The Diaconis Mind Reader

Mark Goresky, May 2008

http://www.math.ias.edu/~goresky/preprints.html

00 = ♣, 01 = ♡, 10 = ♠, 11 = ♢.

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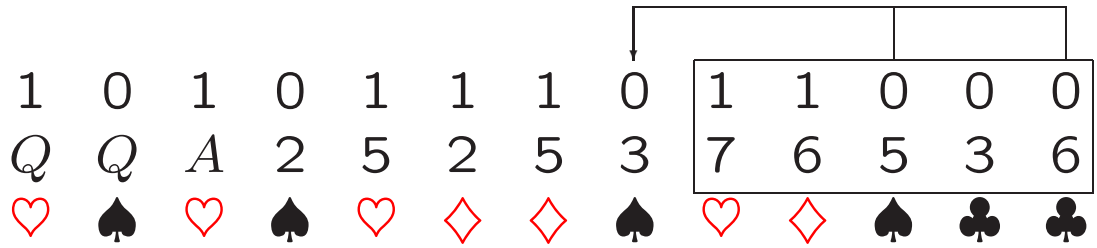
1	0	1	0	1	1	1	0	1	1	0	0	0
Q	Q	A	2	5	2	5	3	7	6	5	3	6
♡	♠	♡	♠	♡	♢	♢	♠	♡	♢	♠	♣	♣

$00 = \clubsuit,\ 01 = \heartsuit,\ 10 = \spadesuit,\ 11 = \diamondsuit.$



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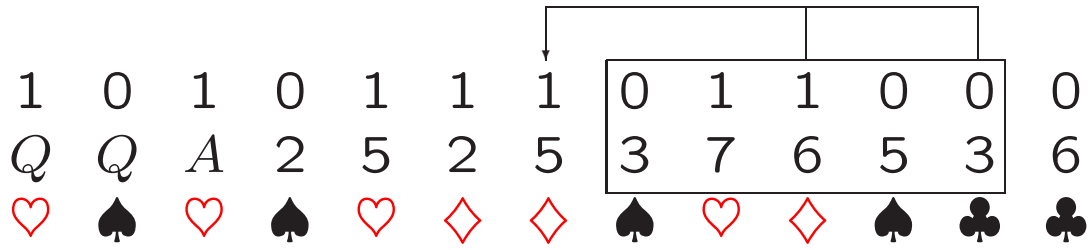

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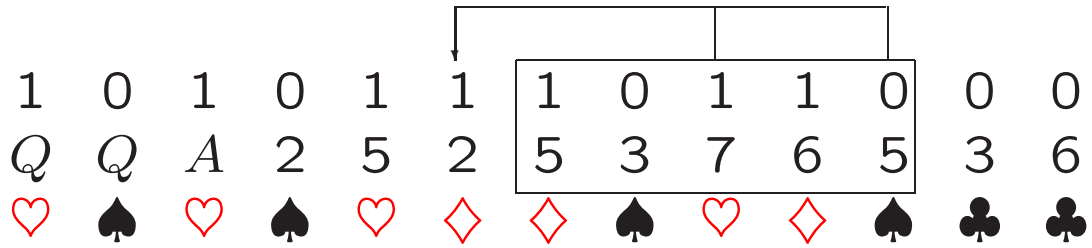

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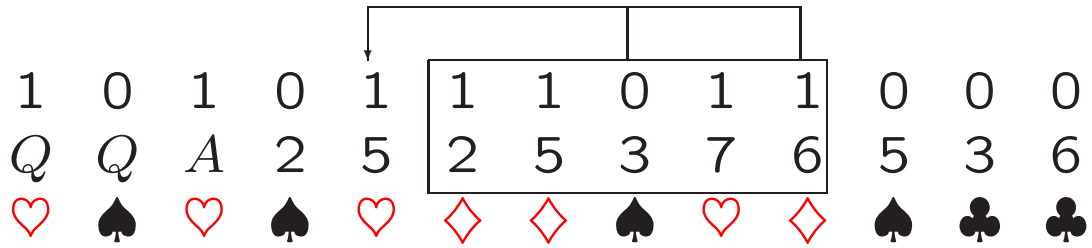

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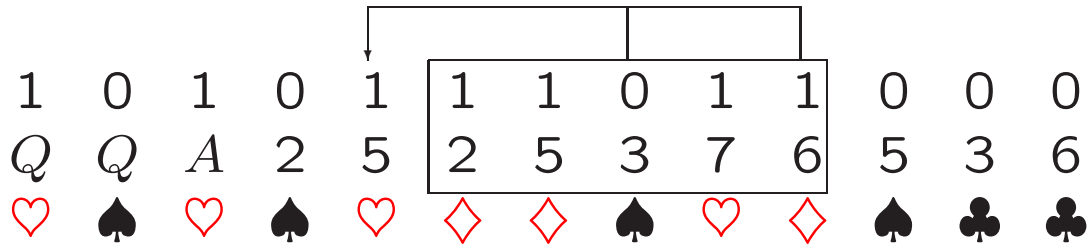
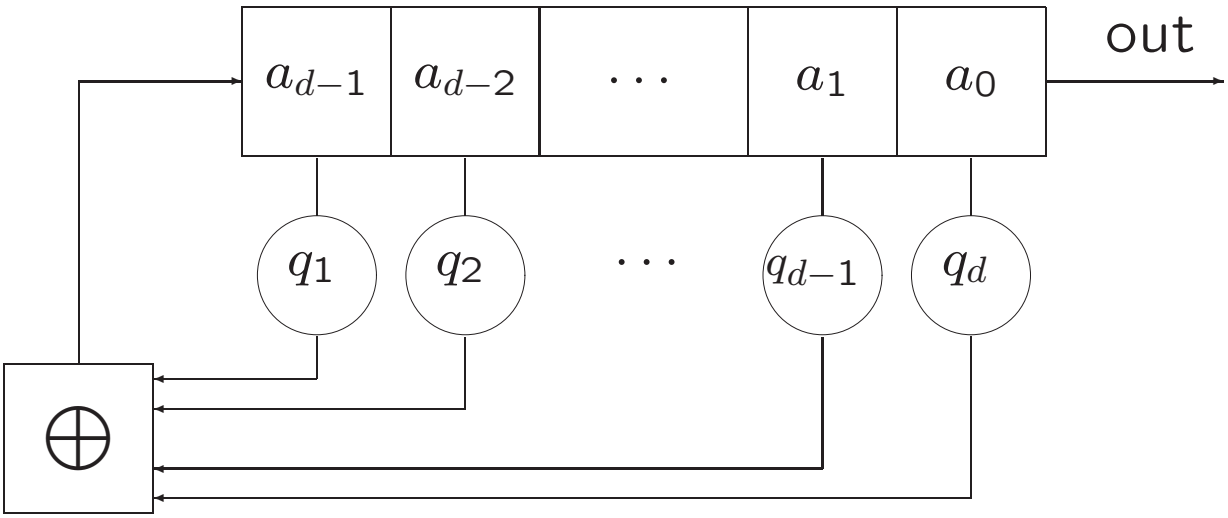
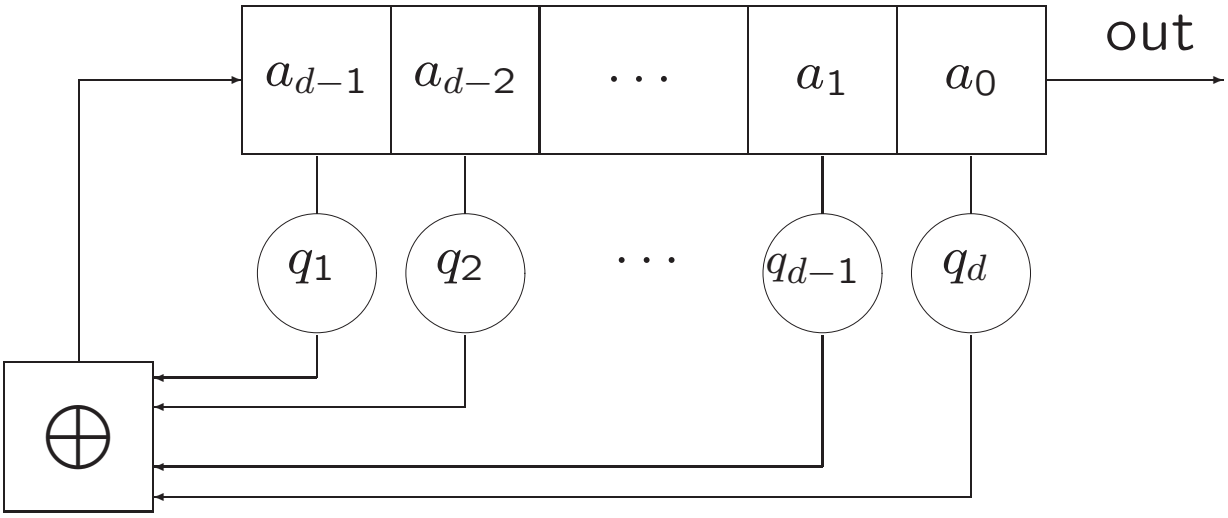
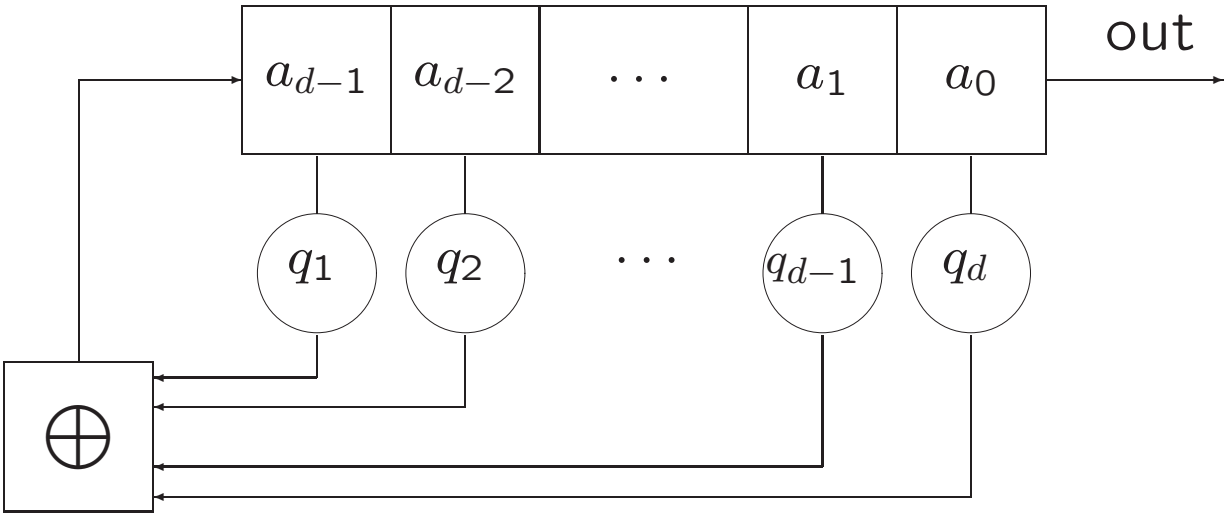






Connection polynomial:

$$q(x) = -1 + q_1 x + q_2 x^2 + \cdots + q_d x^d$$



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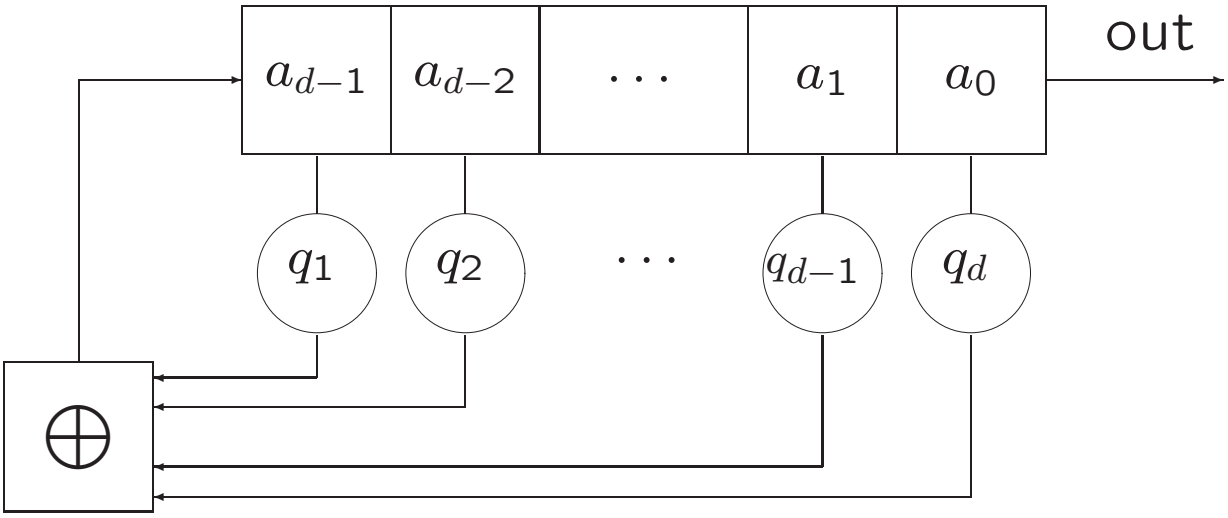
"Initial" polynomial:

$$h(x) = \sum_{n=0}^{d-1} \left(\sum_{i=0}^{n} q_i a_{n-i} \right) x^n$$

$$\left\{ \begin{array}{c} \text{initial} \\ \text{loadings} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{polynomials of} \\ \text{degree} < d \end{array} \right\}$$



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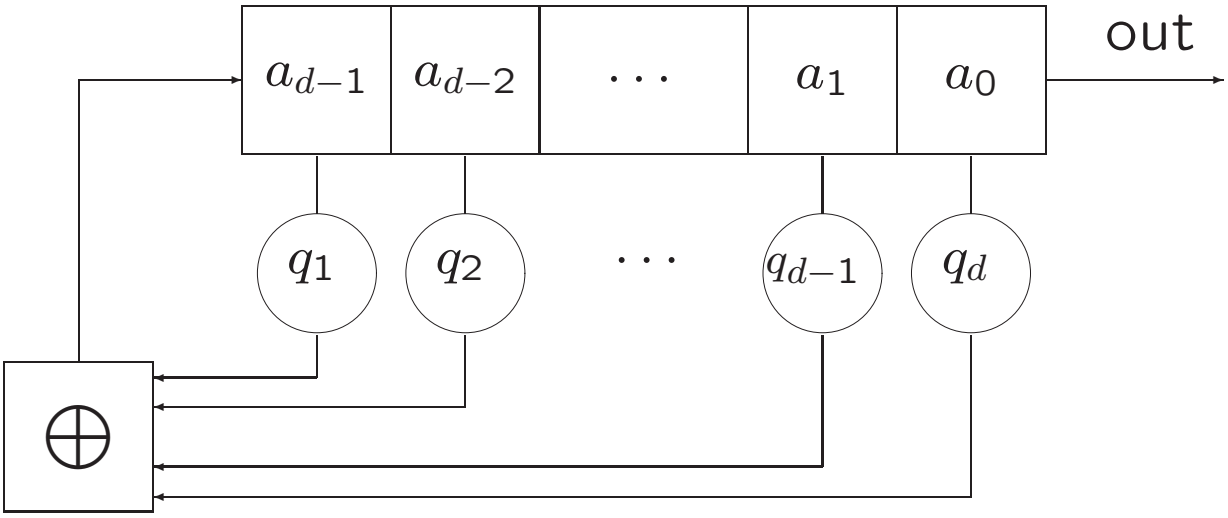
Generating function of output sequence:

$$a(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$



Basic Fact

$$a(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \frac{h(x)}{q(x)}$$

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2. The bit-wise sum of two periodic sequences $a_1(x)$ and $a_2(x)$ corresponds to

$$\frac{h_1(x)}{q_1(x)} + \frac{h_2(x)}{q_2(x)} = \frac{k(x)}{LCM(q_1, q_2)(x)}.$$

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3. Let $\alpha \in \mathbb{F}_{2^d}$ be a root of $q(x)$. Let

$$T : \mathbb{F}_{2^m} \to \mathbb{F}_2$$

be a linear surjective map. Then:

$$a_n = T(A\alpha^{-n})$$

for all $n \geq 0$. ($A \leftrightarrow$ initial loading)

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5. The Massey-Berlekamp algorithm constructs the shift register for a given sequence

$$a(x) = a_0 + a_1 x + a_2 x^2 + \cdots .$$

Continued fraction expansion in $\mathbb{F}_2((x))$.

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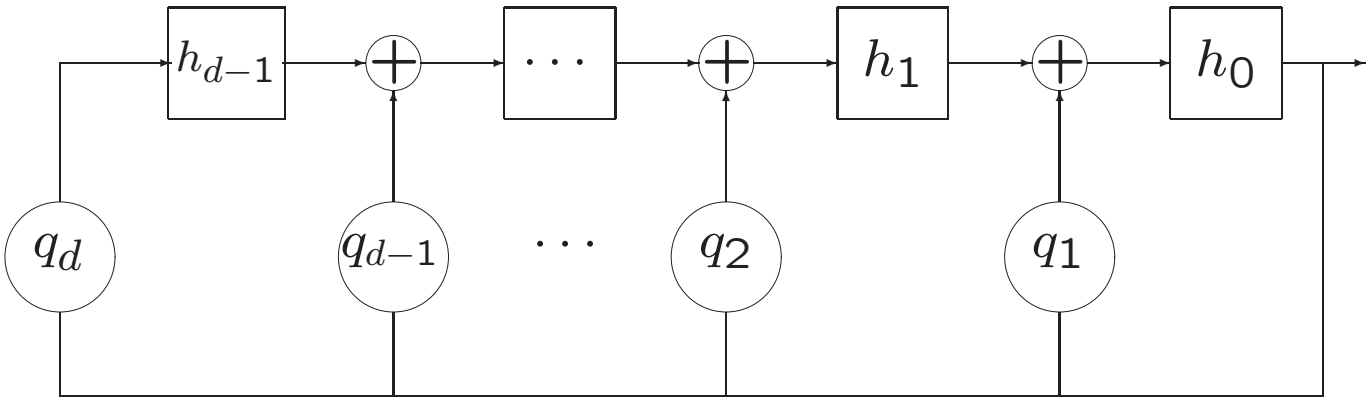
7. Galois mode shift registers



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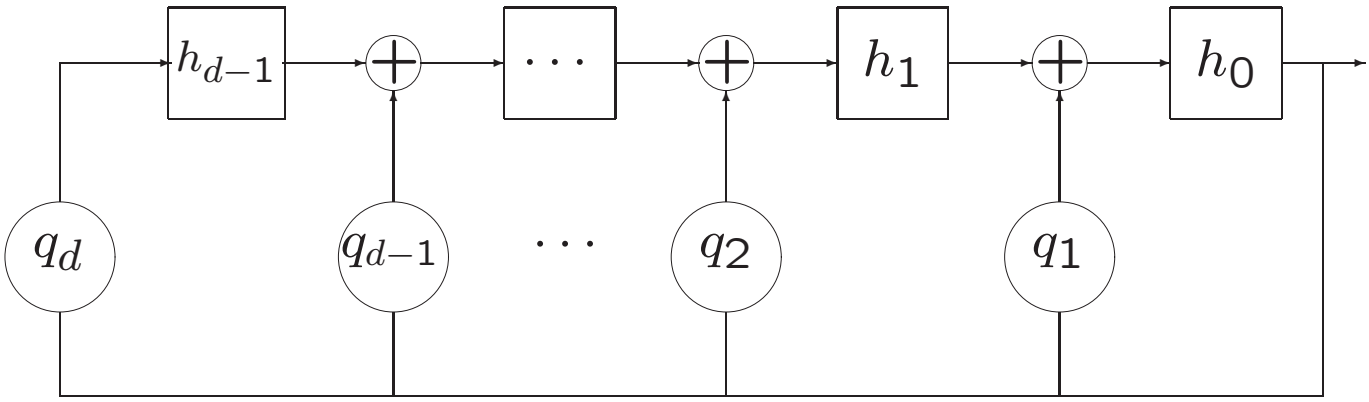
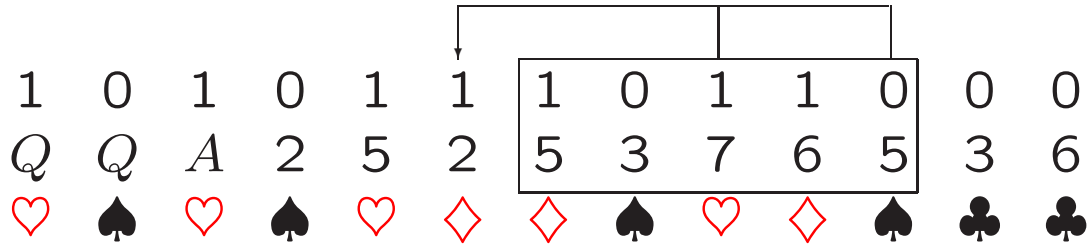


$$h(x) = h_0 + h_1 x + \cdots + h_{d-1} x^{d-1}$$
$$q(x) = -1 + q_1 x + \cdots + q_d x^d$$
$$a(x) = h(x)/q(x)$$

In summary, let

$$q(x) = -1 + x^3 + x^5$$

This is primitive. Use it for feedback taps.



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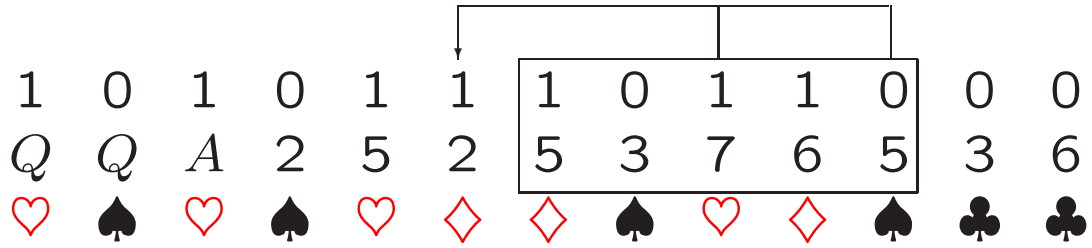

This generates an LFSR sequence of period 31

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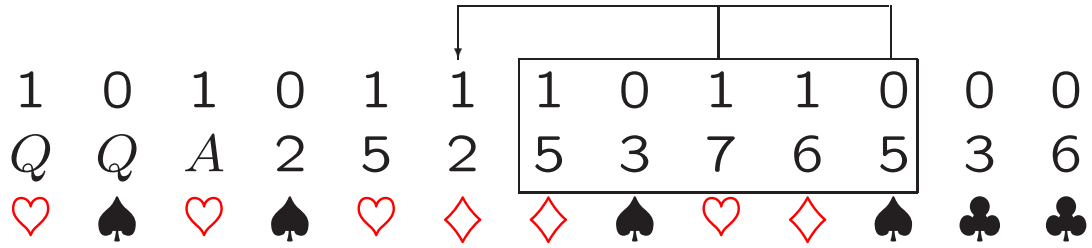

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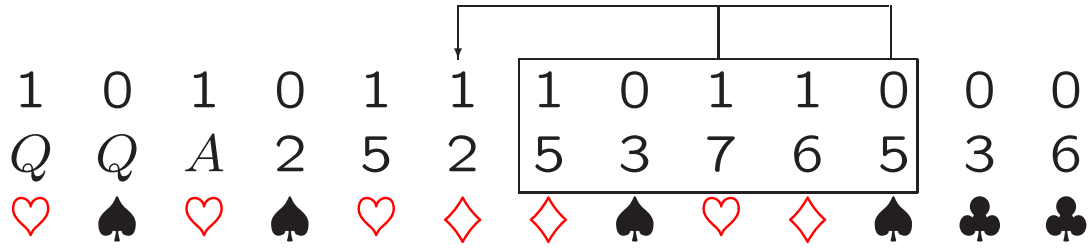

This generates an LFSR sequence of period 31

Good for magic

Bad for cryptography

Summation combiner = sum-with-carry

$$
\begin{array}{cccccccccc}
1 & 1^{1} & 0^{1} & 0^{1} & 0 & 1 & 1^{1} & 0^{1} & 1^{1} & \cdots \\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & \cdots \\
\hline
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & \cdots
\end{array}
$$

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$$a = a_0 2^0 + a_1 2^1 + a_2 2^2 + \cdots \in \mathbb{Z}_2$$

Summation combiner = sum-with-carry

$$
\begin{array}{l}
2^0+2^1+0\ +0\ +0\ +2^5+2^6+0\ +2^8+\cdots \\
2^0+0\ +2^2+0\ +2^4+2^5+0\ +2^7+2^8+\cdots \\
\hline
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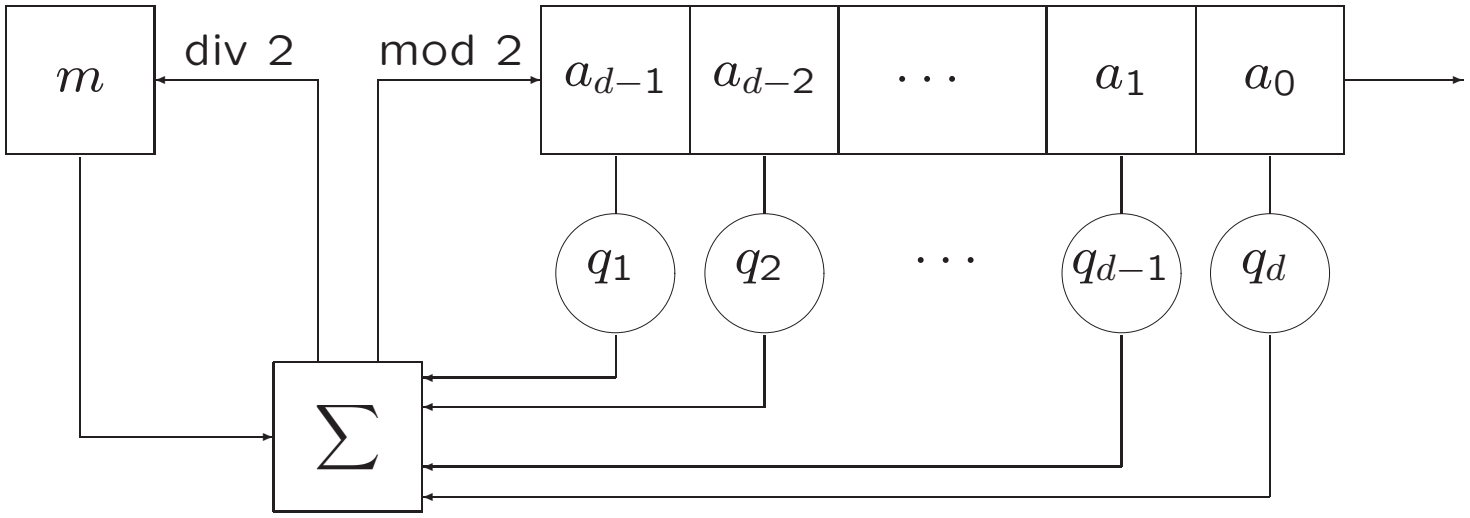
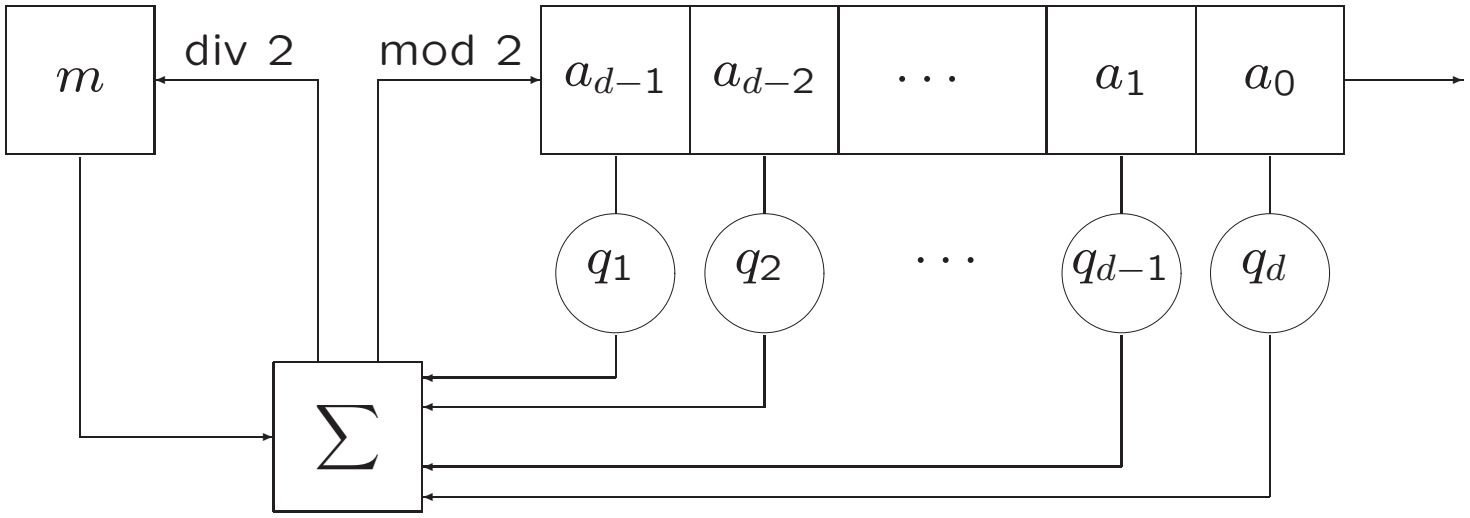
From here on:

Joint work with Andy Klapper (U. Kentucky)



Feedback with carry shift register



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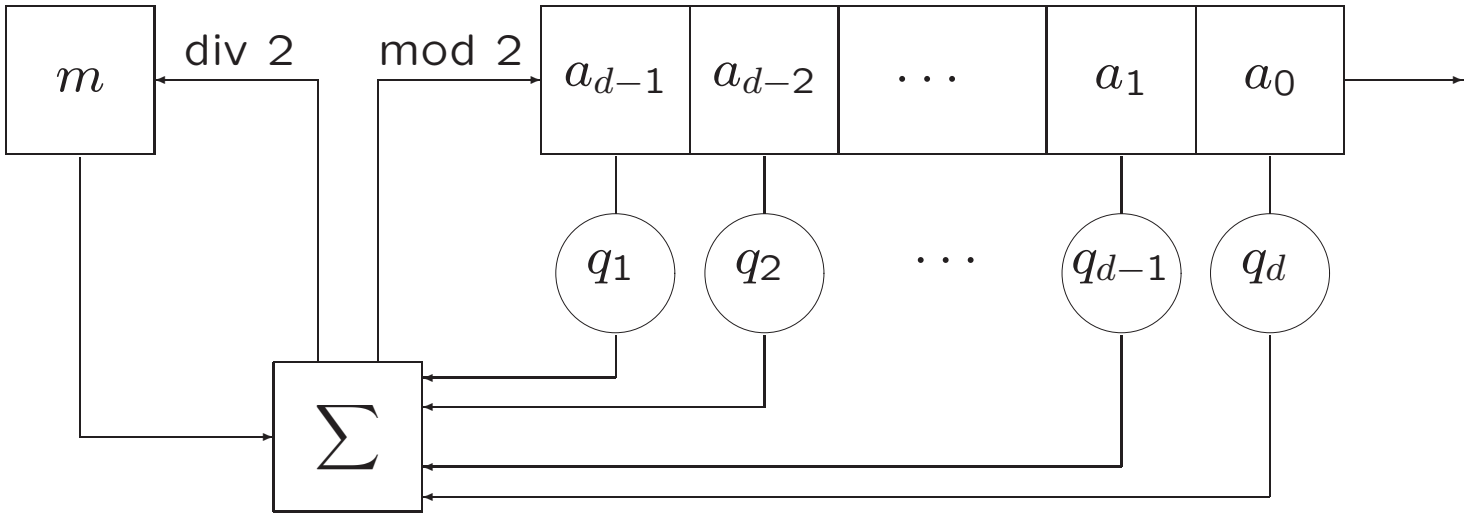

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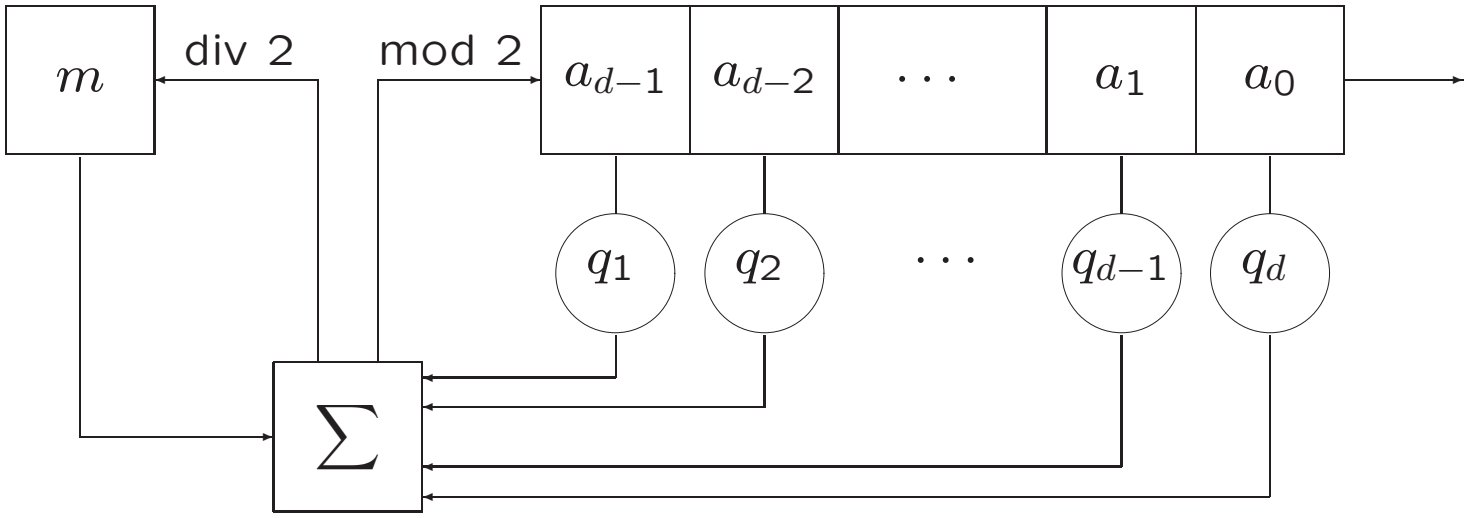

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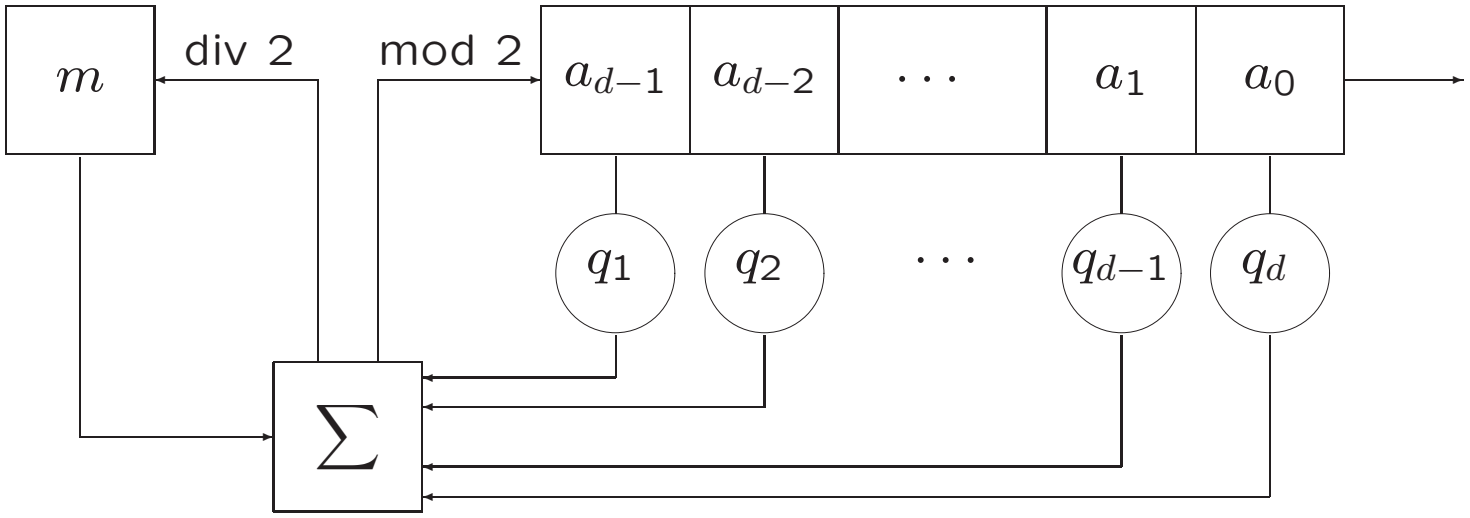
Generating "series":

$$a = a_0 + a_1 2 + a_2 2^2 + \cdots$$



Feedback with carry shift register

Main fact:

$$\mathbf{a} = a_0 + a_1 2 + a_2 2^2 + \cdots = \frac{h}{q}$$

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2. The sum-with-carry of two sequences a_1 and a_2 corresponds to

$$\frac{h_1}{q_1} + \frac{h_2}{q_2} = \frac{k}{LCM(q_1, q_2)}$$

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3. Let $(\mathrm{mod}\, 2) : \mathbb{Z}/(q) \to \mathbb{Z}/(2)$. Then

$$a_n = (A2^{-n}) \ (\mathrm{mod}\, q) \ (\mathrm{mod}\, 2).$$

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4. Maximal period ($= q - 1$) occurs when 2 is a primitive root modulo q.

5. Continued fraction expansion in $\mathbb{Z}_2$ does not converge, but (Mahler-deWeger) approximation lattices may be used to construct the FCSR for a given sequence.

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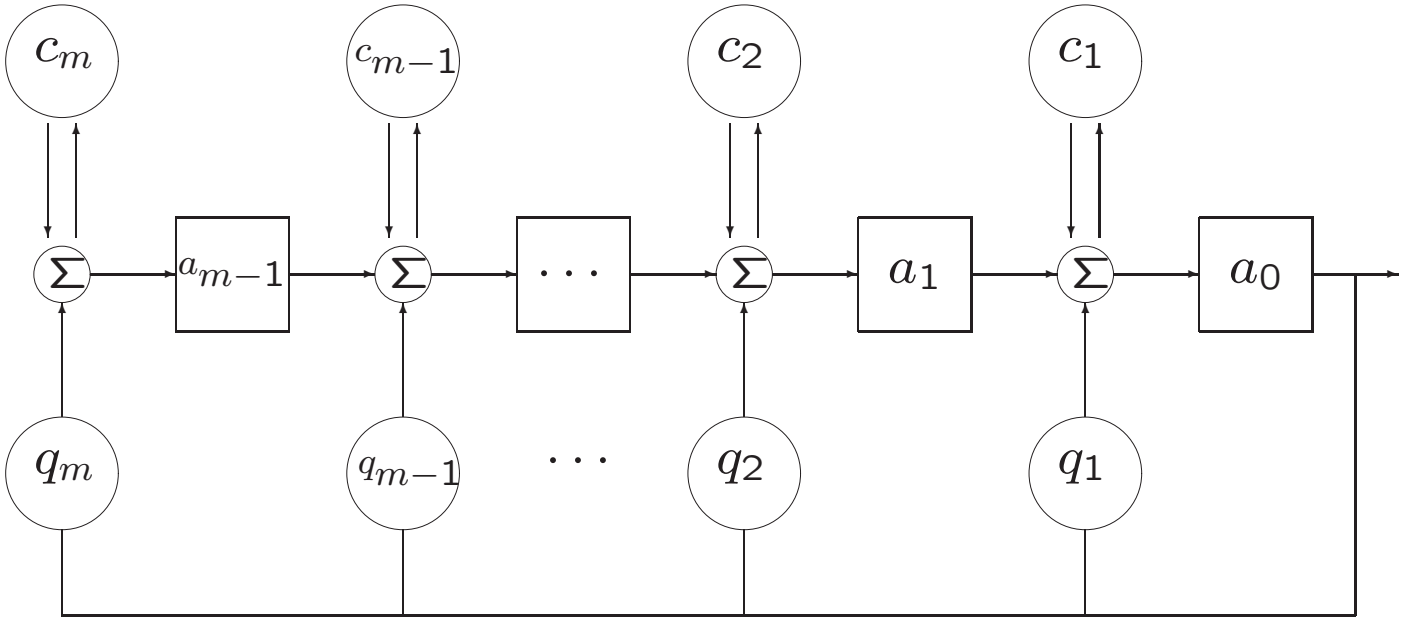
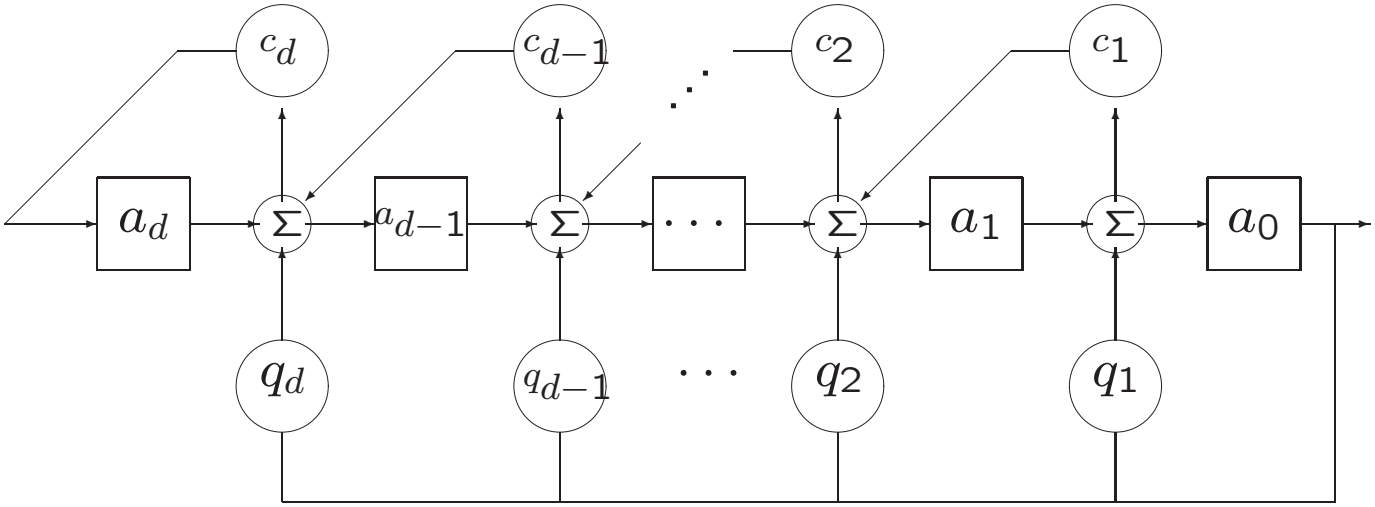
7. Galois mode FCSR



Many further variations, e.g. ramified extension of 2-adic numbers:



Amusing fact: a U.S. deck has 52 cards.
53 is prime and 2 is primitive mod 53.

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So it is possible to carry out the Diaconis mind-dreader, using a full deck of 52 cards, with the FCSR sequence for

$$q = 53 = -1 + 2 + 4 + 16 + 32.$$