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Combinatorial Geometries, Convex Polyhedra, and Schubert Cells

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INTRODUCTION

This paper is a continuation of [GM] which was published in the same journal. We will explore a remarkable connection between the geometry of the Schubert cells in the Grassmann manifold, the theory of convex polyhedra, and the theory of combinatorial geometries in the sense of Crapo and Rota [CR]. The results in this paper were obtained simultaneously and independently by Gelfand and Serganova (in Moscow) and by Goresky and MacPherson (at the I.H.E.S. in Paris) as part of larger programs with different purposes (see below). The geometry of this simple example is so beautiful that we decided to publish it independently of the applications. We believe that combinatorial methods will play an increasing role in the future of geometry and topology.

We consider the Grassmann manifold G^k_{n-k} of all $(n-k)$-dimensional subspaces of $\mathbb{C}^n$. By fixing the standard basis in $\mathbb{C}^n$ we obtain an action of the torus $H=(\mathbb{C}^*)^n$ on G^k_{n-k} which is induced from stretching the coordinate axes in $\mathbb{C}^n$ (see also Sect. 1). We will describe not only the trajectories, but also the "strata" of a new and interesting decomposition of the

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Grassmanian (which is finer than the usual stratification by isotropy subgroup of H). Understanding the geometry of the strata and the quotient space of this action is useful in many situations, and this paper may be considered as an introduction to these other situations: (1) for understanding the generalized hypergeometric functions and the Kostant partition function [G, GG, GZ], (2) for understanding the dilogarithm and the polylogarithms and their functional equations [GM, HM], (3) for the study of combinatorial geometries which are associated to other Lie groups and parabolic subgroups [GS], (4) for construction of combinatorial Chern and Pontrjagin classes [GGL, M], (5) for the study of he representability of matroids [GoM], and (6) for the study of algebraic K-theory [BMS].

According to [GM] the trajectories of the action of $(\mathbb{C}^*)^n$ on the Grassmannian G^k_{n-k} correspond to projective configurations of n points in $\mathbb{P}^{k-1}(\mathbb{C})$. This torus action also gives rise to a moment map $\mu: G^k_{n-k} \to \mathbb{R}^n$ (see [GM] for the case of the Grasmannian, and [A] or [GuS] for an important generalization) with the property that the image of each trajectory is a convex polyhedron. Our main result is that the following three different decompositions of the Grassmannian into strata all coincide:

(1) The set of points in G^k_{n-k} such that the corresponding projective configuration represents a fixed combinatorial geometry (see Sect. 1).

(2) The union of the orbits of $(\mathbb{C}^*)^n$ whose projection under μ is a fixed convex polyhedron (see Sect. 2).

(3) A multi-intersection of translates of Schubert cells which are obtained by permuting the coordinate axes (see Sect. 3).

The equivalence of (1) and (2) establishes a one to one correspondence between representable (over $\mathbb{C}$) combinatorial geometries (or matroids) and certain convex polyhedra. In Section 4 we extend this to a correspondence between all matroids and certain polyhedra which are characterized by a restriction on their vertices and edges (1-dimensional faces). This characterization is equivalent to the Steiner exchange axiom. The marriage of matroid theory and convex set theory should have interesting consequences. The polyhedron corresponding to the Fano plane is particularly beautiful.

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1. The Grassmann Strata and Combinatorial Geometries

1.1. Definitions. Throughout this paper we fix the standard unit vectors $e_1, e_2, ..., e_n$ of $\mathbb{C}^n$ and let G^k_{n-k} denote the Grassmann manifold of

$(n-k)$-dimensional subspaces of $\mathbb{C}^n$. For each plane $P \in G^k_{n-k}$ the projection

$$\pi_p: \quad \mathbb{C}^n \to \mathbb{C}^n/P$$

determines n vectors (some of which may be 0), $\pi_p(e_1)$, $\pi_p(e_2),..., \pi_p(e_n)$ in the quotient $\mathbb{C}^n/P \cong \mathbb{C}^k$. We obtain in this way a (representable over $\mathbb{C}$) *matroid* (or combinatorial geometry) of rank k on the set $\{1, 2, 3,..., n\}$, i.e., a "rank function" defined on subsets $J \subset \{1, 2,..., n\}$, which is given by

$$\operatorname{rank}(J) = \dim_{\mathbb{C}}(\operatorname{span}\{\pi_p(e_j) \mid j \in J\})$$

and which satisfies the following matroid axions: [Wh, VW, CR, W]:

(R1) $\operatorname{rank}(\phi) = 0$,

(R2) $I \subseteq J \Rightarrow \operatorname{rank}(I) \leqslant \operatorname{rank}(J)$,

(R3) $\operatorname{rank}(I \cup J) + \operatorname{rank}(I \cap J) \leqslant \operatorname{rank}(I) + \operatorname{rank}(J)$.

Remark. Given any k-dimensional complex vectorspace V and any n vectors $v_1, v_2,..., v_n$ which span V, there is a plane $P \in G^k_{n-k}$ and an isomorphism $F: \mathbb{C}^n/P \cong V$ such that $F(\pi_p(e_i)) = v_i$ (for $i = 1, 2,..., n$). In fact, F is induced by the surjective homomorphism $\tilde{F}: \mathbb{C}^n \to V$ which is defined by $\tilde{F}(e_i) = v_i$.

1.2. *Grassmann Strata*

DEFINITION. Two points $P_1, P_2 \in G^k_{n-k}$ are said to lie in the same *Grassmann stratum* Γ of G^k_{n-k} if they give rise to the same matroid, i.e., if for each subset $J \subset \{1, 2,..., n\}$ we have,

$$\dim_{\mathbb{C}} \operatorname{span}\{\pi_{P_1}(e_j) \mid j \in J\} = \dim_{\mathbb{C}} \operatorname{span}\{\pi_{P_2}(e_j) \mid j \in J\}.$$

1.3. *Torus Action*

The algebraic torus $H = (\mathbb{C}^*)^n$ acts on $\mathbb{C}^n$ by stretching the coordinate axes, i.e., if $\lambda = (\lambda_1, \lambda_2,..., \lambda_n) \in H$ and if $x \in \mathbb{C}^n$ then

$$\lambda \cdot x = (\lambda_1 x_1, \lambda_2 x_2,..., \lambda_n x_n).$$

The action of each $\lambda \in H$ is linear so it takes subspaces to subspaces and therefore induces an action on G^k_{n-k}. The fixed points of this action are easily described: for each k-element subset $J \subset \{1, 2,..., n\}$ there are coordinate k and $n-k$ planes,

$$R_J = \operatorname{span}\{e_j \mid j \in J\},$$

$$R_J^{\perp} = \operatorname{span}\{e_j \mid j \notin J\}.$$

It is easy to see that the fixed points of the action of H on G^k_{n-k} are precisely the coordinate $n-k$ planes $R_J^\perp$ (for arbitrary k-element subsets J).

Remark. The closure (in G^k_{n-k}) of an orbit of H is a normal algebraic subvariety of G^k_{n-k} which is H-stable and consists of finitely many H-orbits, i.e. it is a toric variety [D].

1.4. LEMMA. *Fix $P \in G^k_{n-k}$ and let Φ denote the corresponding matroid. Let $\overline{H \cdot P}$ denote the closure (in G^k_{n-k}) of the orbit of H which contains P. Then the fixed points of H which lie in $\overline{H \cdot P}$ are precisely those coordinate $n-k$ planes $R_J^\perp$ such that J is a basis (i.e., a maximal independent subset) of Φ.*

Proof. First, suppose that J is a basis of Φ. This means that $\{\pi_P(e_j) \mid j \in J\}$ are linearly independent in $\mathbb{C}^n/P$, i.e., that $P \cap R_J = \{0\}$, where

$$R_J = \operatorname{span}\{e_j \mid j \in J\}.$$

Thus the plane P can be realized as the *graph* of a linear transformation

$$f\colon \quad R_J^\perp \to R_J$$

in the product space $\mathbb{C}^n = R_J^\perp \oplus R_J$. Now consider the action of $\mathbb{C}^* \subset H$ on the Grassmannian G^k_{n-k}, which is induced by the following action on $\mathbb{C}^n$:

$$\lambda \cdot e_j = \begin{cases} \lambda e_j & \text{if } \ j \in J, \\ e_j & \text{if } \ j \notin J. \end{cases}$$

It follows that for any plane $P \in G^k_{n-k}$, the induced action satisfies

$$\lambda \cdot P = \operatorname{graph}(\lambda f),$$

so

$$\lim_{\lambda \to 0} (\lambda \cdot P) = \operatorname{graph}(0) = R_J^\perp,$$

i.e., the coordinate plane $R_J^\perp$ is in the closure of $H \cdot P$.

On the other hand, suppose that J is not a basis of Φ, but suppose there exists a sequence $\lambda_i \in H$ such that $\lambda_i \cdot P \to R_J^\perp$. Then for sufficiently large i we have

$$\lambda_i \cdot P \cap R_J = 0$$

since any such $n-k$ plane which is sufficiently close to $R_J^\perp$ will necessarily be transverse to R_J. However, this implies that J must be an independent set of Φ: if it were dependent then $\{\pi_p(e_j) \mid j \in J\}$ would be linearly dependent which would mean that $P \cap R_J \neq 0$, and so the same would be true for $\lambda \cdot P \cap R_J$.

1.5. *Remarks on Projective Configurations*

For any $r \geqslant k$, we let $C_r^n(\mathbb{P}^{k-1})$ denote the set of maps $c: S \to \mathbb{P}^{k-1}$ from an r-element subset $S \subset \{1, 2, ..., n\}$ to $\mathbb{P}^{k-1}$, whose image $c(S)$ spans $\mathbb{P}^{k-1}$. (Thus an element of $C_r^n(\mathbb{P}^{k-1})$ is r points, not necessarily distinct, and labelled by certain integers between 1 and n). The group $PGl_k(\mathbb{C})$ acts on the space $C_r^n(\mathbb{P}^{k-1})$. A *projective configuration* is an element of the quotient space $C_r^n(\mathbb{P}^{k-1})/PGl_k(\mathbb{C})$.

Fix a plane $P \in G_{n-k}^k$ and let r denote the number of nonzero vectors in the collection $\{\pi_p(e_i) \mid 1 \leqslant i \leqslant n\} \subset \mathbb{C}^n/P$. We thus obtain a configuration $\Lambda(P)$ of r ordered points (which are labelled by r of the integers between 1 and n) in the projective space $\mathbb{P}(\mathbb{C}^n/P) \cong \mathbb{P}^{k-1}$. The following proposition indicates that we may transform questions involving the action of $PGL_k(\mathbb{C})$ on the space of ordered r-tuples of points in $\mathbb{P}^{k-1}$ into questions involving the action of the torus $H = (\mathbb{C}^*)^n$ on G_{n-k}^k:

PROPOSITION [GM]. *The association Λ induces a one-to-one correspondence between the factor spaces*

$$G_{n-k}^k/H \qquad \text{and} \qquad \left(\coprod_{r=k}^{n} C_r^n(\mathbb{P}^{k-1})\right)\Big/PGL_k(\mathbb{C}).$$

Remark. There is a natural (non-Hausdorff) topology on each of these spaces.

Proof of Proposition. We repeat the essential idea behind the proof in [GM]. Choose an r-element subset $J \subset \{1, 2, ..., n\}$. Let

$$\tilde{G}_J = \{P \in G_{n-k}^k \mid \pi_p(e_j) = 0 \Leftrightarrow j \in J\}.$$

It suffices to show that Λ induces a bijection

$$\tilde{\Lambda}_J\colon \quad \tilde{G}_J/H \to C_J(\mathbb{P}^{k-1})/PGL_k(\mathbb{C}),$$

where $C_J(\mathbb{P}^{k-1}) \subset C_r^n(\mathbb{P}^{k-1})$ denotes the set of r-tuples of points in $\mathbb{P}^{k-1}$ which span $\mathbb{P}^{k-1}$ and are labelled by the integers in the set J.

Any element $P \in \tilde{G}_J$ is the kernel of a surjective linear map $\pi: \mathbb{C}^n \to \mathbb{C}^k$ which is uniquely determined up to composition with elements of $GL_k(\mathbb{C})$ because the induced map $\mathbb{C}^n/P \to \mathbb{C}^k$ is an isomorphism. Thus P determines a unique GL_k-equivalence class of r nonzero vectors in $\mathbb{C}^k$. The action of H stretches these vectors but does not change their directions, so the corresponding points in $\mathbb{P}^{k-1}$ are well defined (modulo PGL_k equivalence). Thus $\tilde{\Lambda}_J$ is well defined and we have already remarked (Sect. 1.1) that it is surjective. To see that $\tilde{\Lambda}_J$ is injective, suppose that $P_1, P_2 \in G^k_{n-k}$ are kernels of surjective homomorphisms $\pi_1, \pi_2: \mathbb{C}^n \to \mathbb{C}^k$ and that $\tilde{\Lambda}_J(P_1) = \tilde{\Lambda}_J(P_2)$, i.e., there exists an invertible linear transformation $F: \mathbb{C}^k \to \mathbb{C}^k$ such that, for each $j \in J$ there exists $\lambda_j \in \mathbb{C}^*$ with $F\pi_1(e_j) = \lambda_j \pi_2(e_j)$. If $\Delta: \mathbb{C}^n \to \mathbb{C}^n$ is given by the diagonal matrix

$$\Delta_{ii} = \begin{cases} \lambda_j & \text{if } j \in J, \\ 1 & \text{if } j \notin J. \end{cases}$$

then the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{\pi_1} & \mathbb{C}^k \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle F} \\ \mathbb{C}^n & \xrightarrow{\pi_2} & \mathbb{C}^k \end{array}$$

and therefore $\Delta(P_2) = P_1$.

2. Moment Map

2.1. Definition of the Moment Map. Associated to the torus action (Sect. 1.3) of H on the Grassmannian G^k_{n-k}, there is a moment map

$$\mu: \quad G^k_{n-k} \to \mathbb{R}^n$$

which was defined first (in this case of the Grassmannian) in [M] and [GM], and later, for arbitrary group actions on symplectic manifolds in [A] and [GuS]. In this section we will give an explicit expression for the moment map.

A plane $P \in G^k_{n-k}$ can be realized as the kernel of a surjective homomorphism $F: \mathbb{C}^n \to \mathbb{C}^k$ which corresponds to a matrix M with n columns and k rows. For any subset $J \subset \{1, 2, ..., n\}$ of cardinality k, we obtain a $k \times k$ matrix $M(J)$ consisting of the columns of M which are indexed by J. There are $\binom{n}{k}$ such subsets.

Proposition. *The coordinates $\mu_i: G^k_{n-k} \to \mathbb{R}$ of the moment map are given by*

$$\mu_i(P) = \frac{\sum_{i \in J} |\det M(J)|^2}{\sum_J |\det M(J)|^2},$$

where the summation in the numerator is over all k-element subsets J which contain the index i, and where the summation in the denominator is over all k-element subsets J.

Proof. The association $P \to \{|\det M(J)|\}$ (where J varies over the k element subsets of $\{1, 2, \ldots, n\}$) gives rise to the Plücker embedding

$$G^k_{n-k} \to \mathbb{P}^{\binom{n}{k}-1}$$

on which the moment map is computed as in [K].

2.2. *The Hypersimplex*

For any $P \in G^k_{n-k}$ we have

$$0 \leqslant \mu_i(P) \leqslant 1 \qquad \text{and} \qquad \sum_{i=1}^{n} \mu_i(P) = k.$$

Thus the image of the moment map μ is the hypersimplex Δ^k_{n-k} of [GGL] and [GM], i.e., the set of all points $x \in \mathbb{R}^n$ such that $0 \leqslant x_i \leqslant 1$ and $\sum_{i=1}^{n} x_i = k$. The hypersimplex Δ^k_{n-k} is the convex hull of the $\binom{n}{k}$ vectors $e(J) \in \mathbb{R}^n$ which are indexed by k-element subsets $J \subset \{1, 2, \ldots, n\}$ and are given by

$$e(J)_j = \begin{cases} 1 & \text{if } j \in J \\ 0 & \text{if } j \notin J \end{cases}$$

2.3. CONVEXITY THEOREM. *We recall the convexity theorem of* [GuS], *and* [A]: *Let $\overline{H \cdot P}$ denote the closure in G^k_{n-k} of the orbit of the point P under the action of $H = (\mathbb{C}^*)^n$. Then the image $\mu(\overline{H \cdot P})$ is the convex hull of the points $\mu(Q)$ where Q varies over the fixed points in the closure $\overline{H \cdot P}$.* (*In other words, $\mu(\overline{H \cdot P})$ is the convex hull of a certain subset of the vertices of the hypersimplex.*)

LEMMA. *The preimage of each vertex of the hypersimplex is the H-fixed point $\mu^{-1}(e(J)) = R^{\perp}_J$.*

Proof. By [A] the preimage $\mu^{-1}(e(J))$ of any vertex of Δ^k_{n-k} consists of a single fixed point. However the coordinate $n-k$ plane $R^{\perp}_J$ may be represented as the kernel of a matrix $M: \mathbb{C}^n \to \mathbb{C}^k$ such that the minor $M(J)$ is the identity and the remaining columns of M are all zero. Therefore, for any k-element subset $K \subset \{1, 2, \ldots, n\}$ we have

$$\det M(K) = \begin{cases} 1 & \text{if } K = J \\ 0 & \text{if } K \neq J, \end{cases}$$

so

$$\mu_i(R^{\perp}_J) = \begin{cases} 1 & \text{if } i \in J \\ 0 & \text{if } i \notin J \end{cases}$$

which shows that $\mu(R^{\perp}_J) = e(J)$.

2.4. Second Definition of the Stratification. We shall say that two points $P, Q \in G^k_{n-k}$ are in the same stratum of the second stratification of G^k_{n-k} if the image under the moment map of the closure of the H-trajectory of P coincides with the image under the moment map of the closure of the H-trajectory of Q, i.e. if

$$\mu(\overline{H \cdot P}) = \mu(\overline{H \cdot Q})$$

Theorem. *The second stratification of G^k_{n-k} coincides with the first stratification of G^k_{n-k} which was defined in Sect.* 1.2.

Corollary. *We have therefore assigned, to each representable combinatorial geometry Φ, a unique convex polyhedron*

$$\Delta(\Phi) = \text{closure}(\mu(\Gamma)),$$

where Γ is the stratum in G^k_{n-k} which corresponds to Φ. Moreover, (*by Lemma* 1.4 *and Lemma* 2.3), *the polyhedron $\Delta(\Phi)$ has the simple description as the convex hull of the vectors*

$$\{e(J) \mid J \textit{ is a basis of } \Phi\}.$$

Proof of Theorem. If two points $P, Q \in G^k_{n-k}$ lie in the same stratum Γ (as defined in Sect. 1.2) then they determine the same matroid so (by Lemma 1.4 and the convexity theorem) they have the same bases, so $\mu(\overline{H \cdot P})$ and $\mu(\overline{H \cdot Q})$ are the convex hulls of the same collection of vectors, so they coincide. On the other hand, suppose that P and Q have the property that $\mu(\overline{H \cdot P}) = \mu(\overline{H \cdot Q})$. Then the matroids corresponding to P and Q have the same bases. However the bases of a matroid determine the matroid [W] so P and Q are in the same stratum Γ of G^k_{n-k}.

3. Schubert Cells and Strata in the Grassmannian

3.1. *Schubert Cells*

The standard ordering $\{e_1, e_2, ..., e_n\}$ of the standard basis of $\mathbb{C}^n$ gives rise to the standard flag

$$F^1 \subset F^2 \subset \cdots \subset R^n = \mathbb{C}^n,$$

where $F^i = \text{span}\{e_1, e_2, ..., e_i\}$.

A *Schubert symbol* is a sequence of k numbers,

$$1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$$

and determines the Schubert cell

$$\Omega[i_1 i_2 \cdots i_k] = \left\{ P \in G^k_{n-k} \,\middle|\, \begin{matrix} \dim(P \cap F^{i_j - 1}) < j \\ \dim(P \cap F^{i_j}) = j \end{matrix} \right\},$$

i.e., the numbers i_j label the subspaces for which the dimension of the intersection with P jumps up.

The Schubert cells form a decomposition of the Grassmannian into even-dimensional cells. [MS].

Now let $\sigma \in \Sigma_n$ be a permutation on $\{1, 2,..., n\}$, and consider the new ordering $\{e_{\sigma(1)}, e_{\sigma(2)},..., e_{\sigma(n)}\}$ of the basis vectors of $\mathbb{C}^n$. This gives rise to a new flag

$$F^1_\sigma \subset F^2_\sigma \subset \cdots \subset F^n_\sigma = \mathbb{C}^n,$$

where $F^i_\sigma = \mathrm{span}\{e_{\sigma(1)}, e_{\sigma(2)},..., e_{\sigma(i)}\}$. We obtain a new decomposition of the Grassmannian into Schubert cells,

$$\Omega^\sigma[i_1 i_2 \cdots i_k]$$

by replacing F^i with F^i_σ in the above definition.

3.2. *The Third Stratification of the Grassmannian*

We define the third stratification of the Grassmannian to be the common refinement of the $n!$ decompositions into Schubert cells $\Omega^\sigma[i_1 i_2 \cdots i_k]$, where σ is allowed to vary over all permutations and $i_1 i_2 \cdots i_k$ is allowed to vary over all Schubert symbols.

THEOREM. The third decomposition of the Grassmannian coincides with the decomposition of G^k_{n-k} into the strata of Section 1.2.

Proof. If $P \in \Omega^\sigma[i_1 i_2 \cdots i_k]$ then the rank function r of the corresponding matroid satisfies

$$r(\sigma(1), \sigma(2),..., \sigma(m)) = m - j,$$

where j is uniquely determined by

$$i_j \leqslant m < i_{j+1}$$

because $\dim(F^m_\sigma / F^m_\sigma \cap P) = m - \dim(F^m_\sigma \cap P) = m - j$. In other words, the rank function is not completely determined, however, its value on the particular subsets

$$\{\sigma(1)\}, \{\sigma(1), \sigma(2)\},..., \{\sigma(1), \sigma(2),..., \sigma(n)\}$$

is determined. Now a stratum in the third decomposition of G^k_{n-k} has the form

$$S = \bigcap_{\sigma \in \Sigma_n} \Omega^\sigma[L_\sigma],$$

where each L_σ is some Schubert symbol. (Most such intersections will be empty, of course, and a given stratum may have many such representations.) Thus a point $P \in S$ corresponds to a matroid whose rank function is completely determined: if $J \subset \{1, 2,..., n\}$ is any subset then we can find a permutation σ such that $J = \{\sigma(1), \sigma(2),..., \sigma(|J|)\}$. If $L_\sigma = [i_1 i_2 \cdots i_k]$ then the value of the rank function is $r(J) = |J| - j$, where

$$i_j \leqslant |J| < i_{j+1}.$$

(The permutation σ is not unique. However, if another permutation τ is found such that $J = \{\tau(1), \tau(2),..., \tau(|J|)\}$ and if the resulting computation for $r(J)$ differs from the above, then this will imply $S = \phi$.) This shows that the intersection S is contained in at most one stratum of the stratification from Section 1.2.

On the other hand, suppose that Γ is a stratum of the stratification from Section 1.2. Fix a permutation $\sigma \in \Sigma_n$. For each $P \in \Gamma$ the ranks of the sets

$$\{\sigma(1)\}, \{\sigma(1), \sigma(2)\},..., \{\sigma(1), \sigma(2),..., \sigma(n)\}$$

are determined by the rank function r of the matroid associated to Γ. However,

$$r\{\sigma(1), \sigma(2),..., \sigma(m)\} = \dim(F^m_\sigma / F^m_\sigma \cap P) = m - \dim(F^m_\sigma \cap P)$$

so the dimensions $\dim(F^m_\sigma \cap P)$ are also determined by Γ. This means that P is in a certain Schubert cell of type $\Omega^\sigma[L_\sigma]$ and the Schubert symbol L_σ is determined by the matroid associated to Γ. Thus $\Gamma \subset \Omega^\sigma[L_\sigma]$. If we allow the permutation σ to vary, we conclude that each stratum Γ is contained in a unique intersection,

$$\Gamma \subset \bigcap_{\sigma \in \Sigma_n} \Omega^\sigma[L_\sigma]$$

which completes the proof.

4. Matroids and Convex Polyhedra

4.1. *Introduction.*

We can extend the correspondence (Corollary 2.4) between representable matroids and certain convex polyhedra, to all matroids. Thus, to any matroid Φ (of rank k, defined on the set $\{1, 2, ..., n\}$), we associate the convex polyhedron $\Delta(\Phi)$,

$$\Delta(\Phi) = \text{convex hull}\{e(I) \mid I \text{ is a basis of } \Phi\}$$

In this section we will investigate which polyhedra can occur.

DEFINITION. We will say that a convex polyhedron Δ which is contained in the hypersimplex Δ^k_{n-k} is a *matroid polyhedron* if the vertices of Δ are a subset of the vertices of the hypersimplex Δ^k_{n-k} and if each edge (i.e., 1-dimensional face) of Δ is a translation of one of the vectors $e_i - e_j$ (for $i \neq j$).

4.1. THEOREM. *Suppose Δ is a convex polyhedron which is contained in the hypersimplex Δ^k_{n-k}. Then there exists a matroid Φ such that $\Delta = \Delta(\Phi)$ iff Δ is a matroid polyhedron, and in this case the matroid Φ is uniquely determined.*

Remarks. (1) The vectors $e_i - e_j$ are the "roots" of the group $GL_n(\mathbb{C})$.

(2) Isomorphic matroids determine congruent polyhedra.

(3) This theorem implies, for example, that if Φ_1 and Φ_2 are matroids such that $\Delta(\Phi_1) \subset \Delta(\Phi_2)$ then the edges (and the vertices) of $\Delta(\Phi_1)$ are a subset of the edges (and the vertices) in $\Delta(\Phi_2)$.

(4) The essential observation in the proof is that an edge which is a translate of $e_i - e_j$ joins two bases which are related by a Steiner exchange.

4.3. *Proof of* ($\Rightarrow$)

Fix a matroid Φ. For each basis $B \subset \{1, 2, ..., n\}$ of Φ we denote the corresponding vertex of Δ^k_{n-k} by $e(B)$, i.e.,

$$e(B)_i = \begin{cases} 1 & \text{if } i \in B \\ 0 & \text{if } i \notin B. \end{cases}$$

Now suppose that I and J are bases of the matroid Φ, and that the vertices $e(I)$ and $e(J)$ are joined by an edge in the convex set $\Delta(\Phi)$. By reordering the elements of the matroid, we may suppose that the vectors $e(I)$ and $e(J)$ differ only in the first $2p$ coordinates and that

$$e(I) = (1, 1, ..., 1, 0, 0, ..., 0, \text{other}),$$

$$e(J) = (0, 0, ..., 0, 1, 1, ..., 1, \text{other})$$

(there are p ones and p zeroes in each case). We will show that unless $p = 1$, the midpoint

$$m = (\tfrac{1}{2}, \tfrac{1}{2}, ..., \tfrac{1}{2}, \text{other})$$

of the segment joining $e(I)$ and $e(J)$ is a nontrivial convex combination of other vertices of $\Delta(\Phi)$ and therefore this segment is not an edge of $\Delta(\Phi)$. For this discussion we can ignore the "other" coordinates, i.e., we may take $I=\{1, 2, 3,..., p\}$ and $J=\{p+1, p+1,..., 2p\}$. We will repeatedly apply the Steiner exchange axiom to these two bases.

Step 1 = *Step* 1b. Exchange the element $1 \in I$ with the basis J, obtaining a new basis B_1 of Φ which, by reordering the elements in J, can be assumed to be

$$B_1 = B_{1b} = I - \{1\} + \{p+1\}.$$

Step 2a. Exchange the element $p+1 \in J$ with the basis I, obtaining one of two possibilities (up to a reordering of the elements $\{2, 3,..., p\}$): $B_{2a} = J - \{p+1\} + \{1\}$ or else $J - \{p+1\} + \{2\}$. In the first case we have

$$m = \tfrac{1}{2}[e(B_{2a}) + e(B_{1b})]$$

so we are finished. Thus, we can assume

$$B_{2a} = J - \{p+1\} + \{2\}$$

is a basis of Φ.

Step 2b. Exchange $2 \in I$ with the basis J, obtaining one of two possible bases (up to a reordering of the elements $\{p+2, p+3,..., 2p\}$): $B_{2b} = I - \{2\} + \{p+1\}$ or $I - \{2\} + \{p+2\}$. In the first case,

$$m = \tfrac{1}{2}[e(B_{2b}) + e(B_{2a})]$$

so we are finished. Thus, we can assume

$$B_{2b} = I - \{2\} + \{p+2\}$$

is a basis of Φ.

Continuing in this way, we either prove that m does not lie on an edge, or else we construct a sequence of bases B_{1b}, B_{2a}, B_{2b}, B_{3a},..., of Φ. At the kth step (part a) we exchange $p+k-1 \in J$ with the basis I, obtaining one of k possibilities (up to a reordering of the elements $\{e_k, e_{k+1},..., e_p\}$),

$$B_{ka} = J - \{p+k-1\} + \{i\},$$

where $1 \leqslant i \leqslant k$. However, one checks (by a straightforward but messy computation) that if $i \neq k$ then

$$m = \frac{1}{2i}[e(B_{ka}) + e(B_{(k-1)b}) + e(B_{(k-1)a}) + \cdots + e(B_{(k-i)b})]$$

and so m does not lie on an edge. This leaves only the possibility that

$$B_{ka} = J - \{p+k+1\} - \{k\}$$

is a basis of Φ.

Similarly the kth step (part b) gives a basis

$$B_{kb} = I - \{k\} + \{p+k\}.$$

This process terminates after p steps when we exchange (in step $(p+1)$ part (a)) the element $2p \in J$ with the basis I. There are one of p possible results,

$$B_{(p+1)a} = J - \{p\} + \text{one of } \{1, 2, 3, \dots, p\}$$

and in each case the point m can be written as a nontrivial convex combination of previous vertices, as above. This completes the proof that the edges of any $\Delta(\Phi)$ must be translates of vectors $e_i - e_j$.

4.4. *Proof of* $(\Leftarrow)$

Suppose that Δ is a convex hull of some vertices in the hypersimplex Δ^k_{n-k}, and that each edge of Δ is a translation of some vector $e_i - e_j$. We must show that the vertices of Δ constitute the bases of a matroid. By [W] this is equivalent to verifying the Steiner exchange axiom: if I and J are k-element subsets of $\{1, 2, \dots, n\}$ such that $e(I)$ and $e(J)$ are vertices of Δ, and if $m \in I - J$, then there exists $l \in J - I$ such that the vector $e(I - \{m\} + \{l\})$ is a vertex of Δ. By relabelling the coordinate axes in $\mathbb{R}^n$, we may assume that $e(I)$ and $e(J)$ differ only in the first $2p$ positions, and that

$$e(I) = (1, 1, \dots, 1, 0, 0, \dots, 0, \text{other}),$$
$$e(J) = (0, 0, \dots, 0, 1, 1, \dots, 1, \text{other}).$$

We may further assume that the "other" coordinates are arranged so that all the 1's appear before the 0's. In this way we have divided the set $\{1, 2, \dots, n\}$ into four intervals:

$$A = \{1, 2, \dots, p\}, \qquad B = \{p+1, p+2, \dots, 2p\},$$
$$C = \{2p+1, 2p+2, \dots, p+k\}, \quad D = \{p+k+1, p+k+2, \dots, n\}$$

such that $I = A \cup C$, $J = B \cup C$, and $m \in A$.

Since Δ is convex, the line segment joining $e(I)$ to $e(J)$ is completely contained in Δ, which is in turn contained in the convex cone which is spanned by the edges $E_1, E_2, \dots, E_r$ which emanate from the vertex $e(I)$. Thus there are nonnegative real numbers $a_1, a_2, \dots, a_r$ such that

$$e(J) - e(I) = (-1, -1, \dots, -1; 1, 1, \dots, 1; 0, 0, \dots, 0; 0, 0, \dots, 0) = \sum_{i=1}^{r} a_i E_i \qquad (*)$$

(where the semicolons are used to separate the coordinates which are in A, B, C, and D). By assumption, each such edge vector E which emanates from the vertex $e(I)$ is of the form $e_l - e_k$, for some l and k in the set $\{1, 2, ..., n\}$. Since the vertex $e(I) + E$ lies in the hypersimplex, which is contained in the region $0 \leqslant |x_i| \leqslant 1$ (for $1 \leqslant i \leqslant n$), we must have

$$l \notin A \cup C \quad \text{and} \quad k \notin B \cup D \qquad (**)$$

Furthermore, if such an edge vector $E = e_l - e_k$ appears with nonzero coefficient in the above sum (*), then $l \notin D$: otherwise this would give a positive value to the coordinate x_l, which could not be cancelled by any other terms in the sum, because of the condition (**). Similarly, we must have $k \notin C$. In conclusion, each of the vectors $E = e_l - e_k$ which appear with nonzero coefficient in the sum (*), must satisfy $l \in B$ and $k \in A$.

Now consider the particular coordinate $m \in A$. Since $(e(J) - e(I))_m = -1$, at least one of the vectors (say, E_s) in the sum (*) has -1 in the mth coordinate. For this particular vector we have

$$E_s = e_l - e_m \quad \text{and} \quad l \in B \subset J - I.$$

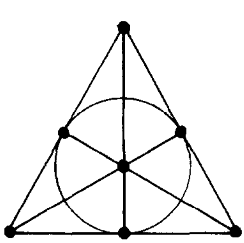
Thus, the vertex of Δ which is given by

$$e(I) + E_s = e(I - \{m\} + \{l\})$$

verifies the desired Steiner exchange.

4.5. *The Fano Polyhedron*

Associated to the Fano configuration (which is not representable over $\mathbb{C}$),



we obtain a beautiful, highly symmetric 6-dimensional convex polyhedron with 28 vertices, 126 edges, 245 2-dimensional faces, 238 3-dimensional faces, 112 four-dimensional faces and 21 5-dimensional faces. The full symmetry group of this polyhedron is the finite simple group $PGL_3(\mathbb{F}_2)$. This example has obvious generalizations to other finite projective spaces.

5. Remarks

5.1. *Topology of the Strata*

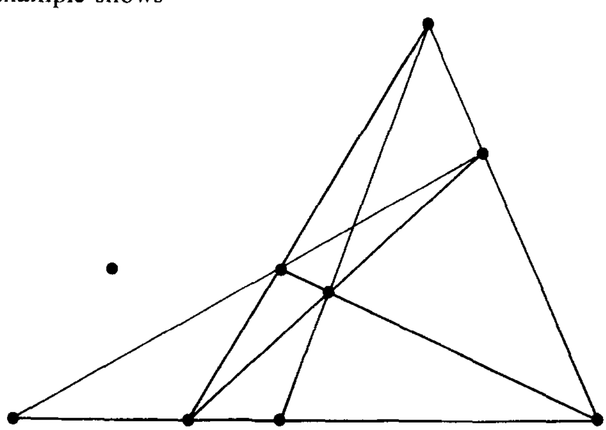
We do not know whether each stratum $\Gamma \subset G^k_{n-k}$ is nonsingular. We do not know whether each stratum Γ is a $K(\pi, 1)$ space.

5.2. *Degeneration of Matroids*

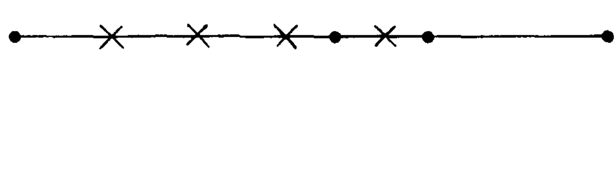
If $\Gamma \subset G^k_{n-k}$ is a stratum and if $P \in \bar{\Gamma} - \Gamma$, we shall say that the matroid corresponding to Γ degenerates to the matroid corresponding to P. In this case, we have for any subset $J \subset \{1, 2,..., n\}$ the following relation on their corresponding rank functions:

$$r_\Gamma(J) \leqslant r_P(J).$$

However, the closure of the stratum Γ is not necessarily a union of strata Γ', and may for example contain a proper subset of a stratum Γ', as the following example shows



A theorem in projective geometry [HC] states that the four points A, B, C, and D are harmonic, i.e., the cross ratio of $(A, C; B, D)$ is -1. However, it is possible to degenerate the above configuration to the following configuration:



but in doing so we will only obtain 8-tuples of points such that A, B, C, and D are harmonic.

5.3. *Other Lie Groups and Parabolics*

For any complex algebraic Lie group G and parabolic subgroup P, the moment map associated to the torus action, $\mu: G/P \to \mathfrak{g}^*$ gives rise to new combinatorial geometries and interesting convex polyhedra. These will be explored in [GS].

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