mathbb{Q}^4$  modulo  $\Gamma(N)$ -equivalence is

$$n_A = \frac{N^4}{2} \prod_{p|N} \left(1 - \frac{1}{p^4}\right) \tag{7.6.3}$$

This gives a formula for  $n_C$ , which is also the number of double cosets  $B(\mathbb{Z}) \backslash \mathbf{Sp}_4(\mathbb{Z}) / \Gamma(N)$ :

$$n_C = \frac{N^6}{4} \prod_{p|N} \left(1 - \frac{1}{p^2}\right) \left(1 - \frac{1}{p^4}\right).$$

(Here  $B$  denotes the standard Borel subgroup.) Finally, the Gauss–Bonnet formula ([H], p. 453) gives

$$\chi(\Gamma(N)) = \zeta(-1)\zeta(-3) \cdot |\mathbf{Sp}_4(\mathbb{Z}/(N))|, \tag{7.6.4}$$

(where  $\zeta$  is Riemann’s zeta function) for which we use

$$|\mathbf{Sp}_4(\mathbb{Z}/(N))| = N^{10} \prod_{p|N} \left(1 - \frac{1}{p^2}\right) \left(1 - \frac{1}{p^4}\right)$$

and  $\zeta(-1)\zeta(-3) = -1/2^5 \cdot 3^2 \cdot 5$ . Note that  $\chi(\Gamma(N))$  is an integer for  $N \geq 3$ . Theorem 7.5 now follows from (7.4.1), (7.6.1), (7.6.3) and (7.6.4).  $\square$

### 7.7. STALK COHOMOLOGY

Now consider the stalk cohomology of the intersection cohomology at a 0-dimensional boundary stratum  $x \in \hat{X}$ . By Corollary 6.6, this is a sum of two weighted cohomology groups of a boundary stratum of type B,

$$IH_x^i \cong WH^i(\overline{\Gamma_B \backslash \mathfrak{h}}; H^0(\mathfrak{A}_P)) \oplus WH^{i-1}(\overline{\Gamma_B \backslash \mathfrak{h}}; H^1(\mathfrak{A}_P)). \quad (7.7.1)$$

A calculation with the root system for  $\mathbf{Sp}_4$  shows that the weight profile for the first factor (with coefficients in the trivial local system  $H^0(\mathfrak{A}_P)$ ) involves no cutoff at all, while the weight profile for the second factor cuts off all the stalk cohomology in degree 1 at each cusp point, so

$$IH_x^i \cong H^i(\Gamma_B \backslash \mathfrak{h}; \mathbb{C}) \oplus IH^{i-1}(\overline{\Gamma_B \backslash \mathfrak{h}}; H^1(\mathfrak{A}_P)). \quad (7.7.2)$$

Since  $\Gamma_B \backslash \mathfrak{h}$  is not compact,  $H^2(\Gamma_B \backslash \mathfrak{h}) = 0$  and

$$\text{rank}(H^1(\Gamma_B \backslash \mathfrak{h})) = 1 - \chi(\Gamma_B). \quad (7.5.3)$$

We will use two tricks to evaluate the second factor in (7.7.2). First, the local system  $H^1(\mathfrak{A}_P)$  has weight  $\chi^{-2}$ , where  $\chi$  is the canonical positive generator of the character group of  $A_P$ . In fact, it is the irreducible three-dimensional representation of  $\mathbf{SL}_2$ . By Looijenga's purity theorem, weight 2 classes cannot occur in the stalk of the intersection cohomology except in degree 2 or more. Therefore  $IH^0(\overline{\Gamma_B \backslash \mathfrak{h}}; H^1(\mathfrak{A}_P)) = 0$ . Also,  $IH^2(\overline{\Gamma_B \backslash \mathfrak{h}}; H^1(\mathfrak{A}_P)) = 0$  since  $IH_x^3 = 0$  by the usual vanishing property for intersection cohomology. Therefore the second factor in (7.7.2) is completely determined by the intersection cohomology Euler characteristic of this (compactified) modular curve, that is,

$$\text{rank}(IH^1(\overline{\Gamma_B \backslash \mathfrak{h}}, H^1(\mathfrak{A}_P))) = -I\chi(\overline{\Gamma_B \backslash \mathfrak{h}}, H^1(\mathfrak{A}_P)). \quad (7.7.4)$$

This Euler characteristic is the sum over strata of the Euler characteristic of the stratum times the Euler characteristic of the stalk of the intersection cohomology sheaf  $\mathbf{IH}^\bullet(\overline{\Gamma_B \backslash \mathfrak{h}}, H^1(\mathfrak{A}_P))$  at a point in the stratum. The stalk of this sheaf at each cusp point is  $\mathbb{C}$  in degree 0, and 0 in all other degrees. So each cusp contributes 1 to the Euler characteristic, and there are  $v_\infty(N)$  cusps. The interior contributes  $\dim(H^1(\mathfrak{A}_P)) \cdot \chi(\Gamma_B) = 3\chi(\Gamma_B)$ . This gives the following theorem:

**7.8. THEOREM.** *The Betti numbers of the stalk intersection cohomology at a 0-dimensional stratum  $x \in \hat{X}$  are given by*

$$\text{rank}(IH_x^0) = 1,$$

$$\text{rank}(IH_x^1) = 1 + \frac{N^3}{12} \prod_{p|N} \left(1 - \frac{1}{p^2}\right),$$

$$\text{rank}(IH_x^2) = \left[\frac{N^3}{4} - \frac{N^2}{2}\right] \prod_{p|N} \left(1 - \frac{1}{p^2}\right).$$

**7.9.** For  $N = 3$  we obtain the local intersection cohomology Betti numbers  $I\beta^0 = 1$ ,  $I\beta^1 = 3$ ,  $I\beta^2 = 2$ . These numbers agree with the intensive computations which were carried out by M. McConnell on the Symbolics computer at Brown University in 1986.

7.10. DISCRETE SERIES MULTIPLICITIES

The computation of the intersection Euler characteristic (7.5.1) gives some information about multiplicities. The group  $\mathbf{Sp}_4(\mathbb{R})$  has two discrete series representations with nonzero  $(\mathfrak{g}, K)$ -cohomology (see [T] for a list of all representations with cohomology and for other facts used below). Let us denote these  $\pi^H$  (belonging to the holomorphic discrete series) and  $\pi^W$  (having a Whittaker model). Via the isomorphisms  $IH^i(\hat{X}) \cong H_{(2)}^i(X) \cong H^i(\mathfrak{g}, K; L_{\text{dis}}^2(\Gamma(N)\backslash G))$  both of these contribute to  $IH^3$  (in bidegrees  $(3, 0)$ ,  $(0, 3)$  and  $(2, 1)$ ,  $(1, 2)$  respectively). Here  $L_{\text{dis}}^2$  stands for the discrete spectrum of the  $L^2$  space. The other representations of  $\mathbf{Sp}_4(\mathbb{R})$  with cohomology contribute in degrees 0, 2, 4, 6. Using the decomposition of the discrete spectrum we write

$$I\chi(\hat{X}) = \sum_{\pi} m_{\text{dis}}(\pi, N) \sum_i (-1)^i \dim H^i(\mathfrak{g}, K; \pi) = \sum_{\pi} m_{\text{dis}}(\pi, N) \chi(\pi),$$

where  $m_{\text{dis}}(\pi, N)$  is the multiplicity of  $\pi$  in the discrete spectrum of  $\Gamma(N)$  and  $\chi(\pi)$  is the  $(\mathfrak{g}, K)$ -Euler characteristic. If  $\pi$  is not either  $\pi^H$  or  $\pi^W$  then  $\chi(\pi)$  is positive while  $\chi(\pi^H) = \chi(\pi^W) = -2$ . It follows that  $2(m_{\text{dis}}(\pi^H, N) + m_{\text{dis}}(\pi^W, N))$  is at least as large as the negative of (7.5.1). Since  $\pi^H$  and  $\pi^W$  are tempered, an observation of Wallach [W] implies that  $m_{\text{dis}}(\pi^H, N) = m_{\text{cusp}}(\pi^H, N)$  (multiplicity in the cuspidal spectrum) and similarly for  $\pi^W$ . So  $m_{\text{cusp}}(\pi^H, N) + m_{\text{cusp}}(\pi^W, N)$  is at least  $-\lfloor \frac{1}{2} \rfloor I\chi(\hat{X})$  which is positive for  $N \geq 5$ . Hence, for  $N \geq 5$ , there are nonzero cusp forms with infinity type either  $\pi^W$  or  $\pi^H$ . This provides examples of Siegel modular forms to which the constructions of Taylor [T], Laumon [La], Harder (unpublished) and Weissauer (unpublished) can be applied to produce Galois representations.

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