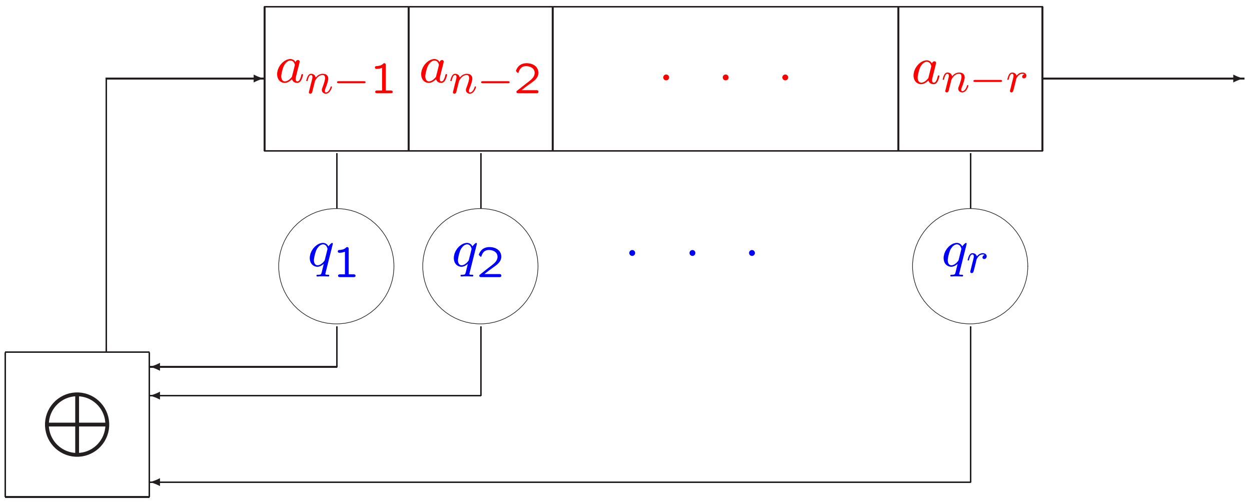
Shift Register Sequences

Mark Goresky



Linear recursion:

$$a_n = q_1 a_{n-1} + q_2 a_{n-2} + \cdots + q_r a_{n-r}$$

or

$$-a_n + q_1 a_{n-1} + q_2 a_{n-2} + \cdots + q_r a_{n-r} = 0$$

Connection Polynomial:

$$q(x) = -1 + q_1 x + q_2 x^2 + \cdots + q_r x^r$$

Theorem. (Golomb, 1955)
Also Dickson, 1903

The output sequence is

$$a_0 + a_1 x + a_2 x^2 + \cdots = \frac{p(x)}{q(x)} \in \mathbb{F}_2[[x]]$$

where

$$q(x) = -1 + q_1 x + q_2 x^2 + \cdots + q_r x^r$$

depends on the taps and

$$p(x) = \sum_{n=0}^{r-1} \left(\sum_{i=0}^{n} q_i a_{n-i} \right) x^n$$

depends on the initial loading.

Applications

• smallest shift register to generate a given sequence: Berlekamp-Massey algorithm $=$ continued fraction expansion in $\mathbb{F}_2[[x]]$.

• sum of several LFSR sequences

• Same analysis works for LFSR over any commutative ring

Fibonacci numbers:

$$1 + 1x + 2x^2 + 3x^3 + 5x^4 + \cdots = \frac{-1}{-1 + x + x^2}$$

Choose a root $\alpha \in \mathbb{F}_{2^n}$, $\quad q(\alpha^{-1}) = 0$

Choose a linear function $T : \mathbb{F}_{2^n} \to \mathbb{F}_2$

The sequence of values

$T(\alpha^{r-1})$	$T(\alpha^{r-2})$	$\cdot\ \cdot\ \cdot$	$T(\alpha)$	$T(1)$

satisfies the linear recursion

Proof.

$$0 = q_0\alpha^0 + q_1\alpha^{-1} + \cdots + q_r\alpha^{-r}$$

$$0 = q_0\alpha^r + q_1\alpha^{r-1} + \cdots + q_r\alpha^0$$

$$T(\alpha^r) = q_1 T(\alpha^{r-1}) + \cdots + q_r T(1)$$

Similarly for any $A \in \mathbb{F}_{2^n}$ the sequence $T(A\alpha^i)$ satisfies the linear recursion.

Example: if $q(x)$ is irreducible of degree r,

$$\mathbb{F}_{2^r} \cong (\mathbb{F}_2)^r$$

$$A \mapsto \Big(T(A\alpha^{r-1}), \cdots, T(A\alpha), T(A)\Big)$$

Recall $\alpha \in \mathbb{F}_{2^r}$ is *primitive* if

$$\mathbb{F}_{2^r} = \left\{0, 1, \alpha, \alpha^2, \cdots, \alpha^{r-2}\right\}$$

Corollary. If $\alpha \in \mathbb{F}_{2^r}$ is a primitive element then the shift register will cycle through all possible nonzero states before it repeats. So $q(x)$ should be a primitive polynomial.

The result is an *m-sequence*
e.g. $q(x) = -1 + x^2 + x^3$

$$\mathbf{a} = 0010111\,0010111 \cdots$$

Spread spectrum communication

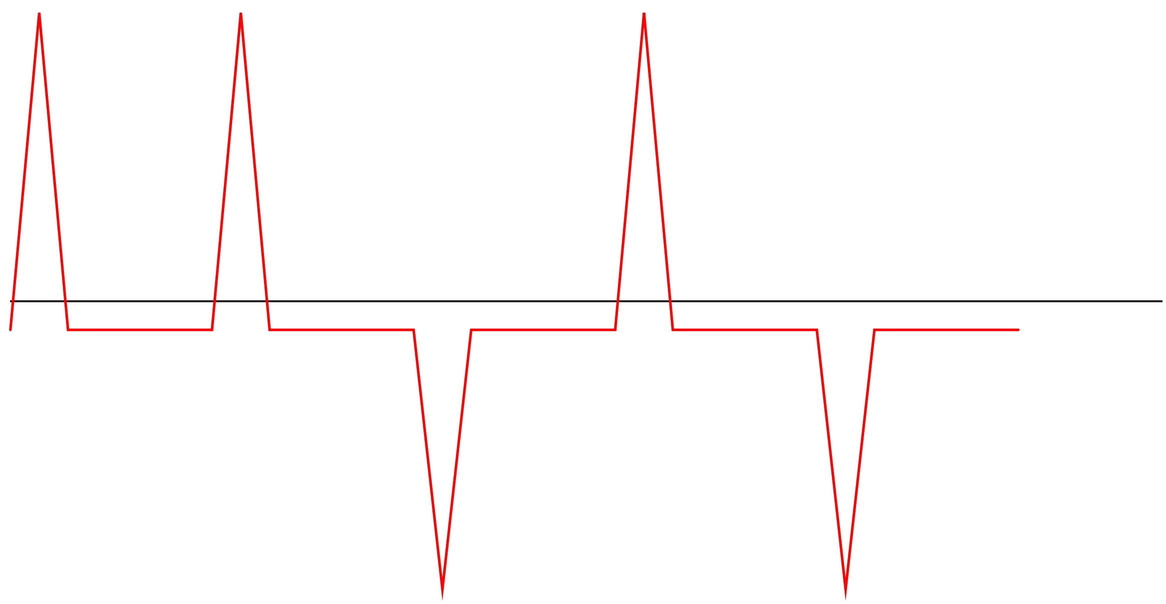
Signature sequence: 0010111
Send 1: 0010111 Send 0: 1101000

00101110010111110100000101111101000
0010111

The correlator counts

#(agreements) - #(disagreements)

in each window



Given periodic sequences of period T
$\mathbf{a} = (a_0, a_1, \cdots)$ and $\mathbf{b} = (b_0, b_1, \cdots)$
Their *cross correlation* is

$$C_{\mathbf{a},\mathbf{b}}(\tau) = \sum_{i=0}^{r-1} (-1)^{a_i}(-1)^{b_{i+\tau}}$$

Autocorrelation $A_{\mathbf{a}}(\tau) = C_{\mathbf{a},\mathbf{a}}(\tau)$

Idea: Each user has a different signature sequence. These sequences have low cross-correlations. The receiver (correlator) knows your sequence. Received signal will look like this:

Theorem. If $\mathbf{a} = (\mathbf{a_0}, \mathbf{a_1}, \cdots)$ is an m-sequence of period $P = 2^r - 1$ then

$$A_{\mathbf{a}}(\tau) = \begin{cases} -1 & \tau \neq 0 \\ P & \tau = 0 \end{cases}$$

Proof.

$$\begin{aligned}
\sum_{i=0}^{P-1} \ (-1)^{T(\alpha^i)} \ (-1)^{T(\alpha^{i+\tau})} \\
&= \sum_{i=0}^{P-1} (-1)^{T(\alpha^i + \alpha^{i+\tau})} \\
&= \sum_{i=0}^{P-1} (-1)^{T(\alpha^i (1 + \alpha^\tau)} \\
&= \sum_{x \in \mathbf{F}_{2^r}} (-1)^{T(Ax)} - 1 \\
&= 0 - 1
\end{aligned}$$

where $A = (1 + \alpha^\tau)$.

Problem. Compute the cross correlation of any two distinct m-sequences:

$$C_{\mathbf{a},\mathbf{b}} = \sum_{i=0}^{2^r-2} (-1)^{T(\alpha^i)}(-1)^{T(\beta^{i+\tau})}$$

where $\alpha, \beta \in \mathbb{F}_{2^r}$ are primitive. Say $\beta = \alpha^k$.

$$= \sum_{i=0}^{2^r-2} (-1)^{T(\alpha^i + \alpha^{ki}\alpha^\tau)}$$

$$= \sum_{x \in \mathbb{F}_{2^r}^*} (-1)^{T(x+Ax^k)}$$

where $A = \alpha^\tau$.

Trigonometric sum (Gauss, Hardy, Littlewood, + about 1000 other number theorists)

Lachaud and Wolfman: For $T : \mathbb{F}_{2^r} \to \mathbb{F}_2$ use the trace

$$Tr(x) = x + x^2 + x^4 + \cdots + x^{2^{r-1}}$$

Lemma. $Tr(x) = 0$ if and only if

$$x = h(y) = y^2 - y$$

for some $y \in \mathbb{F}_{2^r}$.

Proof. If $h(y) = y(y-1) = 0$ then $y \in \mathbb{F}_2$.
And

$$Tr(h(y)) = -y + y^2 - y^2 + y^4 - \cdots + y^{2^r} = 0.$$

So this sequence is exact:

$$0 \longrightarrow \mathbb{F}_2 \longrightarrow \mathbb{F}_{2^r} \xrightarrow{h} \mathbb{F}_{2^r} \xrightarrow{Tr} \mathbb{F}_2 \longrightarrow 0$$

Now consider the equation

$$y^2 - y = Ax + x^k.$$

Let E denote the set of solutions (x, y)
Whenever $Tr(Ax + x^k) = 0$ we get two points in E. So

$$\#E/2 = \#\left\{x | Tr(Ax + x^k) = 0\right\}$$

So

$$\begin{aligned} A(\tau) &= \#E/2 - (2^r - \#E/2) \\ &= \#E - 2^r \ (\pm 1) \end{aligned}$$

Case $k = -1$ (the reverse sequence):

$$y^2 - y = Ax + x^{-1}$$

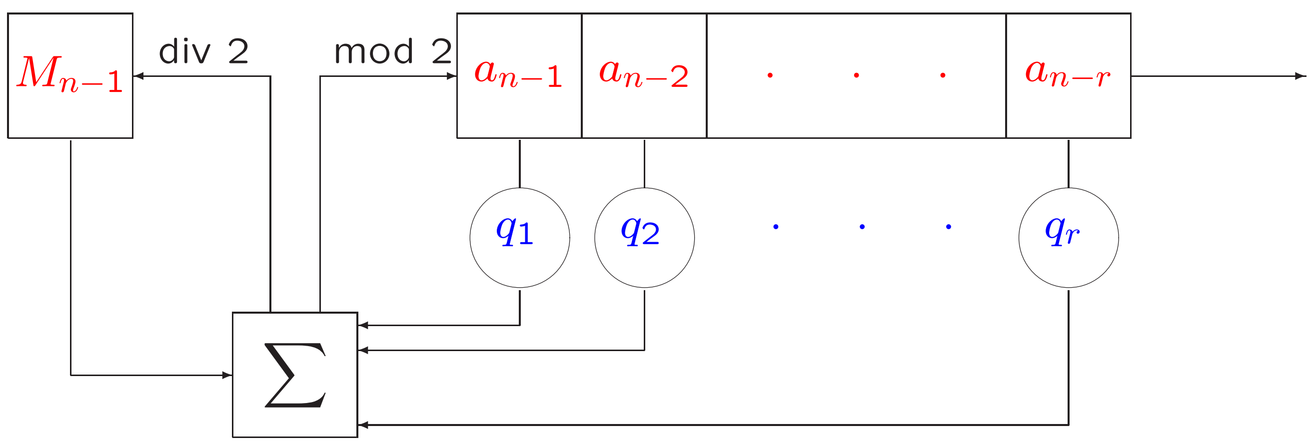
This is an elliptic curve:

$$xy^2 - xy = Ax^2 + 1$$

Problem was completely solved by Honda-Tate.

Feedback with carry

joint with Andrew Klapper



Linear recursion with carry:

$$2M_n + a_n = M_{n-1} + \sum_{i=1}^{r} q_i a_{n-i}$$

Connection integer

$$q = -1 + q_1 2 + q_2 2^2 + \cdots + q_r 2^r$$

Theorem. The output sequence is

$$a_0 2^0 + a_1 2^1 + a_2 2^2 + \cdots = \frac{p}{q} \in \mathbb{Z}_2$$

where

$$p = \sum_{n=0}^{r-1} \sum_{i=0}^{n} q_i a_{n-i} 2^n - M_n 2^n$$

Applications

• smallest FCSR to generate a given sequence continued fraction expansion in $\mathbb{Z}_2$ may fail to converge! Instead, use approximation theory in 2-adic numbers (Mahler, de Weger).

• sum-with-carry of several FCSR sequences (we broke the summation combiner cipher)

• arithmetic cross-correlation analysis.

• Similar analysis for FCSR sequences over other p-adic fields.

• Other related architectures.

Exponential representation:

$$a_i = A2^{-i} \pmod{q} \pmod{2}$$

$$\mathbb{Z}/(q) \xrightarrow{\pmod{2}} \mathbb{Z}/(2)$$

compare LFSR case:

$$a_i = T(A\beta^{-i})$$

or, better: $a_i = Ax^{-i} \pmod{q} \pmod{x}$.

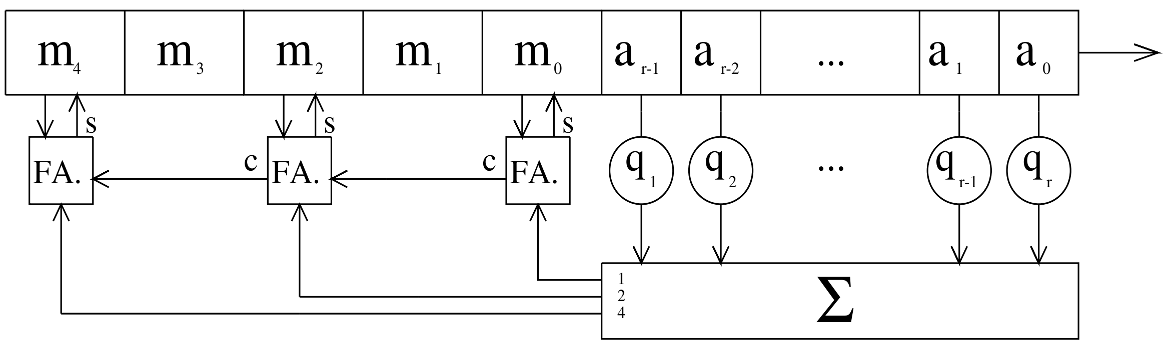
$$\mathbb{F}_2[x]/(q) \xrightarrow{\pmod{x}} \mathbb{F}_2$$

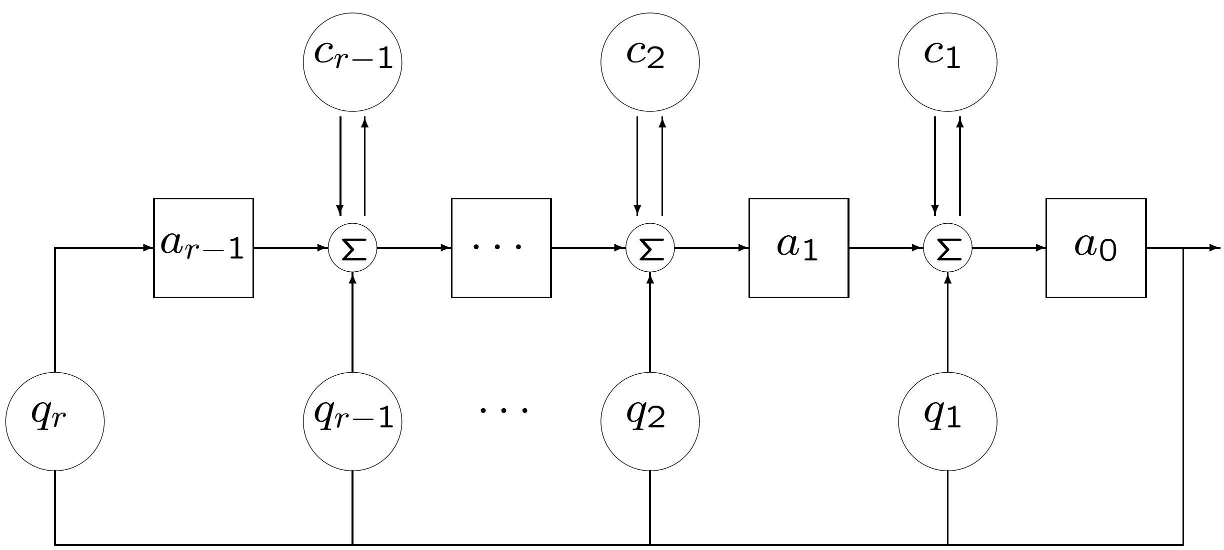
Maximal length sequences:

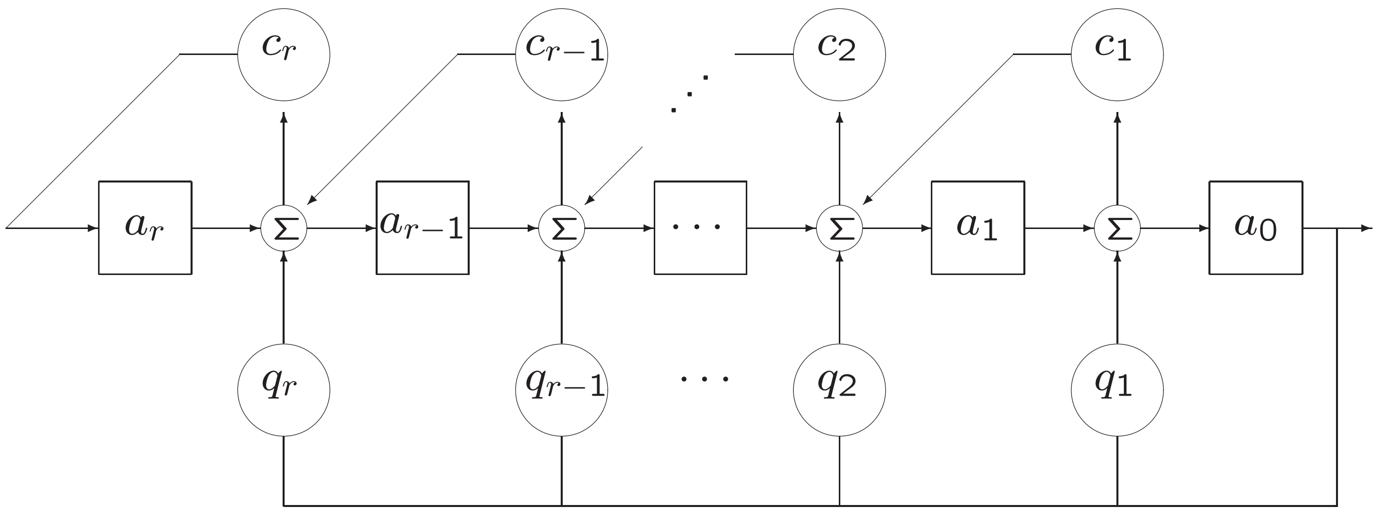
Theorem. If 2 is primitive modulo q the FCSR passes through all (nonzero) periodic states ($q-1$ of them) before it repeats.

Algebraic shift registers

A setting that includes both LFSR and FCSR and others, e.g. the 2-FCSR involves ramified extensions of the 2-adic numbers:







Galois 2-FCSR.