LECTURE NOTES ON SHEAVES AND PERVERSE SHEAVES

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Contents

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Part 1. Sheaves

1. Sheaves: the lightning tour

1.1. **Category theory.** Sheaf theory requires some category theory, as summarized in Appendix A. Don't try to read it all at once. I have added references to this section as needed.

1.2. Let R be a commutative ring (with 1). Let X be a topological space.

A presheaf of R modules on X is a contravariant functor (see §A.1)
S : category of open sets and inclusions $\to$ category of R-modules.

This is a fancy way to say that $V \subset U$ gives $S(U) \to S(V)$ and $W \subset V \subset U$ gives a commutative diagram

$$\begin{tikzcd} S(U) \arrow[rr] \arrow[dr] & & S(V) \arrow[dl] \\ & S(W) & \end{tikzcd}$$

Also, $U \subset U$ gives $S(U) \to S(U)$ identity and $S(\phi) = 0$. Elements $s \in S(U)$ are called sections of S over U (for reasons that will become clear shortly). If $V \subset U$ and $s \in S(U)$, its image in $S(V)$ is denoted $s|V$ and is called the restriction of the section s to V. A morphism $S \to T$ of presheaves is a natural transformation of functors, that is, a collection of homomorphisms $S(U) \to T(U)$ (for every open set $U \subset X$) which commute with the restriction maps. This defines a category of presheaves of R-modules, and it is an abelian category with kernels and cokernels defined in the obvious manner, for example, the kernel presheaf of a morphism $f : S \to T$ assigns to each open set $U \subset X$ the R-module $\ker(S(U) \to T(U))$.

More generally, for any category $\mathcal{C}$ one may define, in a similar manner, the category of presheaves on X with coefficients in $\mathcal{C}$. If $\mathcal{C}$ is abelian then so is the category of presheaves with coefficients in $\mathcal{C}$.

A presheaf S is a *sheaf* if the following *sheaf axiom* holds: Let $\{U_\alpha\}_{\alpha \in I}$ be any collection of open subsets of X, let $U = \cup_{\alpha \in I} U_\alpha$ and let $s_\alpha \in S(U_\alpha)$ be a collection of sections such that

$$s_\alpha|(U_\alpha \cap U_\beta) = s_\beta|(U_\alpha \cap U_\beta)$$

for all $\alpha, \beta \in I$. Then there exists a unique section $s \in S(U)$ such that $s|U_\alpha = s_\alpha$ for all $\alpha \in I$. In other words, if you have a bunch of sections over open sets that agree on the intersections of the

open sets then they patch together in a unique way to give a section over the union of those open sets. [An essential point in this definition is that the index set I may have infinite cardinality.]

The *category of sheaves* is the *full subcategory* (see §A.1 of the category of presheaves, whose objects are sheaves. [This means that $\mathrm{Hom}_{\mathrm{Sh}}(S,T) = \mathrm{Hom}_{\mathrm{preSh}}(S,T)$.]

The *stalk* of a presheaf S at a point $x \in X$ is the R-module

$$S_x = \varinjlim_{U \ni x} S(U)$$

This means, in particular, that for any open set U and for any $x \in U$ there is a canonical mapping $S(U) \to S_x$ which we also refer to as "restriction" and denote by $s \mapsto s|S_x$.

The *leaf space* LS of S is the disjoint union

$$LS = \coprod_{x \in X} S_x \xrightarrow{\pi} X$$

with a topology that is discrete on each S_x and that makes π into a local homeomorphism.

That is, each $U^{open} \subset X$ and each $s \in S(U)$ defines an open set

$$U_s = \{(x,t) | x \in U, t \in S_x \text{ and } t = s|S_x\} \subset \pi^{-1}(U) \subset LS.$$

Then $\pi : U_s \to U$ is a homeomorphism.

Let $\Gamma(U, LS)$ be the set of continuous sections of π over U, that is, the set of continuous mappings $h : U \to LS$ such that $\pi h =$ identity. The restriction maps of S are compatible, giving $S(U) \to S_x$ for any $U \ni x$ and therefore any $s \in S(U)$ defines a continuous section $h \in \Gamma(U, LS)$ by setting $h(x) = s|S_x$.

1.3. Proposition. (exercise) *The presheaf S is a sheaf if and only if the canonical mapping $S(U) \to \Gamma(U, LS)$ is an isomorphism for every open set $U \subset X$. If S and T are sheaves then there are canonical isomorphisms*

$$\mathrm{Hom}_{Sh}(S,T) \cong \mathrm{Hom}_X(LS, LT) \cong \mathrm{Hom}_{preSh}(S,T)$$

where $\mathrm{Hom}_X(LS, LT)$ denotes the R-module of continuous mappings $LS \to LT$ that commute with the projection to X and that consist of R-module homomorphisms $S_x \to T_x$ for all $x \in X$.

1.4. Sheafification. An immediate consequence is that if S is a presheaf then we obtain a sheaf $\widehat{S}$ by defining

$$\widehat{S}(U) = \Gamma(U, LS)$$

to be the R-module of continuous sections of the leaf space of S over the open set U. Then $\widehat{S}$ is called the *sheafification* of S. The *category of sheaves* is the full subcategory of the category of

presheaves whose objects satisfy the above sheaf axiom, in other words,

$$\mathrm{Hom}_{\mathrm{Sh}}(A, B) = \mathrm{Hom}_{\mathrm{PreSh}}(A, B).$$

To simplify notation, if S is a sheaf we drop the notation LS and we write $s \in S(U) = \Gamma(U, S)$

1.5. Proposition. (exercise) *Sheafification is an exact functor from the category of presheaves to the category of sheaves. It is left adjoint (§**??**) to the inclusion functor i : Sheaves $\to$ Presheaves, that is, if A is a presheaf and if B is a sheaf (on X) then*

$$\mathrm{Hom}_{Sh}(\widehat{A}, B) \xrightarrow{\cong} \mathrm{Hom}_{preSh}(A, i(B)).$$

Here is an application of this formula. Following the identity morphisms $B \to B$ through this series of canonical isomorphisms

$$\mathrm{Hom}_{\mathrm{Sh}}(B, B) \cong \mathrm{Hom}_{\mathrm{preSh}}(i(B), i(B)) \cong \mathrm{Hom}_{\mathrm{Sh}}(\widehat{i(B)}, B)$$

gives a canonical isomorphism $\widehat{i(B)} \to B$, that is, if we take a sheaf B, look at it as a presheaf, then sheafifity it, the result is canonically isomorphic to the sheaf B that we started with.

1.6. Caution. If A, B are sheaves then the set of morphisms $A \to B$ is the same whether we consider A, B to be sheaves or presheaves. However, care must be taken when considering the kernel, image, or cokernel of such a morphism. If we consider $f : A \to B$ to be a morphism in the category of presheaves, then $\ker(f)$ is the presheaf which assigns to an open set U the kernel of $f(U) : A(U) \to B(U)$, and this turns out to be a sheaf. But the presheaf

$$U \mapsto \mathrm{Image}(A(U) \to B(U))$$

is (usually) not a sheaf, so it is necessary to define the sheaf $\mathrm{Image}(f)$ to be the sheafification of this presheaf. Similarly for the cokernel. Consequently, *a sheaf mapping $f : A \to B$ is injective (resp. surjective) iff the mapping $f_x : A_x \to B_x$ on stalks is injective (resp. surjective) for all $x \in X$.* In other words, the abelian category structure on the category of sheaves is most easily understood in terms of the leaf space picture of sheaves.

1.7. Caution. If A is a sheaf on X and $s \in A(U)$ is a section over an open set U then (exercise!)

(1) The *support* of s, $\mathrm{spt(s)} := \{\mathrm{x} \in \mathrm{U} |\ \mathrm{s_x} \neq 0\}$ is closed in U
(2) The set $\{x \in U |\ s_x = 0\}$ is open in U

For example consider $s(x) = \begin{cases} 0 & x \leq 0 \\ x & x \geq 0 \end{cases}$ to be a section of the sheaf A of continuous functions on $\mathbb{R}$. Although $s(0) = 0$ the stalk $s_0 \in A_0$ is the *germ* of s at 0, and it is nonzero.

1.8. Examples.

(a). Let $j : U \to S^2$ be the inclusion of the open complement U of the north pole $i : N \to S^2$ and let $j_!(\underline{\underline{\mathbb{Z}}}_U)$ be the constant sheaf on U extended by zero over the point $\{N\}$. Let $i_*(\underline{\underline{\mathbb{Z}}}_N)$ be the skyscraper sheaf on S^2 supported at $\{N\}$. Then $A = j_!(\underline{\underline{\mathbb{Z}}}_U) \oplus i_*(\underline{\underline{\mathbb{Z}}}_N)$ is a sheaf on S^2 whose stalk at every point is $\mathbb{Z}$ but there are no morphisms between A and the constant sheaf $\underline{\underline{\mathbb{Z}}}$ on S^2.

(b). Let $C^0(0,1)$ be the presheaf that assigns to an open set $U \subset (0,1)$ the vector space of continuous functions $f : U \to \mathbb{R}$. Then this is a sheaf because a family of continuous functions defined on open sets that agree on the intersections of those sets clearly patch together to give a continuous function on the union. Similarly, smooth functions, holomorphic functions, algebraic functions etc. can be naturally interpreted as sheaves.

(c). Let S be the sheaf on $(0,1)$ whose sections over an open set U are those C^∞ functions $f : U \to \mathbb{R}$ such that $\int_U f(x)^2 dx < \infty$. This presheaf is not a sheaf because it is possible to patch (infinitely many) L^2 functions (defined on smaller and smaller subintervals) together to obtain a function that grows too fast to be square integrable. In fact, the sheafification of this sheaf is the set of all smooth functions on $(0,1)$.

(d). **Local systems.** Let X be a connected topological space with universal cover $\widetilde{X}$. Let $x_0 \in X$ be a basepoint and let $\pi_1 = \pi_x(X, x_0)$ be the fundamental group of X. This group acts freely on $\widetilde{X}$ (from the right) with quotient X. Let M be an R-module and let $\rho : \pi_1 \to \mathrm{Aut(M)}$ be a homomorphism. (For example, if M is a vector space over the complex numbers then $\rho : \pi_1 \to \mathrm{GL}_n(\mathbb{C})$ is a representation of π_1). Define

$$\mathcal{L} = \widetilde{X} \times_{\pi_1} M$$

to be the quotient of $\widetilde{X} \times M$ by the equivalence relation $(yg, m) \sim (y, \rho(g)m)$ for all $y \in \widetilde{X}$, $m \in M$, and $g \in \pi_1$. The projection $\widetilde{X} \to X$ passes to a projection $\pi : \mathcal{L} \to X$ which makes $\mathcal{L}$ into the leaf space of a sheaf, which is called a *local system*, or *bundle of coefficients*. Its stalk at x_0 is canonically isomorphic to M and whose stalk other points $x \in X$ is isomorphic to M but not in a canonical way. If $U \subset X$ is a contractible subset then there exist trivializations

$$\pi^{-1}(U) \cong U \times M$$

which identify the leaf space over U with the constant sheaf. So the sheaf $\mathcal{L}$ is a *locally constant sheaf* and every locally constant sheaf of R modules on a connected space X arises in this way. If X is a simplicial complex then a simplicial r-chain with values in the local system $\mathcal{L}$ is a finite formal sum $\sum a_i \sigma_i$ where σ_i are (oriented) r-dimensional simplices and where $a_i \in \pi^{-1}(x_i)$ for some (and hence for any) choice of point $x_i \in \sigma_i$. So there is a chain group (or module) $C_r(X; \mathcal{L})$. One checks that the boundary map $\partial_r : C_r(X; \mathcal{L}) \to C_{r-1}(X; \mathcal{L})$ continues to make sense in this setting and so it is possible to define the simplicial homology group $H_r(X; \mathcal{L}) = \ker(\partial_r)/\mathrm{Image}(\partial_{\mathrm{r+1}})$. In other words, locally constant sheaves have homology.

(e). **Flat vector bundles** Let $\pi : \mathcal{E} \to M$ be a smooth vector bundle (of $\mathbb{R}$-vector spaces) on a smooth manifold. The sheaf of smooth sections assigns to any open set $U \subset M$ the vector space of smooth functions $\Gamma(U, \underline{\underline{C}}^\infty(\mathcal{E})) = \{s : U \to \mathcal{E} |\ \pi \circ s(x) = x \text{ for all } \ x \in U\}$. Its leaf space is infinite dimensional. Every smooth vector bundle admits a connection. Suppose $\mathcal{E}$ admits a *flat* connection ∇. A flat section ($\nabla s = 0$) is determined locally by its value at a single point. So the subsheaf of flat sections of $\mathcal{E}$ form a local system which realize $\mathcal{E}$ as its leaf space.

(f). Let R be a commutative ring (with 1). There is a topological space, $\mathrm{Spec}(R)$ which consists of all prime ideals in R. The topology on this set was constructed by O. Zariski. For any subset $E \subset R$ let $V(E) = \{\mathfrak{p} | E \subset \mathfrak{p}\}$ be the set of prime ideals, each of which contains E. These form the closed sets in a basis for a topology, that is, the open sets in this basis are the sets $X - V(E)$. The topology generated by these open sets is called the Zariski topology. If M is an R-module then it defines a sheaf on this space in the following way. [to be completed]

(g). Let $K \subset \mathbb{R}$ be the Cantor set and let $\mathbb{Z}_K$ be the constant sheaf (with value equal to the integers, $\mathbb{Z}$) on K. Let $j : K \to \mathbb{R}$ be the inclusion. Then $j_*(\mathbb{Z}_K)$ is a sheaf on $\mathbb{R}$ (see the definition of f_* below) that is supported on the Cantor set. So "bad" sheaves exist on "good" spaces.

(h). Fix $r \geq 0$. For any topological space Y let $C_r(Y; \mathbb{Z})$ be the group of singular r-dimensional simplices on Y. (It is the set of finite formal sums of pairs (σ, f) where σ is an oriented r-dimensional simplex and $f : \sigma \to Y$ is a continuous map.) Now let X be a topological space. The presheaf of r-dimensional singular cochains C^r on X assigns to any open set $U \subset X$ the group $C^r(U) := \mathrm{Hom}(C_r(U; \mathbb{Z}), \mathbb{Z})$. If $V \subset U$ then $C_r(V; \mathbb{Z})$ is included in $C_r(U; \mathbb{Z})$ which gives a (surjective) restriction mapping $C^r(U) \to C^r(V)$. This presheaf is also a sheaf.

1.9. Sheaf Hom. If A is a sheaf on X with leaf space $\pi : LA \to X$ and if $U \subset X$ is an open set let $A|U$ be the restriction of the sheaf A to the subset U, in other words, the sheaf on U whose leaf space is $\pi^{-1}(U) \to U$. In other words, If A, B are sheaves of R modules then $\mathrm{Hom}(A, B)$ is again an R module that consists of all sheaf mappings $A \to B$. However there is an associated presheaf, perhaps we will denote it by $\underline{\underline{\mathrm{Hom}}}(A, B)$, which assigns to any open set the R module of homomorphism

$$\mathrm{Hom}_{Sh(U)}(A|U, B|U))$$

of sheaf mappings $A|U \to B|U$. This presheaf is a sheaf (exercise) for which the group of global sections is the original module of all sheaf homomorphisms, that is,

$$\Gamma(X, \underline{\underline{\mathrm{Hom}}}(A, B)) = \mathrm{Hom}_{\mathrm{Sh}(X)}(A, B)$$

1.10. Functoriality. Let $f : X \to Y$ be a continuous map, let T be a sheaf on Y, let S be a sheaf on X. Define $f_*(S)$ to be the presheaf on Y given by

$$f_*(S)(U) = S(f^{-1}(U)).$$

This presheaf is a sheaf (exercise). Define $f^*(T)$ to be the sheaf on X whose leaf space is the pull back of the leaf space of T, that is,

$$Lf^*(T) = f^*(LT) = X \times_Y LT = \{(x,\xi) \in X \times LT |\ f(x) = \pi(\xi)\}.$$

Then this defines a sheaf. It is the sheafification of its presheaf of sections,

$$\Gamma(U, f^*T) = \varinjlim_{V \supset f(U)} \Gamma(V, T). \tag{1.10.1}$$

(Although $f(U)$ may fail to be open, we take a limit over open sets containing $f(U)$.) The functor f^* is exact (exercise; see §A.5).

The pushforward with proper[1] support, $f_!S$ is defined to be the presheaf on Y,

$$f_!(S)(U) = \left\{ s \in S(f^{-1}(U)) |\ \text{spt(s)} \to \text{U is proper} \right\}$$

Then $f_!(S)$ is a sheaf (exercise). (Recall from §1.7) that the support of $s \in S(f^{-1}(U))$ is a closed subset of $f^{-1}(U)$.) If $f : X \to Y$ is the inclusion of a subspace whose closure is compact, then $(f_!S)(U)$ consists of sections $s \in \Gamma(U \cap X, S)$ whose support is compact and its stalks are

$$(f_!S)_x = \begin{cases} S_x & \text{if } x \in X \\ 0 & \text{if } x \in Y \setminus X. \end{cases}$$

In this case the sheaf $f_!(S)$ is called the extension by zero of S.

1.11. Exercise. Suppose the space X is locally compact. If $j : K \subset X$ denotes the inclusion of a closed subset, and if S is a sheaf on K then $j_*(S) = j_!(S)$.

1.12. Adjunction. (see §A.4.) Let $f : X \to Y$ be a continuous mapping, let A be a sheaf on X, let B be a sheaf on Y. Then there exist natural sheaf morphisms

$$f^*f_*(A) \to A \quad \text{and} \quad B \to f_*f^*(B).$$

To see this, for the first one, let us consider sections over an open set $U \subset X$. Then

$$\Gamma(U, f^*f_*A) = \varinjlim_{W \supset f(U)} \gamma(f^{-1}(W), A).$$

If $W \supset f(U)$ then $f^{-1}(W) \supset U$ so we get a mapping from this group to $\Gamma(U, A)$. One verifies that these mappings are compatible when we shrink U, and so this gives a sheaf morphism $f^*f_*(A) \to A$. For the second morphism, again we look at sections over an open set $V \subset Y$. If t is a section of LB over V then, pulling it back by f gives a section $f^*(t)$ of the leaf space of $f^*(B)$ over the set $f^{-1}(V)$, in other words, we have defined a map

$$\Gamma(V, B) \to \Gamma(f^{-1}(V), f^*(B)) = \Gamma(V, f_*f^*(B))$$

[1]Recall that a continuous map is proper if the pre-image of every compact set is compact

which again is compatible with restrictions to smaller open sets. In other words, this defines a sheaf morphism $B \to f_* f^* B$. The following statement says that the functor f^* is *left adjoint* (see §A.4) to the functor f_*:

1.13. Proposition. *The adjunction maps determine a canonical isomorphism*

$$\mathrm{Hom}_{Sh(X)}(f^*B, A) \cong \mathrm{Hom}_{Sh(Y)}(B, f_*A)).$$

Given $f^*B \to A$, apply f_* and adjunction to obtain $B \to f_* f^* B \to f_* A$. Given $B \to f_* A$ apply f^* and adjunction to obtain $f^*B \to f^* f_* A \to A$. This gives maps back and forth between the Hom groups in the proposition. It is an exercise to check that they are inverses to each other.

1.14. $f^!$ for locally closed embeddings. Suppose $f : X \to Y$ is the inclusion of a locally closed[2] subset. Let S be a sheaf on Y. Define $S_{\{X\}}$ to be the sheaf whose sections are supported on X:

$$\Gamma(U, S_{\{X\}}) = \{\sigma \in \Gamma(U, S) |\ \mathrm{spt}(\sigma) \subset \mathrm{X}\}\,.$$

Define $f^! S = f^* S_{\{X\}}$. Thus, if $W \subset X$ is open then (see equation (1.10.1))

$$\Gamma(W, i^!(S)) = \varinjlim_{U \supset W} \Gamma((U, S_{\{X\}})$$

(the limit is over open sets $U \subset Y$ containing W). The cohomology $H^*(X, j^! S) = H^*_{\{X\}}(Y, S)$ may be identified with the relative cohomology $H^*(Y, Y - X; S)$.

As with f^*, the functor $f^!$ is exact (exercise) and is a right adjoint to the pushforward with compact support $f_!$, that is,

$$\mathrm{Hom}_{Sh(X)}(A, f^!(B)) \cong \mathrm{Hom}_{Sh(Y)}(f_!(A), B). \tag{1.14.1}$$

In fact, $\mathrm{Hom}_{Sh(X)}(A, f^! B)$ consists of mappings of the leaf space $LA \to LB|X$ that can be extended by zero to a neighborhood of X in Y, and this is the same as $\mathrm{Hom}_Y(i_! A, B)$.

The definition of $f^!$ will be extended to more general mappings in §11.4.

2. Cohomology

2.1. Simplicial sheaves. ([110]) This is a "toy model" of sheaves. See also §2.5, §29.2). Let K be a (finite, for simplicity) simplicial complex. Each (closed) simplex σ is contained in a naturally defined open set, $St^o(\sigma)$, the open star of σ. It has the property that $\sigma < \tau \implies St^o(\tau) \subset St^o(\sigma)$. Using these open sets to define a presheaf, and assigning the values of the presheaf to the simplex itself, gives the following definition:

[2] A subset $X \subset Y$ is locally closed if each point $x \in X$ has a neighborhood $U_x \subset Y$ in Y so that $U_x \cap X$ is closed in U_x. Equivalently, X is the intersection of an open subset and a closed subset of Y.

2.2. Definition. A simplicial sheaf S (of abelian groups, or R-modules, etc.) on K is an assignment of an abelian group $S(\sigma)$ for the interior of each simplex and a *restriction* homomorphism $S(\sigma) \to S(\tau)$ whenever $\sigma < \tau$, in such a way that whenever $\sigma < \tau < \omega$ then the resulting triangle of groups and morphisms commutes.

To make some explicit notation, let $j_{\sigma,\tau} : \sigma \to \tau$ denote the inclusion whenever $\sigma < \tau$ and let $S_{\sigma,\tau} : S(\sigma) \to S(\tau)$ be the corresponding homomorphism (so that S becomes a covariant functor from the category of simplices and inclusions to the category of abelian groups).

In this setting there is no distinction between a sheaf and a presheaf. A simplicial sheaf determines an actual sheaf whose leaf space of S is easily constructed as the union

$$LS = \coprod_{\sigma} \sigma^0 \times S(\sigma)$$

with a topology defined using $S_{\sigma,\tau}$ to glue these pieces together whenever $\sigma < \tau$. This gives a natural functor from the category of simplicial sheaves to the category of sheaves on K that are locally constant on the interior of each simplex.

2.3. Cohomology of simplicial sheaves. Let K be a finite simplicial complex and let S be a simplicial sheaf (of abelian groups, or of R-modules). The cohomology $H^*(K, S)$ can be constructed the same way as ordinary simplicial cohomology $H^*(K)$. First choose orientations of the simplices. (An orientation of a simplex is determined by an ordering of its vertices, two orderings giving the same orientation iff they differ by an even permutation.) The simplest method of orienting all the simplices, is to choose an ordering of the vertices of K and to take the induced ordering on the vertices of each simplex. Assume this to be done.

Fix $r \geq 0$. An r-chain with values in the simplicial sheaf S is a function F that assigns, to each (oriented) r-dimensional simplex σ an element of $S(\sigma)$. The collection of all r-chains is denoted $C^r(K; S)$. The coboundary $\delta F \in C^{r+1}(K; S)$ is defined as follows. If τ is an $r+1$ simplex with vertices $v_0, v_1, \cdots, v_{r+1}$ (in ascending order) we write $\tau = \langle v_0, v_1, \cdots, v_{r+1} \rangle$ and we denote its i-th face by $\partial_i \tau = \langle v_0, \cdots, \widehat{v_i}, \cdots, v_{r+1} \rangle$. Then

$$(\delta F)(\tau) = \sum_{\sigma < \tau} [\tau : \sigma] S_{\sigma,\tau}(F(\sigma))$$

where the sum is over codiimension one faces $\sigma < \tau$. One checks that $\delta\delta F = 0$ (it is the same calculation that is involved in proving that $\partial\partial = 0$ for simplicial homology), so we may define the cohomology, $H^r(K; S) = \ker(\delta)/\operatorname{Im}(\delta)$ to be the cohomology of the cochain complex

$$\cdots \longrightarrow C^{r-1}(K; S) \xrightarrow{\delta} C^r(K; S) \xrightarrow{\delta} C^{r+1}(K; S) \longrightarrow \cdots$$

This combinatorial construction of $H^*(K; S)$ is easily implemented on a computer. The identification with other constructions of sheaf cohomology is explained in §4.4 below.

By reversing the arrows one has the analogous notion of a simplicial cosheaf and a similar construction of the *homology* of a simplicial cosheaf.

2.4. Historical interlude. Many different techniques have been developed for exploring the properties of cohomology of sheaves, the most elegant being the methods associated with the derived category. Any one of these methods may be used as a "definition" of the cohomology of a sheaf, and although the historical methods are the most accessible, they are also the most cumbersome. We will use the method of injective resolutions. First we mention a few milestones in the development of cohomology and sheaves.

(1873) B. Riemann and later, E. Betti, consider the number of "cuts" of varying dimensions that are needed in order to reduce a space into contractible pieces.
(1892) H. Poincaré, in *Analysis Situs* constructs homology of a "variété" using cycles that consist of the zeroes of smooth functions.
(1898) P. Heegaard publishes a scathing scriticism of Poincareś article for its lack of rigor.
(1900) H. Poincaré publishes *Supplement* to his *Analysis Situs* in which he essentially describes simplicial homology for a space that has been decomposed into simplices.
(1912) F. Hausdorff publishes the general definition of a topological space and interprets continuity purely in terms of the open sets.
(1925) H. Hopf develops the general notion of a chain complex.
(1926) Alexander, Hopf give precise definition of simplicial complex.
(1928) H. Hopf, E. Noether describe homology as a group.
(1930) E. Cartan, G. deRham formalize notion of differential forms, Poincaré lemma, de Rham theorem.
(1933) The drive to develop singular homology theory, with contributions by Dehn, Heegard, Lefschetz, others.
(1934) E. Čech develops his approach to cohomology using the open sets in a space; cohomology with coefficients in a ring.
(1935) H. Whitney develops the abstract theory of differentiable manifolds and their embeddings into Euclidean space.
(1935) H. Reidemeister develops theory of homology with local coefficients.
(1935-40) Products in cohomology, modern formulation of Poincaré duality, Stiefel Whitney classes, differential forms. (Until this period, differential forms were "expressions".)
(1942) S. Eilenberg and S. MacLane: Category theory
(1945-46) J. Leray: sheaves and their cohomology, spectral sequence of a map
(1946) S. S. Chern: Chern classes
(1950) A. Borel's notes on Leray's theory, vastly increasing its accessibiliy
(1950) Čech cohomology of sheaves
(1956) A. Borel, J. C. Moore, the dual of a complex of sheaves; Borel-Moore homology
(1956) H. Cartan, S. Eilenberg: injective resolutions and derived functors
(1957) A. Grothendieck: Tohoku paper on homological algebra

(1958) D. Kan: notion of adjoint functors
(1961) J. L. Verdier, derived categories, Verdier duality (published in 1996).

Given the complexity of this history, we will describe several ways to define the cohomology of a sheaf, and leave the proof that they all give the same answer until Proposition 3.3. Further details on sheaves, the derived category and perverse sheaves may be found in any of the wonderful textbooks on these subjects, including [33, 37, 38, 66, 74, 77, 19, 96].

2.5. An injective object (see §A.5) in the category of sheaves is called an injective sheaf.

Exercise. In the category of simplicial sheaves (§2.1, §29.2) over a field k, every injective (resp. every projective) sheaf is a direct sum of the following *elementary injective* sheaves I_σ (resp. *elementary projective* sheaves P_σ) where (cf. [110]):

$$I_\sigma(\tau) = \begin{cases} k & \text{if } \tau \leq \sigma \\ 0 & \text{otherwise} \end{cases} \qquad P_\sigma(\tau) = \begin{cases} k & \text{if } \tau \geq \sigma \\ 0 & \text{otherwise} \end{cases}.$$

A sheaf S is *flabby* if $S(U) \to S(V)$ is surjective, for all open subsets $V \subset U$. The sheaf S is *soft (sheaf)* if $\Gamma(X, S) \to \Gamma_K(S)$ is surjective, for every closed subset $K \subset X$. The sheaf S is *fine (sheaf)* if, for every open cover $X = \cup_{\alpha \in I} U_\alpha$ there exists a family of morphisms $h_i : S \to S$ such that $\mathrm{spt}(\mathrm{h}_\alpha) \subset \mathrm{U}_\alpha$ and $\sum_\alpha h_\alpha = I$. (Usually the h_α are just a partition of unity with respect to the coefficient ring.) For completeness we include here the following definition, which actually requires having previously defined cohomology: The sheaf S is *acyclic* if $H^r(X, S) = 0$ for all $r \geq 1$. The following fact will not be used

2.6. Proposition. *For any sheaf S on a locally compact space X,*

$$\text{injective} \implies \text{flabby} \implies \text{soft} \implies \text{acyclic} \ \textit{and} \ \text{fine} \implies \textit{soft} \implies \textit{acyclic}.$$

Of these notions, *injective* and *acyclic* are categorical, and we will concentrate on them. However, in order that a sheaf be injective, it must have certain topological properties and certain algebraic properties. For example, if the ring R is an integral domain, then the constant sheaf on a single point is injective iff R is a field. The ring $\mathbb{Z}$ is not injective but it has an injective resolution $\mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$. Recall from §A.15 that

2.7. Definition. An injective resolution of a sheaf S on X is an exact sequence

$$0 \to S \to I^0 \to I^1 \to \cdots$$

where each I^r is an injective sheaf.

The category of modules over a commutative ring R has enough injectives, and the category of sheaves of R modules on any topological space X has enough injectives. Consequently (see §A.15)

every sheaf of R modules has an injective resolution. However, there is a canonical and functorial injective resolution of any sheaf, namely the *Godement* resolution, see §4.1.

2.8. Cohomology of a sheaf. First definition of cohomology Let S be a sheaf on a topological space X and let $0 \to S \to I^0 \to \cdots$ be the Godement injective resolution of S. Then the cohomology of the complex of global sections

$$0 \to \Gamma(X, I^0) \to \Gamma(X, I^1) \to \Gamma(X, I^2) \cdots$$

is called the cohomology of S, denoted $H^r(X, S)$.

Second definition of cohomology In fact: You will get the same answer, up to unique isomorphism, if you use any injective resolution, or any fine, flabby, soft, or acyclic resolution instead of an injective resolution. (See §4: injectives and Čech cohomology) (See §A.23.)

3. Complexes of sheaves

3.1. Recall from §A.6 that a *complex* $S^\bullet$ of sheaves is a bounded below sequence

$$\cdots \longrightarrow S^{r-1} \xrightarrow{d^{r-1}} S^r \xrightarrow{d^r} S^{r+1} \xrightarrow{d^{r+1}} \cdots$$

where $d \circ d = 0$. Its cohomology is a sheaf: $\underline{\underline{\mathrm{H}}}^r(S^\bullet) = \ker d^r / \operatorname{Im} d^{r-1}$ whose stalk coincides with the cohomology of the stalks (exercise), that is,

$$\underline{\underline{\mathrm{H}}}^r_x(S^\bullet) := \underline{\underline{\mathrm{H}}}^r(S^\bullet)_x \cong H^r(S^\bullet_x).$$

A morphism $\phi : S^\bullet \to T^\bullet$ is a *quasi-isomorphism* (see §A.6) if it induces isomorphisms on the cohomology objects $H^r(S^\bullet) \to H^r(T^\bullet)$ for all r. For complexes of sheaves on a space X this means that ϕ induces isomorphisms $\underline{\underline{\mathrm{H}}}^r_x(S^\bullet) \to \underline{\underline{\mathrm{H}}}^r_x(T^\bullet)$ of the stalk cohomology at each point $x \in X$. A quasi-isomorphism of complexes of sheaves induces isomorphisms on cohomology (see Lemma 3.3).

In §A.15 it is proven that

> Every complex of sheaves $A^\bullet$ admits a quasi-isomorphism $A^\bullet \to I^\bullet$ with a complex $I^\bullet$ of injective sheaves.

Such a quasi-isomorphism is called an *injective resolution* of $A^\bullet$. This fits beautifully with the notion of a resolution of a single sheaf S: it is a quasi-isomorphism,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots \\ & & \downarrow & & & & & & \\ 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \cdots \end{array}$$

[A messy technical point: a complex of injective objects is not necessarily an injective object in the category of complexes. So the somewhat ambiguous terminology of "injective resolution" could be misleading and some authors refer to these as "K-injective resolutions". Fortunately we will not be required to consider injective objects in the category of complexes.]

3.2. Cohomology of a complex of sheaves. The cohomology of a sheaf S (§2.8) on a topological space X is defined to be the cohomology of the complex of global sections of an injective resolution. The same applies if $S^\bullet$ is a complex of sheaves, see §A.6: define the cohomology of X with coefficients in $S^\bullet$,

$$H^i(X; S^\bullet) := H^i(\Gamma(X, I^\bullet))$$

where $S^\bullet \to I^\bullet$ is an injective resolution, that is, $H^i = R^i\Gamma$. (This is sometimes referred to as the hypercohomology of $S^\bullet$, to distinguish it from the cohomology sheaves $\underline{\underline{\mathrm{H}}}^i(S^\bullet)$.)

As before, flabby, soft or fine resolutions may be used instead of injective resolutions.

3.3. Lemma. *Let $A^\bullet \to B^\bullet$ be a quasi-isomorphism of complexes of sheaves on a topological space X. Then it induces an isomorphism on cohomology $H^r(U, A^\bullet) \cong H^r(U, B^\bullet)$ for any open set $U \subseteq X$.*

Proof. The morphism $\phi : A^\bullet \to B^\bullet$ induces a mapping on the spectral sequence for cohomology whose E^2 page is

$$H^p(U; \underline{\underline{H}}^q(A^\bullet)) \implies H^{p+q}(U; A^\bullet).$$

Since ϕ is a quasi-isomorphism it induces an isomorphism on th E^2 page and on all other pages. The spectral sequence comparison theorem gives isomorphism $H^{p+q}(U; A^\bullet) \cong H^{p+q}(U; B^\bullet)$. □

Here is an alternate proof that does not require spectral sequences. Hypercohomology is defined in terms of sections of an injective resolution. So we may assume that $A^\bullet$ and $B^\bullet$ are injective. By Lemma A.8 the cone $C(\phi)$ is homotopic to zero. Let h be such a homotopy. Now take sections over U. The sections of the cone coincides with the cone o global sections, that is, we have a triangle of groups:

$$\begin{tikzcd} \Gamma(U, A^\bullet) \arrow[rr] & & \Gamma(U, B^\bullet) \arrow[dl] \\ & \Gamma(U, C(\phi)) \arrow[ul] & \end{tikzcd}$$

The homotopy h also gives a homotopy on sections so that $H^n(\Gamma(U, C(\phi))) = 0$. So the long exact sequence on cohomology implies that $H^n(U, A^\bullet) \to H^n(U, B^\bullet)$ is an isomorphism. □

3.4. Examples.

1. Let M^n be a smooth manifold. Let $x \in M$ and let U_x be a neighborhood of x that is diffeomorphic to an n-dimensional ball. The Poincaré lemma says that if ξ is a closed (i.e. $d\xi = 0$) differential r-form ($r \geq 1$) defined in U_x then there is a differential $r-1$ form η so that $d\eta = \xi$. The sheaf of smooth differential forms is fine, so the Poincaré lemma says that this complex of sheaves is a fine resolution of the constant sheaf,

$$\begin{tikzcd} 0 \arrow[r] & \underline{\underline{\mathbb{R}}} \arrow[r] \arrow[d] & 0 & & & \\ 0 \arrow[r] & \underline{\underline{\Omega}}^0_M \arrow[r, "d"] & \underline{\underline{\Omega}}^1_M \arrow[r, "d"] & \underline{\underline{\Omega}}^2_m \arrow[r, "d"] & \cdots \end{tikzcd}$$

Therefore the cohomology $H^r(M,\mathbb{R})$ is canonically isomorphic to the cohomology of the complex of global sections of Ω^*_M, that is, the de Rham cohomology.

2. Let X be the 2-dimensional simplex. In the category of simplicial sheaves, suppose that S is a sheaf on X that assigns the value $\mathbb{Q}$ to the interior of the 2-simplex and assigns 0 to simplices on the boundary. Find an injective resolution of S. Determine the global sections of each step in the resolution. Show that the cohomology of S is $\mathbb{Q}$ in degree 2 and is 0 in all other degrees.

3. Let X be a triangulation of the 2-sphere, which may be taken, for example to be the boundary of a 3-simplex. . Find an injective resolution of the constant sheaf $\underline{\underline{\mathbb{Q}}}$, in the category of simplicial sheaves, and compute the cohomology of its global sections.

4. Let X be a triangulation of the 2-sphere, with 10^8 simplices. Describe an injective resolution of X.

5. Let X be a topological space, let $x_0 \in X$ and let $S = S(x_0, \mathbb{Q})$ be the presheaf that assigns to any open set U

$$S(U) = \begin{cases} \mathbb{Q} & \text{if } x_0 \in U \\ 0 & \text{else} \end{cases}$$

Show that S is injective and that its leaf space LS is a skyscraper, that is, it consists of a single group $\mathbb{Q}$ at the point x_0 and zero everywhere else.

6. In the above example, fix $x_0 \in X$ and let $T^\bullet$ be the complex of sheaves $S(x_0,\mathbb{Q}) \to S(x_0,\mathbb{Q}/\mathbb{Z})$. Show that this complex is an injective resolution of the skyscraper sheaf that is $\mathbb{Z}$ at the point x_0.

4. Godement and Čech

These examples show that injective sheaves must be sums of sheaves with tiny support. This leads one to the following:

4.1. Godement resolution. Given a sheaf A on a topological space X it embeds in a flabby sheaf $\mathrm{God}(A)$ with sections

$$\Gamma(U, \mathrm{God}(A)) = \prod_{x \in U} A_x$$

the product of all the stalks at points in U. It is sometimes called the sheaf of totally discontinuous sections. If we start with the constant sheaf $\underline{\underline{\mathbb{Z}}}$ then a section $s \in \Gamma(U, \mathrm{God}(\underline{\underline{\mathbb{Z}}}))$ assigns to each point $x \in U$ an integer, without any regard to continuity or compatibility. It is the sort of sheaf that you definitely do not want to meet in a dark alley. The Godement *resolution* $\underline{\underline{\mathrm{God}}}^\bullet(A)$ is obtained by applying this construction to the cokernel of $A \to \mathrm{God}(A)$ and iterating:

$$\begin{tikzcd}
A \arrow[r, hook] & \mathrm{God}(A) \arrow[rr] \arrow[dr] & & \mathrm{God}(\mathrm{coker}\,) \arrow[rr] \arrow[dr] & & \mathrm{God}(\mathrm{coker}\,) \\
& & \mathrm{coker} \arrow[ur, hook] & & \mathrm{coker} \arrow[ur, hook] &
\end{tikzcd}$$

If A_x is injective for all $x \in X$ (for example, if the coefficient ring R is a field) then this is an injective resolution of A. The Godement resolution is functorial: a morphism $f : A \to B$ induces a morphism of complexes $\mathrm{God}(f) : \underline{\underline{\mathrm{God}}}^\bullet(A) \to \underline{\underline{\mathrm{God}}}^\bullet(B)$ in such a way that $\mathrm{God}(f \circ g) = \mathrm{God}(f) \circ \mathrm{God}(g)$.

In summary, injective sheaves are huge, horrible objects and maybe we use them to prove things but never to compute with. A much more efficient computational tool is the Čech cohomology.

4.2. Čech cohomology of sheaves. Let A be a sheaf on X. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be a collection of open sets that cover X. Fix a well ordering on the index set I and for any finite subset $J \subset I$, $J = \{i_0 < i_1 < \cdots < i_q\}$ let $U_J = \cap_{\alpha \in J} U_\alpha$ be the corresponding intersection. The Čech cochain complex is

$$\begin{array}{ccc} \check{C}^r(X, A) & = & \displaystyle\prod_{|J|=r+1} \Gamma(U_J, A) \\ \Big\downarrow d & & \Big\downarrow d \\ \check{C}^{r+1}(X, A) & = & \displaystyle\prod_{|K|=r+2} \Gamma(U_K, A) \end{array}$$

where d is defined as follows: suppose that $K = \{k_0, k_1, \cdots, k_{r+1}\} \subset I$ and $\sigma \in \check{C}^r(X, A)$. Then

$$d\sigma(U_K) = \sum_{i=0}^{r+1} (-1)^i \sigma(U_{k_0} \cap \cdots \cap \widehat{U}_{k_i} \cap \cdots \cap U_{k_{r+1}})|U_K.$$

For example, if $\sigma \in \check{C}^1$ and if $U = U_0 \cap U_1 \cap U_2$ then

$$d\sigma(U) = \sigma(U_1 \cap U_2)|U - \sigma(U_0 \cap U_1)|U + \sigma(U_0 \cap U_2)|U.$$

Then one checks that $d \circ d = 0$ so the cohomology of this resulting complex is defined:

$$\check{H}^r_{\mathcal{U}}(X, A) = \ker d / \operatorname{Im} d.$$

Notice, in particular, that $\check{H}^0_{\mathcal{U}}(X, A)$ consists of sections σ_α over U_α that agree on each intersection $U_\alpha \cap U_\beta$ so it coincides with the global sections: $\check{H}^0_{\mathcal{U}}(X, A) = \Gamma(X, A)$ for any covering $\mathcal{U}$.

4.3. Theorem. *Suppose A is a sheaf on X and an open cover $\mathcal{U}$ has the property that $H^r(U_J, A) = 0$ for every $J \subset I$ and for all $r > 0$. Then there is a canonical isomorphism $\check{H}^i(X, A) \cong H^i(X, A)$ for all i. In particular, $H^0(X, A) \cong \Gamma(X, A)$.*

Theorem 4.3 is an incredibly useful result because it says that we can use possibly very few and very large open sets when calculating sheaf cohomology, and it even tells us how to tailor the open sets to take advantage of the particular sheaf, whereas the original theorem of Čech assumed that all the multi-intersections of the open sets were contractible.

4.4. Example. Let $f : |K| \to X$ be a triangulation of a topological space X. This means that K is a simplicial complex with geometric realization $|K|$ and f is a homeomorphism. Let A be a sheaf on X and suppose that K is sufficiently fine that A is constant on $f(\sigma)$ for each (closed) simplex σ of K. Then A corresponds to a simplicial sheaf S on K in a canonical way. The open stars of simplices form an open cover $\mathcal{U}$ of X that satisfies the condition of Theorem 4.3. The Čech complex $\check{C}^\bullet_{\mathcal{U}}(A)$ agrees with the complex $C^\bullet(K, S)$ of simplicial cochains (§2.1), so they have the same cohomology: $H^*(K, S) = \check{H}^*_{\mathcal{U}}(X, A)$.

Proof. Let A be a sheaf on a topological space and let $\mathcal{U} = \{U_\alpha\}_{\alpha \in K}$ be an open covering of X. The Čech complex $\check{C}^\bullet(A) = \check{C}_{\mathcal{U}}{}^\bullet(A)$ is

$$\cdots \to \prod_{|J|=r+1} \Gamma(U_J, A) \to \prod_{|J|=r+2} \Gamma(U_J, A) \to \cdots .$$

Sheafify this construction by defining

$$\mathbf{C}^p(A) = \mathbf{C}^p_{\mathcal{U}}(A) = \prod_{|J|=p+1} i_*(A|U_J) \to \mathbf{C}^{p+1}(A) \to \cdots$$

so that $\mathbf{C}^p(A)(V) = \prod_{|J|=p+1} \Gamma(V \cap U_J, A)$. This is functorial in A. There is a little combinatorial argument to show that

$$0 \to A \to \mathbf{C}^0(A) \to \mathbf{C}^1(A) \to \cdots$$

is exact. (To check exactness of the stalks at a single point x, we need to consider the combinatorics of having $p + 1$ open sets whose multi-intersection contains x. Then exactness comes down to proving that the homology of the p-simplex is trivial.)

Now let $A \to I^0 \to I^1 \to I^2 \to \cdots$ be an injective resolution of A, and apply the Čech resolution to each term in this sequence, which gives a double complex of sheaves:

$$\begin{tikzcd}
\mathbf{C}^0(I^2) \arrow[r] & \mathbf{C}^1(I^2) \arrow[r] & \mathbf{C}^2(I^2) \arrow[r] & {} \\
\mathbf{C}^0(I^1) \arrow[r] \arrow[u] & \mathbf{C}^1(I^1) \arrow[r] \arrow[u] & \mathbf{C}^2(I^1) \arrow[r] \arrow[u] & {} \\
\mathbf{C}^0(I^0) \arrow[r] \arrow[u] & \mathbf{C}^1(I^0) \arrow[r] \arrow[u] & \mathbf{C}^2(I^0) \arrow[r] \arrow[u] & {}
\end{tikzcd}$$

whose rows are exact. Let $\mathbf{T}^\bullet$ denote the associated single complex (§A.9) of sheaves. If we augment the left column with the column $I^0 \to I^1 \to I^2 \to \cdots$ then Lemma A.10 says that the resulting total complex is acyclic, so the map on cohomology sheaves $\mathbf{H}^*(I^\bullet) \to \mathbf{H}^*(T^\bullet)$ is an isomorphism, which is to say that $I^\bullet \to T^\bullet$ is a quasi-isomorphism. So it induces an isomorphism on cohomology, $H^*(I^\bullet) = H^*(X, A) \cong H^*(T^\bullet)$.

On the other hand, let us take global sections to obtain a double complex of groups. The r-th column now reads (from the bottom up)

$$\prod_{|J|=r+1} \Gamma(U_J, I^0) \to \prod_{|J|=r+1} \Gamma(U_J, I^1) \to \cdots$$

which is a complex that computes the product of hypercohomology groups $\prod_{|J|=r+1} H^*(U_J, A) = 0$ by hypothesis. The kernel of the zeroth vertical map is exactly the Čech cochains $\check{C}^r(A)$. Therefore, if we augment the bottom row with the complex $\check{C}^0(A) \to \check{C}^1(A) \to \check{C}^2(A) \cdots$ of Čech cochains then (as in the previous paragaph) we will obtain a quasi-isomorphism of this complex with the total complex of this double complex, namely $\Gamma(X, T^\bullet)$. Hence, the cohomology of the Čech complex of groups coincides with the cohomology of this total complex, which was shown above to coincide with the hypercohomology of the sheaf A as computed using injective resolutions. □

5. The sheaf of chains

5.1. The problem. Most of the complexes of sheaves that were discussed until now have the property that their cohomology sheaves live only in degree zero. The sheaf of chains, however, is a naturally occurring entity with complicated cohomology sheaves. But it is not so obvious how to construct a sheaf on X that corresponds to (say) the singular chains. How do you define the restriction of a singular simplex that might wrap many times around the whole space? One might define a presheaf C_r with sections $\Gamma(U, C_r) = C_r(U)$ to be the group of singular chains on U, with restriction mapping $C_r(U) \to C_r(V)$ that assigns zero to every singular simplex that is not completely contained in V but this map may fail preserve the boundary homomorphism (but see §5.3 below).

It is necessary to confront the problem of restricting (to V) a singular simplex $\sigma \subset U$ that is not completely contained in V. The solution is to divide $\sigma \cap V$ into (possibly) infinitely many simplices and to call this the restriction $\sigma|V$. So the sheaf of chains requires the use of non-compact chains that may be infinite sums of simplices. If the space X is paracompact then we may restrict to sums that are locally finite.

5.2. Borel-Moore homology. Borel and Moore defined a sheaf $\underline{\underline{C}}^\bullet_{BM}$ (that is, a complex of sheaves) whose presheaf of sections $\Gamma(U, \underline{\underline{C}}^r_{BM})$ is the "locally finite r dimensional chains in U". The cohomology of this sheaf is called the Borel-Moore homology of the space [18]. There are a number of rigorous constructions of this sheaf see §7.2 and §11.1. For the moment, let us take it on faith that such a sheaf (or rather, complex of sheaves) exist. The impatient reader may jump to §11.1 for the general definition, where the sheaf of chains becomes an essential actor. The usual (e.g. singular) homology of X is the compact support cohomology:

$$H_c^{-i}(X; \underline{\underline{C}}^\bullet_{BM}) = H_i(X).$$

The stalk cohomology of the sheaf of chains is the local homology: $\underline{\underline{H}}_x^{-r}(\underline{\underline{C}}^\bullet_{BM}) \cong H_r(X, X-x)$. We use negative degrees in the superscript so that the boundary map on chains raises degree, as with all complexes of sheaves.

If X is a (topological) manifold of dimension n then the top degree local homology group (say, with integral coefficients), $H_n(X, X-x; \mathbb{Z})$, is isomorphic to $\mathbb{Z}$ but such an isomorphism involves a choice of local orientation of X near x. This local homology sheaf $\mathcal{O} = \underline{\underline{H}}^{-n}({}^{\bar{p}}C^\bullet_{BM})$ is the *orientation sheaf* of X. An orientation of X (if one exists) is a global section of the orientation sheaf.

5.3. Finite type. Let us say that a topological space X has *finite type* if it is homeomorphic to $K - L$ where K is a finite simplicial complex and L is a closed subcomplex. In this case $H_r^{BM}(X) \cong H_r(K, L)$ coincides with the relative homology. Therefore it can then be expressed as the homology of a chain complex formed by the simplices that are contained only in K, and by defining the differential so as to ignore all components of the boundary that may lie in L. This gives a simple combinatorial and computable construction of Borel-Moore homology for spaces of finite type, although it does not describe the Borel-Moore *sheaf*.

6. Homotopy and injectives

6.1. Homotopy theory. Recall from §A.11 that two morphisms $f, g : A^\bullet \to B^\bullet$ of complexes of sheaves are said to be *homotopic* if there is a collection of morphisms $h : A^r \to B^{r-1}$ so that $hd_A + d_B h = f - g$. This is an equivalence relation. Equivalence classes are referred to as homotopy classes of maps; the set of which is denoted $[A^\bullet, B^\bullet]$. From Appendix A we have the following facts.

Let $\phi : A^\bullet \to B^\bullet$ be a quasi-isomorphism of (bounded below) complexes of sheaves. The following statements hold.

(1) (Lemma A.14 For any complex $J^\bullet$ of injectives, the induced map $[B^\bullet, J^\bullet] \to [A^\bullet, J^\bullet]$ on homotopy classes is an isomorphism. If $A^\bullet$ is injective then there exists a homotopy inverse $g : B^\bullet \to A^\bullet$ (meaning that $g\phi \sim I_A$ and $\phi g \sim I_B$).
(2) (Lemma A.13) Suppose that the cohomology sheaves $\mathbf{H}^r(A^\bullet) = 0$ for all r. Let $J^\bullet$ be a complex of injective sheaves. Then any morphism $f : A^\bullet \to J^\bullet$ is homotopic to zero, meaning that there exists $h : A^\bullet \to J^\bullet[-1]$ such that $d_J h + hd_C = f$.
(3) (Proposition A.17) if $A \to I^\bullet$ and $B \to J^\bullet$ are injective resolutions then the morphism ϕ admits a lift $\tilde{\phi} : I^\bullet \to J^\bullet$ and any two such lifts are homotopic.

7. The derived category

Let X be a topological space and let R be a commutative ring. The category of $\mathcal{S}(X)$ of sheaves of R-modules on X is Abelian. In §A.20 the corresponding (bounded) derived category $D^b(X)$ is defined. Recall that an object in this category is a (bounded below) complex of sheaves on X.

Let $K^b(X)$ denote the homotopy category of (bounded below) complexes of sheaves on X. From §A.20 we have:

- The mapping $K^b(X) \to D^b(X)$ is a functor (that is, a morphism between complexes determines a morphism in the derived category also).
- If $A^\bullet \to B^\bullet$ is a quasi-isomorphism of complexes of sheaves then it becomes an isomorphism in $D^b(X)$ (so it has an inverse in $D^b(X)$).
- if $A^\bullet$ is a complex of sheaves such that $H^m(A^\bullet) = 0$ for all m then $A^\bullet$ is isomorphic to the zero sheaf.
- if $\mathcal{B}$ is an Abelian category with enough injectives then any left exact functor $F : \mathcal{S}(X) \to \mathcal{B}$ passes to a right derived functor $RF : D^b(X) \to D^b(\mathcal{B})$.

7.1. Examples. 1 If $f : X \to Y$ is a continuous map between topological spaces, then $f_* : \mathcal{S}(X) \to \mathcal{S}(Y)$ takes injectives to injectives. Taking $Y = \{pt\}$ we get that the global sections functor Γ takes injectives to injectives. Then, for any complex of sheaves $A^\bullet$,

$$H^m(X, A^\bullet) = R^m\Gamma(X, A^\bullet) = H^m(\Gamma(X, I^\bullet))$$

where $A^\bullet \to I^\bullet$ is an injective resolution. (see also §**??**.)

2. Let $f : X \to Y$ be a continuous map and let $\underline{\underline{\mathbb{Z}}}$ be the constant sheaf on X. If f is surjective and its fibers are connected then $f_*(\underline{\underline{\mathbb{Z}}})$ is again the constant sheaf, because as a presheaf, $f_*(\underline{\underline{\mathbb{Z}}}(U) = \underline{\underline{\mathbb{Z}}}(f^{-1}(U)) = \mathbb{Z}$ for any connected open set $U \subset Y$. Although we cannot hope to understand $Rf_*(\underline{\underline{\mathbb{Z}}})$ we can understand its cohomology sheaves:

$$R^m f_*(\underline{\underline{\mathbb{Z}}})(U) = H^m(\Gamma(f^{-1}(U), I^\bullet))$$

where $I^\bullet$ is an injective resolution (or perhaps the canonical injective resolution) of the constant sheaf. But this is exactly the definition of the hypercohomology $H^m(f^{-1}(U), \underline{\underline{\mathbb{Z}}}) = H^m(f^{-1}(U, \mathbb{Z}))$. If f is proper then the stalk cohomology (of the cohomology sheaf of $Rf_*(\underline{\underline{\mathbb{Z}}})$) at a point $y \in Y$ is equal to $H^m(f^{-1}(y); \mathbb{Z})$, the cohomology of the fiber. In other words, the sheaf $\underline{\underline{\mathbf{H}}}^m(Rf_*(\underline{\underline{\mathbb{Z}}}))$ is a sheaf on Y which, whose stalk at each point is the cohomology of the fiber.

3. We can also determine the global cohomology of the complex $Rf_*(\underline{\underline{\mathbb{Z}}})$, for it is the cohomology of the global sections $\Gamma(X, I^\bullet)$, that is, the cohomology of X. More generally, the same argument shows that: for any complex of sheaves $A^\bullet$ on X, the complex $Rf_*(A^\bullet)$ is a sheaf on Y whose global cohomology is

$$H^*(Y, Rf_*(A^\bullet)) \cong H^*(X, A^\bullet).$$

This complex of sheaves therefore provides data on Y which allows us to compute the cohomology of X. It is called the Leray Sheaf (although historically, Leray really considered only its cohomology sheaves $R^m f_*(A^\bullet)$). In particular we see that *the functor Rf_* does not change the hypercohomology.* For $f : X \to \{\mathrm{pt}\}$, if S is a sheaf on X then $f_*(S) = \Gamma(X, S)$ is the functor of global sections (or

rather, it is a sheaf on a single point whose value is the global sections), so $R^i f_*(A^\bullet) = H^i(X, A^\bullet)$ is the hypercohomology.

4. The m-th derived functor of Hom is called Ext^m, (cf. Proposition **??**) i.e., it is the group

$$\mathrm{Ext}^m(A^\bullet, B^\bullet) = H^m(\mathrm{RHom}(A^\bullet, B^\bullet)) = H^m(\mathrm{Hom}^\bullet(A^\bullet, J^\bullet)) = H^0(\mathrm{Hom}^\bullet(A^\bullet, J^\bullet[m]))$$

where $B^\bullet \to J^\bullet$ is an injective resolution. (We consider $\mathrm{Hom}(A^\bullet, B^\bullet)$ to be a functor of the $B^\bullet$ variable and derive it by injectively resolving. The same result can be obtained by *projectively* resolving $A^\bullet$.) As in §1.9, there is a sheaf version of Hom, which also gives a sheaf version of Ext:

$$\underline{\underline{\mathrm{Ext}}}^m(A^\bullet, B^\bullet)) = \underline{\underline{H}}^m(\underline{\underline{\mathrm{RHom}}}(A^\bullet, B^\bullet))$$

Exercise. Let G, H be abelian groups, as complexes in degree zero. Show that $\mathrm{Ext}^1(G, H)$ coincides with the usual definition of $\mathrm{Ext}_{\mathbb{Z}}(G, H)$,

7.2. The sheaf of chains in the derived category. Suppose X is a piecewise-linear space, that is, a topological space together with a family of piecewise-linearly related triangulations by locally finite countable simplicial complexes. Let $U \subset X$ be an open subset, let T be a locally finite triangulation of U, and let $C_r^T(U)$ be the group of r-dimensional simplicial chains with respect to this triangulation. Then the *sheaf of piecewise linear chains* is the sheaf $\underline{\underline{\mathbf{C}}}^\bullet_{PL}$ with sections

$$\Gamma(U, \underline{\underline{\mathbf{C}}}^{-r}_{PL}) = \varinjlim_T C_r^T(U)$$

for $r \geq 0$. Such a section is a locally finite but possibly infinite linear combination of oriented simplices. (We place the chains in negative degrees so that the differentials will increase degree; it is a purely formal convention.)

The sheaf $\underline{\underline{\mathbf{C}}}^{-r}_{PL}$ is *soft*, and the resulting complex

$$\underline{\underline{\mathbf{C}}}^0_{PL} \longleftarrow \underline{\underline{\mathbf{C}}}^{-1}_{PL} \longleftarrow \underline{\underline{\mathbf{C}}}^{-2}_{PL} \longleftarrow \cdots$$

is quasi-isomorphic to the sheaf of Borel-Moore chains.

If the space X has a real analytic (or semi-analytic or subanalytic or $\mathcal{O}$-minimal) structure then one similarly has the sheaf of locally finite subanalytic or $\mathcal{O}$-minimal chains, which gives another quasi-isomorphic "incarnation" of the sheaf of chains. See also §11.1.

7.3. Exact sequence of a pair. Let Z be a closed subspace of a topological space X, and let $U = X - Z$, say $Z \xrightarrow{i} X \xleftarrow{j} U$.

If S is a sheaf on X then there is a short exact sequence of sheaves $0 \to j_! j^* S \to S \to i_* i^* S \to 0$. The morphisms are obtained by adjunction, and exactness can be checked stalk by stalk: If $x \in Z$ then the sequence reads $0 \to 0 \to S_x \to S_x \to 0$. If $x \in U$ then the sequence reads $0 \to S_x \to S_x \to 0 \to 0$. Consequently if $A^\bullet$ is a complex of sheaves then there is a distinguished triangle

(7.3.1)
$$\begin{tikzcd} Rj_!j^*(A^\bullet) \arrow[rr] & & A^\bullet \arrow[dl] \\ & Ri_*i^*(A^\bullet) \arrow[ul, "{[1]}"] & \end{tikzcd}$$

Recall from Lemma A.8 (which is an exercise) that the cokernel of an injective morphism is quasi-isomorphic to the mapping cone. If X is compact the triangle gives an exact sequence

$$\cdots \to H^r(X; A^\bullet) \to H^r(Z; A^\bullet) \to H_c^{r+1}(U; A^\bullet) \to \cdots$$

but observe that the relative cohomology $H_c^*(U; A^\bullet) = H^*(X, Z; A^\bullet)$ is the cohomology of cochains on X that vanish on Z. If $A^\bullet$ is the sheaf of chains this gives an exact sequence $H_r(U) \to H_r(X) \to H_r(X, U) \to \cdots$ because $i^*(\text{chains})$ is the limit over open sets containing Z of Borel-Moore chains on that open set.

7.4. Proposition. *Let $A^\bullet$ be a complex of sheaves on X. The natural morphism $i_!i^!A^\bullet \to A^\bullet$ gives a distinguished triangle, Verdier dual to (7.3.1),*

$$\begin{tikzcd} Ri_*i^!(A^\bullet) \arrow[rr] & & A^\bullet \arrow[dl] \\ & Rj_*j^*(A^\bullet) \arrow[ul, "{[1]}"] & \end{tikzcd}$$

Consequently the *cohomology supported in Z* is:

$$H_Z^j(S^\bullet) := H^j(Z; i^!S^\bullet) = H^j(X; Ri_*i^!S^\bullet) = H^j(X, X - Z; S^\bullet)$$

and the triangle gives the long exact cohomology sequence for the pair (X, U). For the (Borel-Moore) sheaf of chains, $i^!C_{-r}$ gives the homology of a regular neighborhood of Z in X which, for most nice spaces, will be homotopy equivalent to Z itself. The sheaf $j^*(C^{-r})$ gives the Borel-Moore homology of U, which is the relative homology $H_r(X, Z)$. So this triangle gives the long exact sequence for the *homology* sequence of the pair (X, Z).

Caution. For a complex of sheaves $A^\bullet$ on X and inclusion $j_x : x \to X$ of a point, the stalk cohomology of $A^\bullet$ is $H_x^r(A^\bullet) = H^r(j_x^*A^\bullet)$ but the cohomology supported at x is $H_{\{x\}}^r(A^\bullet) = H^r(j_x^!A^\bullet)$.

8. Stratifications

8.1. The plan. The plan is to decompose a reasonable space into a locally finite union of smooth manifolds (called strata) which satisfy the axiom of the frontier: the closure of each stratum should

be a union of lower dimensional strata. If $Y \subset \overline{X}$ are strata we write $Y < X$. But this should be done in a locally trivial way. In Whitney's example below, it is not enough to divide this figure

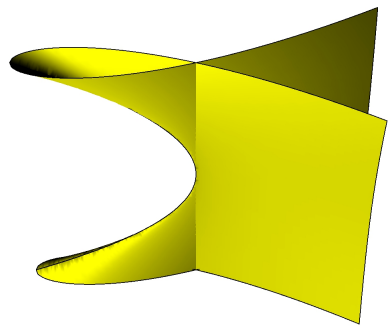


FIGURE 1. Three strata are required.

into 1- and 2-dimensional strata, even though this gives a decomposition into smooth manifolds. If the origin is not treated as another stratum then the stratification fails to be "locally trivial". Whitney proposed a condition that identifies the origin as a separate stratum in this example. Let us say that a stratification of a closed subset W of some smooth manifold M is a locally finite decomposition $W = \coprod_\alpha S_\alpha$ into locally closed smooth submanifolds $S_\alpha \subset M$ (called strata) so as to satisfy the axiom of the frontier.

For convenience let us say that strata are connected. A proper continuous mapping $f : W_1 \to W_2$ between two stratified sets is a *stratified mapping* if it takes strata to strata submerisvely, that is, if $X \subset W_1$ is a stratum then $f(X)$ is contained in a stratum Y of W_2 and $f : X \to Y$ is a smooth proper submersion. This implies that $f(X) = Y$ and that $f^{-1}(Y)$ is a union of strata.

8.2. Definition. Let $Y \subset \overline{X}$ be strata in a stratification of a closed set $W \subset M$. The pair (X, Y) *satisfies Whitney's condition B* at a point $y \in Y$ if the following holds. Suppose that $x_1, x_2, \cdots \in X$ is a sequence that converges to y, and suppose that $y_1, y_2, \cdots \in Y$ is a sequence that also converges to y. Suppose that (in some local coordinate system near y) the secant lines $\ell_i = \overline{x_i, y_i}$ converge to some limiting line ℓ. Suppose that the tangent planes $T_{x_i}X$ converge to some limiting plane τ. Then $\ell \subset \tau$.

We say the pair (X, Y) satisfies condition B if it does so at every point $y \in Y$. The decomposition into strata is a *Whitney stratification* if every pair of strata $Y < X$ satisfies condition B at every point in the smaller stratum Y. If condition B is satisfied, and if the tangent planes $T_{y_i}Y$ also converge to some limiting plane η then $\eta \subset \tau$ as well, which Whitney had originally proposed as an additional condition, which he called Condition A.

It turned out that Whitney's condition B was just the right condition to guarantee that a stratification is locally trivial, but the verification involved the full development of stratification theory by René Thom ([115, 116]) and John Mather ([94]). The problem is that stratifications

satisfying Condition B may still exhibit certain pathologies, such as infinite spirals, so there is a very delicate balance between proving that local triviality holds while avoiding a host of counterexamples to similar sounding statements.

8.3. Suppose $W \subset M$ has a stratification that satisfies condition B. Let Y be a stratum and let $y \in Y$. Let $N_y \subset M$ be a *normal slice*, that is, a smooth submanifold of dimension $\dim(N_y) = \dim(M) - \dim(Y)$ that intersects Y transversally in the single point $\{y\}$. (cf. §17.2) It follows from Condition A that N is also transverse to all strata $Z > Y$ in some neighborhood of $\{y\}$. Define the *link of the stratum* Y at the point $y \in Y$,

$$L_Y(y, \epsilon) = (\partial B_\epsilon(y)) \cap N_y \cap W$$

where $B_\epsilon(y)$ is a ball of radius ϵ (measured in some Riemannian metric on M) centered at the point y. The following result, due to R. Thom and J. Mather, is a *local structure theorem* for Whitney stratified spaces:

8.4. Theorem. *If ϵ is chosen sufficiently small then*

1. *The closed set $L_Y(y, \epsilon)$ is stratified by its intersection with the strata Z of W such that $Z > Y$.*
2. *This stratification satisfies condition B.*
3. *The stratified homeomorphism type of $L_Y(y, \epsilon)$ is independent of the choice of N_y, ϵ, and the Riemannian metric so we may denote it by $L_Y(y)$.*
4. *If the stratum Y is connected then the stratified homeomorphism type of $L_Y(y)$ is also independent of the point y so we may denote it by L_Y. (So it is notationally convenient to assume henceforth that all strata are connected.)*

Moreover, the point y has a basic neighborhood *$U_y \subset W$ and a stratified homeomorphism*

$$U_y \cong c^o(L_Y) \times B_1$$

where $c^o(L_Y)$ denotes[a] *the open cone on L_Y (with its obvious stratification) and where B_1 denotes the open ball of radius 1 in $\mathbb{R}^{\dim(Y)}$.*

This homeomorphism preserves strata in the obvious way: it takes

1. $\{y\} \times B_1 \to Y \cap U$ with $\{y\} \times \{0\} \to \{y\}$
2. $(L_Y \cap X) \times (0, 1) \times B_1 \to X \cap U$ for each stratum $X > Y$

[a] here, we interpret $c^o(L_Y) = \{y\}$ if $L_Y = \phi$ is empty, that is, if y is a point in a maximal stratum.

Theorem 8.4 says that the set W does not have infinitely many holes or infinitely much topology as we approach the singular stratum Y and it says that the normal structure near Y is locally trivial as we move around in Y. In particular, the collection of links $L_Y(y)$ form the fibers of a stratified fiber bundle over Y. It also implies that (for any $r \geq 0$) the local homology $H_r(W, W - y; \mathbb{Z})$ forms

a local coefficient system on Y with stalk

$$H_r(W, W - y; \mathbb{Z}) \cong H_{r-\dim(Y)-1}(L_Y; \mathbb{Z}).$$

These statements are consequences of the *first isotopy lemma* of R. Thom, which is best explained and proven in Mather's notes [94].

8.5. Lemma. *Suppose M, P are smooth manifolds and $f : M \to P$ is a smooth proper map. Let $W \subset M$ be a Whitney stratified (closed) subset, and suppose that $f|X : X \to P$ is a submersion for each stratum X of W. Then the mapping $f|W : W \to P$ is locally trivial: each point $p \in P$ has a neighborhood basis consisting of open sets U_p for which there exists a stratum preserving homeomorphism*

$$f^{-1}(U_p) \cong U_p \times f^{-1}(p)$$

that is smooth on each stratum.

8.6. In fact, Thom [115] and Mather [94] proved that a Whitney stratified W set admits a system of *control data* consisiting of a triple (T_Z, π_Z, ρ_Z) for each stratum Z, where T_Z is a neighborhood of Z in W, where $\pi_Z : T_Z \to Z$ is a "tubular projection", $\rho_Z : T_Z \to [0, \epsilon)$ is a "tubular distance function" so that for each stratum $Y > Z$ the following holds:

(1) $\pi_Z \pi_Y = \pi_Z$ in $T_Z \cap T_Y$
(2) $\rho_Z \pi_Y = \rho_Z$ in $T_Z \cap T_Y$
(3) the pair $(\pi_Z, \rho_Z)|Y \cap TZ : Y \cap T_Z \to Z \times (0, \epsilon)$ is a smooth submersion.

Theorem 8.4 then follows by applying the isotopy lemma 8.5 to the mapping (π_Z, ρ_Z).

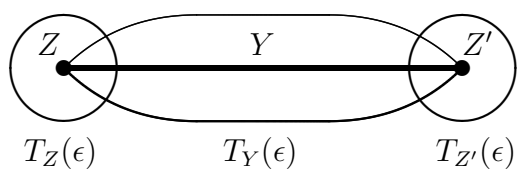


FIGURE 2. Tubular neighborhoods

8.7. The local triviality statements of the isotopy lemma (Lemma 8.5) are proven in [115, 94], using the control data to construct *controlled vector fields* whose flow traces out the local triviality of the stratification. It is a delicate construction: a controlled vector field is not necessarily continuous but it has a continuous flow. Such a flow may "spin" around a small stratum, faster and faster for points that are close to the small stratum.

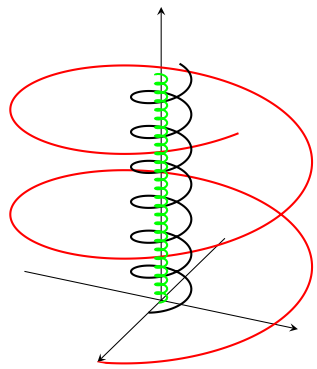


FIGURE 3. Controlled flow near the Z axis

8.8. For each stratum S in a Whitney stratified set the union $U_S = \bigcup_{R \leq S} T_R$ form a neighborhood of the closure $\overline{S}$ and the inclusion $\overline{S} \to U_S$ is a homotopy equivalence. In fact there is a deformation retraction, $r_1 : U_S \to \overline{S}$ that is the time $= 1$ value of a one parameter family of maps $r : U_S \times [0,1] \to U_S$ such that $r_0 = I$ (identity) and each r_t $(t < 1)$ is stratum preserving and smooth on each stratum, see Figure 4. [40, 41, 43] The mapping r_1 is a "weak" deformation retraction: its restriction to $\overline{S}$ is the identity outside of a suitable neighborhood V_S of $\overline{S} - S$ (for example, $V_S = T_R(2\epsilon)$ in Figure 4) but it is only homotopic to the identity within $V_S \cap \overline{S}$.

8.9. Whitney himself outlined a procedure for proving that any closed subset W of Euclidean space defined by analytic equations admits a Whitney stratification. The idea is to start with the open, nonsingular part W^0 of W as the "top" stratum, and then to look at the set of points in the singular set $\Sigma = W - W^0$ where condition B *fails*. He proves that this is an analytic subset of codimension two, whose complement in Σ is therefore the first singular stratum, W^1. Now, carry both W^0 and W^1 along, looking at the set of points (in what remains) where condition B fails, and continue in this way inductively. Since Whitney's early work, many advances have been made in the subject (see [127]). Theorem 8.10 below is a partial summary of the work of many people, including [62], [63] [64], [95], [83], [128], [117], [16], [40], [41], [67], [84].

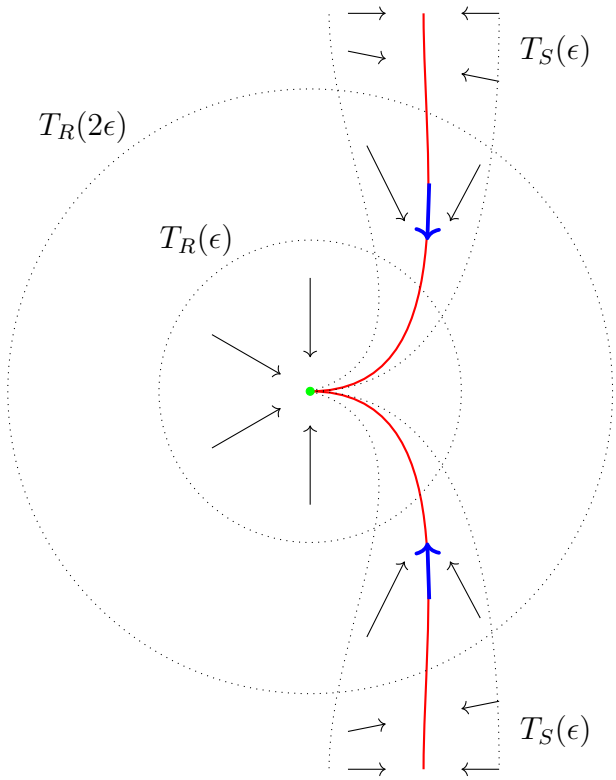


FIGURE 4. Tubular neighborhoods and retraction for $R < S$

8.10. Theorem. *The following sets admit Whitney stratifications: real and complex algebraic varieties, real and complex analytic varieties, semi-algebraic and semi-analytic varieties, subanaltyic sets, and sets with o-minimal structure. Given such an algebraic (resp. analytic etc.) variety W and a locally finite union Z of algebraic (resp. analytic etc.) subvarieties, the stratification of W can be chosen so that Z is a union of strata. Given an algebraic (resp. proper analytic etc.) mapping $f : W \to W'$ of algebraic (resp. analytic etc.) varieties, it is possible to find algebraic (resp. analytic) Whitney stratifications of W, W' so that the mapping f is stratified (see §8.1). Whitney stratified sets can be triangulated by a triangulation that is smooth on each stratum, such that that the closure of each stratum is a subcomplex of the triangulation.*

A subanalytic set is the image under a projection (for example, a linear projection $\mathbb{R}^m \to \mathbb{R}^n$) of an analytic or a semi-analytic set. O-minimal structures allow for certain non-analytic functions to be included in the definition of the set. Whitney stratifications also make sense for algebraic varieties defined over fields of finite characteristic. Given an algebraic mapping $f : W \to W'$

between complex algebraic varieties, it is not generally possible to choose triangulations of W, W' so that f becomes a simplicial mapping.

9. Constructible sheaves

9.1. A topological space X with locally infinite topology, such as the Cantor set or the Alexander horned sphere, behaves very badly with respect to homological duality. If $j : X \to M$ is an embedding in a smooth manifold then $j_*(\mathbb{Z}_X)$ is a "bad" sheaf on M with similarly bad behavior. Historically, various technical conditions of "constructibility" for sheaves were proposed[3] to eliminate this sort of pathology. In this section we describe the derived category $D^b_c(W)$ of complexes whose cohomology sheaves are bounded and constructibe. It forms a "paradise", in the words of Verdier[4], because it is closed under pullback, proper push forward, Hom, and Verdier duality.

9.2. Constructible sheaves. Fix a Whitney stratification of a closed subset $W \subset M$ of some smooth manifold. A sheaf S (of abelian groups, or of R modules) on W is *constructible with respect to this stratification* if the restriction $S|X$ to each stratum X is a locally constant sheaf and the stalks S_x are finitely generated. A complex of sheaves $A^\bullet$ on W is *cohomologically constructible* with respect to this stratification if its cohomology sheaves are bounded (that is, $\underline{\underline{H}}^r(A^\bullet) = 0$ for $|r|$ sufficiently large) and constructible.

If W is an complex algebraic (resp. complex analytic, resp. real algebraic etc.) variety then a complex of sheaves $A^\bullet$ on W is *algebraically construcible* (resp. analytically constructible, etc.) if its cohomology sheaves are bounded and constructible with respect to some algebraic (resp. analytic etc.) Whitney stratification.

In each of these constructibility settings (that is, constructible with respect to a fixed stratification, or algebraically constructible, etc.) the two constructions of the derived category make sense (as the homotopy category of injective complexes, or as the category of complexes and equivalence classes of roofs), which is then (ambiguously) referred to as the bounded constructible derived category and denoted $D^b_c(W)$.

9.3. Theorem. *Suppose that W is a compact subset of some smooth manifold M and suppose that $A^\bullet$ is a complex of sheaves that is cohomologically constructible with respect to some Whitney stratification of W. Then the hypercohomology groups $H^r(W, A^\bullet)$ are finitely generated. If U_x is a basic neighborhood of $x \in W$ then the stalk cohomology $\underline{\underline{H}}^r(A^\bullet)_x$ coincides with the cohomology $H^r(U_x, A^\bullet)$ for all r (and so the limit over open sets containing x is essentially constant). If $i : Z \to W$ is a closed union of strata with open complement $j : U \to W$ then $Ri_*i^*(A^\bullet)$ and $Rj_*j^*(A^\bullet)$ are also cohomologically constructible. If $A^\bullet, B^\bullet$ are cohomologically constructible then so is $\mathrm{RHom}^\bullet(A^\bullet, B^\bullet)$. If $f : W \to W'$ is a proper stratified mapping and $A^\bullet$ is CC on W then $Rf_*(A^\bullet)$ is CC on W'.*

[3] such as condition (P,Q) of Wilder [129], condition clc and hlc of Borel-Moore [18]

[4] who assured the author, before constructibility was well understood, that such a paradise did not exist

Proof. Let X be the top stratum and let $\Sigma = W - X$ be the singular set. Let U be the union of the tubular neighborhoods of the strata in Σ. Then $X - (X \cap U)$ is compact, and as U shrinks these form a sequence of diffeomorphic compact manifolds with corners that exhaust X. If $\mathcal{L}$ is a local system on X then (since W is compact)

$$H_c^*(X, \mathcal{L}) = H^*(X - (X \cap U), \mathcal{L})$$

is finitely generated. From the spectral sequence for cohomology of a complex, the same holds for $H_c^*(X, A^\bullet)$ since the cohomology sheaves of $A^\bullet$ are local systems on X.

Now consider the exact triangle

$$Rj_! j^* A^\bullet \to A^\bullet \to Ri_* i^*(A^\bullet) \to \cdots .$$

The cohomology of $Rj_! j^* A^\bullet$ is $H_c^*(X, A^\bullet)$ which is finitely generated as just shown. The complex $i^*(A^\bullet)$ is a constructible complex on Σ, which has smaller dimension, so its cohomology is finitely generated by induction. The long exact sequence implies the cohomology of $A^\bullet$ is finitely generated.

The stalk cohomology coincides with the cohomology of U_x because the family of these basic neighborhoods are cofinal in the set of all neighborhoods of x but as they shrink there are stratified isomorphisms $h : U_x \to U'_x$ with the inverse given by inclusion. Since the cohomology sheaves of $A^\bullet$ are locally constant on each stratum there is a quasi-isomorphism $h_*(A^\bullet) \to A^\bullet$ which induces isomorphisms on cohomology. In other words, $H^*(U_x, A^\bullet)$ is independent of the choices, so the limit stabilizes. Constructibility of $Ri_* i^* A^\bullet$ is obvious but constructibility of $Rj_* j^* A^\bullet$ takes some work. Here is the key point:

9.4. Lemma. *Let $A^\bullet$ be a cohomologically constructible complex of sheaves on W. Let $Z \subset W$ be a closed subset with complement $j : V \to W$. Let X be the largest stratum of Z. Then the stalk cohomology at $x \in X$ of $Rj_* j^*(A^\bullet)$ is*

$$\underline{\underline{H}}^i(Rj_* j^* A^\bullet)_x \cong H^i(L_X(x), A^\bullet). \tag{9.4.1}$$

Here, $L_X(x)$ denotes the link of the stratum X in W at the point x, cf §8.3. The Lemma follows from the fact that the stalk cohomology is $\underline{\underline{H}}^i(Rj_* j^* A^\bullet)_x = H^i(j^{-1}(U_x \cap V); A^\bullet)$ where $U_x \cong c^o(L_X(x)) \times B^{\dim(X)}$ is a basic open neighborhood. Since X is the largest stratum of Z, we have:

$$U_x \cap V \cong L_X(x) \times (0,1) \times B^{\dim(X)},$$

and the cohomology sheaves of $A^\bullet$ are constant in the Euclidean directions of this product.

Since $L_X(x)$ is compact this cohomology is finitely generated. It is locally constant as $x \in X$ varies because the same is true (Lemma 8.5) of $L_X(x)$ and of the cohomology sheaves of $A^\bullet | L_X(x)$.

The constructibility of $\mathrm{RHom}^\bullet(A^\bullet, B^\bullet)$ follows from the spectral sequence for its stalk cohomology in terms of the stalk cohomology of $A^\bullet$ and $B^\bullet$. The last statement in Theorem 9.3 follows from the isotopy lemma, Lemma 8.5. □

9.5. If W is complex algebraic, analytic, real algebraic, subanalytic, or definable, then the corresponding category $D^b_c(W)$ is defined to have objects that are complexes of sheaves whose cohomology sheaves are locally constant along strata of *some* complex algebraic (resp. analytic, resp. real algebraic, resp. subanalytic, resp. definable) stratification of W. In these cases we may refer to a complex $A^\bullet \in D^b_c(W)$ without ever referring to a particular stratification.

9.6. Attaching sheaves. Let us examine the triangle for $Rj_*j^*A^\bullet$ for $i \leq \mathrm{cod(X)} - 1$ and its stalk cohomology:

$$\begin{tikzcd} Ri_*i^!(A^\bullet) \arrow[rr] & & A^\bullet \arrow[dl, "\alpha"] \\ & Rj_*j^*(A^\bullet) \arrow[ul, "\beta", "{[1]}"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} H_c^{i-\dim(X)}(U_x; A^\bullet) \arrow[rr] & & \underline{\underline{H}}^i(A^\bullet)_x \arrow[dl, "\alpha"] \\ & H^i(L_x, A^\bullet) \arrow[ul, "\beta", "{[1]}"'] & \end{tikzcd} \tag{9.6.1}$$

The *attaching map* α goes from information $(\underline{\underline{H}}^i(A^\bullet)_x)$ living on the small stratum to information $(H^i(L_x, A^\bullet))$ living completely in the larger strata and so it represents the degree to which the sheaf $A^\bullet$ is "glued" across the strata.

Exercise. Suppose $W = X < U$ consists of two strata. Let $B^\bullet, C^\bullet$ be sheaves on X and on U respectively and let $A^\bullet = Ri_*(B^\bullet) \oplus Rj_!(C^\bullet)$ so that $A^\bullet$ consists of just these two sheaves with no relation between them. Show that the attaching homomorphism α is zero. For example, if $B^\bullet = \mathbb{Z}_X$ and if $C^\bullet = \mathbb{Z}_U$ then $A^\bullet$ is a sheaf whose stalk at each point is $\mathbb{Z}$ however it is not the constant sheaf. Show that if $A^\bullet = \mathbb{Z}_W$ is the constant sheaf then the attaching homomorphism α is injective.

10. Duality

10.1. Pseudomanifolds and Poincaré duality. A pseudomanifold of dimension n is a purely n dimensional (Whitney) stratified space that can be triangulated so that every $n-1$ dimensional simplex is a face of exactly two n-dimensional simplices. This implies that the $n-1$ dimensional simplices can be combined with the n-dimensional simplices to form an n-dimensional manifold, and that the remainder (hence, the "singularity set") has dimension $\leq n-2$. If this manifold is orientable then an orientation defines a fundamental class $[W] \in H_n(W;\mathbb{Z})$. Cap product with the fundamental class defines the Poincaré duality map $H^r(W;\mathbb{Z}) \to H_{n-r}(W;\mathbb{Z})$ which is an isomorphism if W is a manifold (or even a homology manifold) but which, in general, is not an isomorphism.

There is a sheaf-theoretic way to say this. If W is oriented and n-dimensional then a choice of orientation determines a sheaf map $s : \underline{\underline{\mathbb{Z}}}_W \to \underline{\underline{C}}_n^{BM}$ from the constant sheaf to the sheaf of n-chains, cf. §5: Let $U \subset M$ be an open set and choose a triangulation of U. If U is sufficiently small it is possible to orient all of the n dimensional simplices in U in a compatible way, so that $\partial \sum_{\sigma \subset U} \sigma = 0$ where the sum is over those oriented n-dimensional simplices in U. Then set

$$s(m) = m \sum_{\sigma \subset U} m\sigma.$$

(Recall that a PL chains are identified under subdivision.) For example, if W is an n-dimensional homology manifold, that is, if $H_r(W, W-x) = 0$ for all $0 \le r < n$ and $H_n(W, W-x; \mathbb{Z}) = \mathbb{Z}$ then the composition

$$\underline{\underline{\mathbb{Z}}}_W[n] \to \underline{\underline{C}}_n^{BM} \to \underline{\underline{C}}_W^{\bullet}$$

is a quasi-isomorphism. This simple statement is the Poincaré duality theorem. For, it says that this quasi-isomorphism induces an isomorphism on cohomology, that is,

$$H^r(W; \mathbb{Z}) \cong H_{n-r}^{BM}(W; \mathbb{Z})$$

and an isomorphism on cohomology with compact supports, that is,

$$H_c^r(W; \mathbb{Z}) \cong H_{n-r}(W; \mathbb{Z}).$$

[Actually, from this point of view, the deep fact is that $H^i(W; k)$ and $H_i(W; k)$ are dual over any field k, but this is not a fact about manifolds. Rather, it is a fact about the sheaf of chains.]

More generally if W is a homology manifold but not necessarily orientable then the *orientation sheaf* $\mathcal{O}_W$ is the local system whose stalk at $x \in W$ is the top local homology $H_n(W, W-x)$ and the mapping $O_W \to \underline{\underline{C}}_n^{BM} \to \underline{\underline{C}}_W^{\bullet}$ is a quasi-isomorphism. So, for any local coefficient system $\mathcal{L}$ on W the mapping $\mathcal{L} \otimes \mathcal{O}_W \to \underline{\underline{C}}^{\bullet}(\mathcal{L})$ is a quasi-isomorphism, giving an isomorphism on cohomology,

$$H^r(W; \mathcal{L} \otimes \mathcal{O}_W) \cong H_{n-r}^{BM}(W; \mathcal{L})$$

and on cohomology with compact supports,

$$H_c^r(W; \mathcal{L} \otimes \mathcal{O}_W) \cong H_{n-r}(W; \mathcal{L}).$$

So this quasi-isomorphism statement includes the Poincaré duality theorem for orientable and non-orientable manifolds, for non-compact manifolds, and for manifolds with boundary, and with possibly nontrivial local coefficient systems.

11. Verdier duality

11.1. The dual of a complex of sheaves. Suppose X is a paracompact Hausdorff space and R is a commutative ring with finite cohomological dimension. Let $A^\bullet$ be a complex of sheaves of R-modules on X. A. Borel and J. C. Moore defined ([18]) its dual $\mathbf{D}(A)^\bullet$ as follows. First, choose a soft or flabby resolution $A^\bullet \cong S^\bullet$. Then $\mathbf{D}(A)^\bullet$ is the sheafification of the complex of presheaves

$$\mathbf{D}(A)^{-j}(U) = \mathrm{RHom}_{R-mod}(\Gamma_c(U, S^j), R) = \mathrm{Hom}_{R-mod}(\Gamma_c(U, S^j), I^\bullet)$$

where $I^\bullet$ is an injective resolution[5] of (of finite length) of the ring R. This means that $\mathbf{D}^\bullet(U)$ is the single complex obtained from the double complex $\mathrm{Hom}^\bullet_{R-mod}(\Gamma_c(U, S^\bullet), I^\bullet)$ by adding along the diagonals in the usual way. If R is a field this is just

$$\mathrm{Hom}_{R-mod}(\Gamma_c(U, S^j), R).$$

[5]For $R = \mathbb{Z}$ take $I^\bullet$ to be $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ and more generally if R is an integral domain take $F \to F/R$ where F is the fraction field of R.

For example when $R = \mathbb{R}$, the sheaf of *currents* on a smooth manifold is the dual of the sheaf of smooth differential forms. For $R = \mathbb{Z}$, Borel and Moore proved that there are exact cohomology sequences

$$0 \to \mathrm{Ext}(H_c^{n+1}(X, S^\bullet), \mathbb{Z}) \to H^{-n}(X, \mathbf{D}(S^\bullet)) \to \mathrm{Hom}(H_c^n(X, S^\bullet), \mathbb{Z}) \to 0$$

in analogy with the universal coefficient theorem for cohomology:

$$0 \to \mathrm{Ext}(H_{n-1}(X, \mathbb{Z}), \mathbb{Z}) \to H^n(X, \mathbb{Z}) \to \mathrm{Hom}(H_n(X, \mathbb{Z}), \mathbb{Z}) \to 0.$$

The Borel-Moore sheaf of chains $\underline{\underline{C}}^\bullet_{BM}$ is the dual of the constant sheaf but first we must replace the constant sheaf by the (quasi-isomorphic) flabby sheaf of singular cochains. So the sheaf of chains is the sheafification of the complex of presheaves

$$C^{-j}_{BM}(U) = \mathrm{RHom}(C^j_c(U; R), R)$$

where $C^j_c(U; R)$ is the subgroup of the group of singular cochains $\mathrm{Hom}_R(C^{sing}_j(U; R), R)$ with compact support. In short, any reasonable injective model of the complex $\underline{\underline{C}}^\bullet_{BM}$ is a mess since we have "dualized" the (cosheaf of) singular chains twice. For most applications we do not require a precise injective model for this sheaf: it is enough to know that one exists.

The Borel-Moore homology (in degree j) is the cohomology (in degree $-j$) of the complex $\underline{\underline{C}}^\bullet_{BM}$. We have previously seen that this is the homology theory of locally finite chains. The problem that Borel and Moore failed to resolve is that the double dual of $S^\bullet$ does not equal $S^\bullet$; rather, it appears to be something far more complicated.

11.2. Dualizing sheaf. Later, Verdier[120], [8] discovered that the Borel-Moore dual could be interpreted sheaf theoretically:

$$\mathbf{D}(S^\bullet) = \underline{\underline{\mathrm{RHom}}}^\bullet(S^\bullet, \mathbb{D}^\bullet)$$

where $\mathbb{D}^\bullet$ is a particular, universal sheaf called the *dualizing complex*. He then showed (assuming $S^\bullet$ is constructible, see below) that there is a canonical quasi-isomorphism (see §12.3) in the derived category,

$$\mathbf{DD}(S^\bullet) \cong S^\bullet.$$

So double duality is restored. And what is this magic dualizing complex? There is no choice: for sheaves of abelian groups, for example, we have:

$$\mathbb{D}^\bullet = \underline{\underline{\mathrm{RHom}}}^\bullet(\underline{\underline{\mathbb{Z}}}, \mathbb{D}^\bullet) = \mathbf{D}(\underline{\underline{\mathbb{Z}}})$$

is the Borel-Moore dual of the constant sheaf, so it is the sheaf of chains! More precisely, it is the quasi-isomorphism class of the complex of sheaves $\underline{\underline{C}}^\bullet_{BM}$ of Borel-Moore chains.

Remark. Dualizing complexes exist in many other categories and Verdier duality is now recognied as the natural categorical statement of duality.

FIGURE 5. Armand Borel (photo by Bill Casselman)

11.3. For the rest of this chapter we assume the coefficient ring R has finite cohomological dimension and we assume all complexes of sheaves are (cohomologically) constructible, a technical condition that will be explained in §9.2, which rules out horrible sheaves like the Cantor sheaf of §1.8(g). Let $S^\bullet, T^\bullet$ be constructible complexes of sheaves of R-modules on a finite dimensional locally compact space X. A pairing $S^\bullet \otimes T^\bullet \to \mathbf{D}^\bullet$ is said to be a *Verdier dual pairing* if the resulting morphism $S^\bullet \to \underline{\underline{\mathrm{RHom}}}^\bullet(T^\bullet, \mathbf{D}^\bullet) = \mathbf{D}(T^\bullet)$ is an isomorphism in $D^b_c(X)$. In particular, this means that if R is an integral domain then for any open set $U \subset X$ there is a short exact sequence,

$$0 \longrightarrow \mathrm{Ext}^1_R(H_c^{-i+1}(U, S^\bullet), R) \longrightarrow H^i(U, \mathbf{D}(S^\bullet)) \longrightarrow \mathrm{Hom}_R(H_c^{-i}(U, S^\bullet), R) \longrightarrow 0.$$

11.4. Definition. If $f : X \to Y$ is a continuous map and $S^\bullet$ is a complex of sheaves on Y define $f^!(S^\bullet) = \mathbf{D}_X f^* \mathbf{D}_Y(S^\bullet)$.

If $f : X \to Y$ is the inclusion of a locally closed subset then this agrees with $f^!$ of §1.14 because both constructions have the same adjointness properties with respect to Rf_*, see (1.14.1) and (12.3.1).

12. Verdier's paradise

12.1. In the following blue boxes we summarize the "calculus" of the derived category. Complete proofs may be found in [19] p. 97-192 but proofs are outlined in this section.

12.2. Lemma. *Let $f : X \to Y$ be a stratified mapping between Whitney stratified spaces. Then*

(1) $f^*(A^\bullet \overset{L}{\otimes} B^\bullet) \cong f^*(A^\bullet) \overset{L}{\otimes} f^*(B^\bullet)$ *for any* $A^\bullet, B^\bullet$ *in* $D^b(X)$

(2) $\underline{\underline{\mathrm{RHom}}}^\bullet(A^\bullet, \underline{\underline{\mathrm{RHom}}}^\bullet(B^\bullet, C^\bullet)) \cong \underline{\underline{\mathrm{RHom}}}^\bullet(A^\bullet \overset{L}{\otimes} B^\bullet, C^\bullet)$ *for any* $A^\bullet, B^\bullet, C^\bullet$ *in* $D^b(X)$.

(3) $f_!(A^\bullet \overset{L}{\otimes} f^*(B^\bullet)) \cong f_!(A^\bullet) \overset{L}{\otimes} B^\bullet$ *for any* $A^\bullet$ *in* $D^b(X)$, *and* $B^\bullet$ *in* $D^b(Y)$

(4) $\mathrm{RHom}(A^\bullet, B^\bullet) \overset{L}{\otimes} C^\bullet \cong \mathrm{RHom}(A^\bullet, B^\bullet \overset{L}{\otimes} C^\bullet)$

Proof. (outlne) Part (1) is an exercise: prove it for sheaves then use that f^* is exact. For part (2) start with

$$\mathrm{Hom}(A, \mathrm{Hom}(B, C)) \cong \mathrm{Hom}(A \otimes B, C) \tag{12.2.1}$$

for sheaves A, B, C since the same holds for sections over open sets. If B is flat and C is injective then the right side of (12.2.1) is an exact functor of A, hence (§2.5) $\mathrm{Hom}(B, C)$ is injective. So, if we replace $B^\bullet$ with a flat complex and replac $C^\bullet$ with an injective complex then

$$\underline{\underline{\mathrm{RHom}}}^\bullet(A^\bullet, \underline{\underline{\mathrm{RHom}}}^\bullet(B^\bullet, C^\bullet)) = \underline{\underline{\mathrm{Hom}}}(A^\bullet, \underline{\underline{\mathrm{Hom}}}(B^\bullet, C^\bullet)) \cong \underline{\underline{\mathrm{Hom}}}(A^\bullet \otimes B^\bullet, C^\bullet).$$

For part (3), first prove that $f_!(A \otimes f^*(B)) = f_!A \otimes B$ for soft or flabby sheaves A and flat sheaves B. In fact the stalk at $y \in Y$ of the left side is

$$\Gamma_c(f^{-1}(y); A \otimes f^*(B)) = \Gamma_c(f^{-1}(y); A \otimes B_y) = \Gamma_c(f^{-1}(y); A) \otimes B_y$$

which agrees with the stalk of the right side, cf. [19] §V, 10.8 and §VI, 2.6. To prove (3) we may assume that $A^\bullet$ is flabby or soft and that $B^\bullet$ is flat, so the result follows. Part (4) is similar. □

12.3. Theorem. *Let $f : X \to Y$ be a stratified mapping between Whitney stratified spaces. Let $A^\bullet$, $B^\bullet$ and $C^\bullet$ be constructible sheaves of abelian groups on X, Y and Y respectively. Then f^*, $f^!$, Rf_* and $Rf_!$ take distinguished triangles to distinguished triangles. There are canonical isomorphisms in $D^b_c(X)$ (the bounded derived category of constructible sheaves) as follows:*

(1) $\mathbf{DD}(A^\bullet) \cong A^\bullet$
(2) $\mathbb{D}_X^\bullet \cong f^!\mathbb{D}_Y^\bullet$
(3) $f^!(B^\bullet) = \mathbf{D}_X f^* \mathbf{D}_Y(B^\bullet)$
(4) $Rf_!(A^\bullet) = \mathbf{D}_Y Rf_* \mathbf{D}_X(A^\bullet)$
So $f^!$ *is the dual of* f^* *and* $Rf_!$ *is the dual of* Rf_*.
(5) $\underline{\underline{\mathrm{RHom}}}^\bullet(B^\bullet, C^\bullet) \cong \mathbf{D}_Y\left(\mathbf{D}_Y C^\bullet \overset{L}{\otimes} B^\bullet\right)$
(6) $f^!\underline{\underline{\mathrm{RHom}}}^\bullet(B^\bullet, C^\bullet) \cong \underline{\underline{\mathrm{RHom}}}^\bullet(f^*(B^\bullet), f^!(C^\bullet))$
(7) $f^*\underline{\underline{\mathrm{RHom}}}^\bullet(B^\bullet, C^\bullet) \cong \underline{\underline{\mathrm{RHom}}}^\bullet(f^*(B^\bullet), f^*(C^\bullet))$
(8) $Rf_*(\underline{\underline{\mathrm{RHom}}}^\bullet(A^\bullet, f^!B^\bullet)) \cong \mathrm{RHom}^\bullet(Rf_!A^\bullet, B^\bullet)$ *[Verdier duality theorem]*
(9) $Rf_*\underline{\underline{\mathrm{RHom}}}^\bullet(f^*B^\bullet, A^\bullet) \cong \underline{\underline{\mathrm{RHom}}}^\bullet(B^\bullet, Rf_*A^\bullet)$
(10) If $f : X \to Y$ *is the inclusion of an open subset then* $f^!(B^\bullet) \cong f^*(B^\bullet)$.
(11) If $f : X \to Y$ *is the inclusion of a closed subset then* $Rf_!(A^\bullet) \cong Rf_*(A^\bullet)$.
(12) If $f : X \to Y$ *is the inclusion of an oriented submanifold in another, and if* $B^\bullet$ *is cohomologically locally constant on* Y *then* $f^!(B^\bullet) \cong f^*(B^\bullet)[\dim(Y) - \dim(X)]$.

Proof. (outline) It is an exercise to see that distinguished triangles are preserved by the functors Rf_*, $Rf_!$, $f^!$, f^*.

It is an exercise to verify (1) for the category of simplicial sheaves using the canonical model for the (simplicial) sheaf of chains. It comes down to a statement about the second barycentric subdivision. This statement of double duality escaped Borel and Moore since they did not have the derived category, but Borel later made a point of proving it in his notes [19] Thm. 8.10. First check it for a locally constant sheaf (of finitely generated Abelian groups) on a manifold, hence also for a complex of sheaves on a manifold with cohomology in a single degree. Use induction and the triangle defined by truncation (see §18.1) to prove it for any complex of sheaves (on a manifold) whose cohomology sheaves are locally constant and finitely generated. Use induction and the distinguished triangle (**??**) of a pair to prove it for general stratified spaces.

Parts (2) and (3) are immediate. Part (4) follows from part (8) (below). Part (5) follows from Lemma 12.2 part (2). When $C^\bullet$ is the dualizing sheaf, part (6) is immediate from biduality. Next consider the case that $C^\bullet$ is the dual of some sheaf $E^\bullet$, so

$$\begin{aligned} f^!\underline{\underline{\mathrm{RHom}}}^\bullet(B^\bullet, C^\bullet) &\cong f^!\underline{\underline{\mathrm{RHom}}}^\bullet(B^\bullet, \underline{\underline{\mathrm{RHom}}}^\bullet(E^\bullet, \mathbb{D}_Y) \cong f^!\underline{\underline{\mathrm{RHom}}}^\bullet(B^\bullet \overset{L}{\otimes} E^\bullet, \mathbb{D}_Y) \\ &\cong \underline{\underline{\mathrm{RHom}}}^\bullet(f^*(B^\bullet \overset{L}{\otimes} E^\bullet), \mathbb{D}_Y) \cong \underline{\underline{\mathrm{RHom}}}^\bullet(f^*(A^\bullet) \overset{L}{\otimes} f^*(B^\bullet), \mathbb{D}_Y) \\ &\cong \underline{\underline{\mathrm{RHom}}}^\bullet(f^*B^\bullet, \mathbf{D}f^*E^\bullet)) \cong \underline{\underline{\mathrm{RHom}}}^\bullet(f^*B^\bullet, f^!C^\bullet). \end{aligned}$$

Part (7) is an exercise: prove it first for sheaves, then use that f^* is exact.

The proof of (8) is technical, see [19] §7.17 (which follows [66]). It was first proven in [8]. The main point is that $f^!$ is right adjoint to $Rf_!$, that is,

$$\mathrm{Hom}_{D^b_c(X)}(A/b, f^! B/b) \cong \mathrm{Hom}_{D^b_c(Y)}(Rf_! A^\bullet, B^\bullet) \tag{12.3.1}$$

from which (8) follows (exercise) by restricting these sheaves to open sets in Y.

Part (9) follows from Proposition 1.13 by restricting the sheaves to open sets in Y. For (10), let $f : U \to X$ be an open inclusion. Then

$$f^*(\mathbf{D}(A^\bullet)) = f^* \underline{\underline{\mathrm{RHom}}}(A^\bullet, \mathbb{D}^\bullet_X) = \underline{\underline{\mathrm{RHom}}}(f^*(A^\bullet), f^*(\mathbb{D}^\bullet_X)) = \underline{\underline{\mathrm{RHom}}}(f^*(A^\bullet), \mathbb{D}^\bullet_U) = \mathbf{D} f^*(A^\bullet)$$

So $f^!(A^\bullet) = \mathbf{D} f^* \mathbf{D}(^\bullet) = \mathbf{D}\mathbf{D} f^*(A^\bullet) = f^*(A^\bullet)$. Part (11) is straightforward. Part (12) follows from the fact that if X is an n-dimensionial oriented manifold then the orientation defines a quasi-isomorphism $\mathbb{D}_X \cong \underline{\underline{\mathbb{Z}}}[n]$. □

13. Constructible functions

13.1. Euler characteristic and constructible functions. Let k be a field. In this section all sheaves are sheaves of k-vector spaces and all cohomology is taken with coefficients in k.

Suppose Y is a (closed) subcomplex of a finite simplicial complex X and $A^\bullet$ is a complex of sheaves of k-vector spaces on X, constructible with respect to the stratification by interiors of simplices. For such a sheaf $A^\bullet$ its stalk Euler characteristic at a point $x \in X$ is

$$\chi_x(A^\bullet) = \sum_{i \geq 0} (-1)^i \dim H^i_x(A^\bullet) = \sum_{i \geq 0} (-1)^i \dim(\underline{\underline{H}}^i(A^\bullet)_x)$$

and its global Euler characteristic is

$$\chi(X; A^\bullet) = \sum_{i \geq 0} \dim H^i(X; A^\bullet) = \sum_{\sigma} (-1)^{\dim \sigma} \chi_{\hat{\sigma}}(A^\bullet)$$

(sum over simplices σ in X, where $\hat{\sigma}$ denotes the barycenter) from which the long exact sequence §**??** for the pair (X, Y) implies (setting $U = X \setminus Y$) that

$$\chi_c(U; A^\bullet) = \chi(X, Y; A^\bullet) = \sum_{\sigma^o \subset U} (-1)^{\dim(\sigma)} \chi(\underline{\underline{H}}^*(A^\bullet)_{\hat{\sigma}}) = \sum_{\sigma^o \subset U} \chi_c(A^\bullet | \sigma^o). \tag{13.1.1}$$

The sum is over simplices σ whose interior σ^o is contained in U. Here, χ_c denotes the Euler characteristic with compact supports, $\chi_c(S) = \sum_i \dim H^i_c(S; k)$. It coincides with the Borel-Moore homology Euler characteristic. Thus equation (13.1.1) says that χ_c *is additive*: it is the sum over the interiors σ^o of simplices of the Euler characteristic with compact supports of each σ^o.

As in the previous chapter, fix a Whitney stratification of a closed subset $W \subset M$. (For convenience, as mentioned in §8.4, we assume the strata are connected.) A function $f : W \to k$ is *constructible* if it is constant on each stratum. The Euler characteristic of a constructible function f is defined to be $\chi(W; f) = \sum_S \chi_c(S) f(x_S)$ where the sum is over strata S of W, and $x_S \in S$. The above additivity remarks imply that this Euler characteristic does not change when

the stratification is refined and that the global Euler characteristic of $A^\bullet$ is the sum over strata of the stalk Euler characteristics:

$$\chi(X; A]b) = \sum_S \chi_c(S)\chi_{x_s}(A^\bullet)$$

where $x_S \in S$.

13.2. Lemma. *(cf.* [52] *§11.1) Let $A^\bullet \in D^b_c(W)$ and let $\Phi : A^\bullet \to A^\bullet$ be an automorphism whose* local Lefschetz number

$$\ell(\Phi)(x) = \sum_{q\geq 0}(-1)^q \mathrm{tr}(\Phi^* : H^q_x(A^\bullet) \to H^q_x(A^\bullet))$$

is a constructible function. Then the global Lefschetz number

$$L(\Phi) = \sum_{i\geq 0}(-1)^i \mathrm{tr}(\Phi^* : H^i(W; A^\bullet) \to H^i(W; A^\bullet))$$

is the Euler characteristic $\chi(W; \ell(\Phi))$ of the constructible function $\ell(\Phi)$.

Proof. The spectral sequence for hypercohomology has E_2 term

$$E_2^{iq} = H^i(W; \mathbf{H}^q(A^\bullet)) \implies H^{i+q}(W; A^\bullet).$$

If $\Phi^q : \mathbf{H}^q(A^\bullet) \to \mathbf{H}^q(A^\bullet)$ denotes the induced map on the cohomology sheaves of $A^\bullet$ then

$$L(\Phi) = \sum_q (-1)^q \sum_i (-1)^i \mathrm{tr}(H^i(\Phi^q) \circlearrowleft H^i(W; \mathbf{H}^q(A^\bullet)))$$

By refining the stratification if necessary we may assume that the sheaves $\mathbf{H}^q(A^\bullet)$ are locally constant on each stratum and that the local Lefschetz number is constant on each stratum. The inner sum is the Lefsechetz number of the sheaf morphism Φ^q, which may be seen to equal the sum over strata

$$L(\Phi^q) = \sum_S \chi_c(S)\mathrm{tr}\left((\Phi^q)^* \circlearrowleft \mathbf{H}^q_{x_S}(A^\bullet)\right)$$

(where $x_S \in S$) by using the same arguments as in §13.1. Summing over q gives

$$L(\Phi) = \sum_S \chi_c(S)\ell(\Phi)(x_S) = \chi(W; \ell(\Phi)). \qquad \square$$

If $A^\bullet \in D^b_c(W)$ then its stalk Euler characteristic is a constructible function $x \mapsto \chi_x(A^\bullet)$ which we denote by $\chi_\square(A^\bullet)$. If the total cohomology $H^*(W; A^\bullet)$ is finite dimensional then (using either a spectral sequence argument or the above commments about additivity applied to a triangulation of W), its Euler characteristic $\chi(W; A^\bullet)$ is given by

$$\chi(W; A^\bullet) = \sum_i (-1)^i \dim H^i(W; A^\bullet) = \chi(W; \chi_\square(A^\bullet)) = \sum_S \chi_c(S)\chi_\square(A^\bullet).$$

13.3. Calculus of constructible functions. If $\pi : W \to Y$ is a proper, weakly stratified mapping (meaning that it takes each stratum of W submersively to a stratum of Y) and if $f : W \to k$ is a constructible function then its *pushforward* $\pi_*(f)$ is the constructible function $\pi_*(f)(y) = \chi_c(\pi^{-1}(y); f)$. Consequently, if $A^\bullet \in D^b_c(W)$ then $\chi_\square(R\pi_* A^\bullet) = \pi_*\chi_\square(A^\bullet)$. If $\pi : X \to \{pt\}$ it is common to write ([121, 5])

$$\pi_*(f) = \int_X f(x)d\chi(x).$$

If W is compact and $h : W \to \mathbb{R}$ is a Morse function (cf. §15.2 below) with critical points p_i $(1 \le i \le r)$ and critical values $v_1 < \cdots < v_r$ then $\int_X f(x)d\chi(x)$ may be computed using Morse theory: The constructible function $h_*(f)$ vanishes at $t = -\infty$ and it only changes at the critical values of h. By Theorem 15.6 below, the contribution at each critical value v_i is localized near the critical point $p_i \in W$ and is given by the Euler characteristic $\chi_i = \chi(V_+, V_-)$ of the local Morse data (15.6.2) so that for any $t \in (v_s, v_{s+1})$,

$$h(t) = \sum_{i=1}^{s} \chi_i \text{ with } h(\infty) = \int_X f(x)d\chi(x).$$

A less obvious fact ([74, 105]) is that Verdier duality (see §11.1) passes to constructible functions: If f is a constructible function on W then there is a dual function $D(f)$ with the property that for any sheaf $A^\bullet \in D^b_c(W)$ with Verdier dual $\mathbf{D}(A^\bullet)$,

$$D(\chi_\square(A^\bullet)) = \chi_\square(\mathbf{D}(A^\bullet)). \tag{13.3.1}$$

The dual of a general constructible function f is given by

$$D(f)(x) = \lim_{\epsilon \to 0} \chi_c(B^o_\epsilon(x) \cap W; A^\bullet) = \chi(j^!_x; A^\bullet)$$

where $j_x : x \to W$ is the inclusion, and $B^o_\epsilon(x)$ denotes an open ball of radius ϵ centered at x. As in Theorem 8.4 this limit stabilizes for ϵ sufficiently small. Equation (13.3.1) then follows from the fact that $j^*_x(\mathbf{D}(A^\bullet)) = \mathbf{D}(j^!_x A^\bullet)$. (Consequently $DD(f) = f$ for any constructible function f, and $D\pi_* f = \pi_* D(f)$ for any $\pi : W \to Y$ as above.)

14. Fixed points

14.1. Applications of constructible functions. The k-th *Hadwiger invariant* or *intrinsic volume* (see, for example, [79]) of a compact subanalytic subset $W \subset \mathbb{R}^n$ is given ([6, 121]) by

$$\mu_k(W) = c(n,k) \int_{\mathrm{Gr_{n-k}(\mathbb{R}^n)}} \int_{\mathbb{R}^n/P} (\pi_P)_*(\mathbf{1}_W) dx_P dP$$

where $c(n,k) \in \mathbb{R}$ are specific constants (see [6]), $\mathbf{1}_W$ is the characteristic function of W, $\pi_P : \mathbb{R}^n \to \mathbb{R}^n/P$ is the projection, the Grassmannian $\mathrm{Gr_{n-k}(\mathbb{R}^n)}$ is provided with the volume one invariant measure dP and dx_P denotes Lebesgue measure on $\mathbb{R}^n/P$. The intrinsic volume $\mu_k(f)$ of a constructible function f is similarly defined. (See [5] and references therein).

In [88] the Chern class $c_*(f) \in H_*(X)$ of a constructible function f on a complex algebraic variety X is constructed so as to have the property that $c_*(\pi_* f) = \pi_*(c_*(f))$ for any proper algebraic map $\pi : X \to Y$. The class $c_*(\mathbf{1}_X)$ was later ([22]) identified with the Schwartz Chern class ([108, 109]).

15. Sheaves and Morse theory

15.1. Conormal vectors. Throughout this section W denotes a Whitney stratified closed subset of a smooth manifold M. Let X be a stratum of W and let $p \in X$. A cotangent vector $\xi \in T_p^*M$ is said to be *conormal* to X if its restriction vanishes: $\xi|T_pX = 0$. The collection of all conormal vectors to X in M is the *conormal bundle* T_X^*M. It is a smooth conical Lagrangian locally closed submanifold of T^*M. An orientation of M induces [106] an orientation of every T_X^*M (whether or not X is orientable).

A subspace $\tau \subset T_pM$ will be said to be a *limit of tanget spaces from* W if there is a stratum $Y > X$ ($Y \neq X$) and a sequence of points $y_i \in Y$, $y_i \to p$ such that the tangent spaces $T_{y_i}Y$ converge to τ. A conormal vector $\xi \in T_X^*M$ at p is *nondegenerate* if $\xi(\tau) \neq 0$ for every limit $\tau \subset T_pM$ of tangent spaces from larger strata $Y > X$. The set of nondegenerate conormal vectors is denoted Λ_X. Evidently,

$$\Lambda_X = T_X^*M - \bigcup_{Y>X} \overline{T_Y^*M}|X \tag{15.1.1}$$

where the union is over all strata $Y > X$ (including the case $Y = M - W$ because T_M^*M is the zero section, and elements of Λ_X are necessarily nonzero). If M is an analytic manifold and W is a subanalytically stratified subanalytic subset then each Λ_X has finitely many connected components.

The *conormal variety* $T_W^*M = \bigcup_X T_X^*M$ is the union of the conormal bundles of all strata. Suppose M is a real analytic manifold. If W is subanalytic then so is T_W^*M and it admits a stratification whose n-dimensional stratum is the disjoint union $\coprod_X \Lambda_X$.

15.2. Morse functions. Let $f : M \to \mathbb{R}$ be a smooth function and let $W_{\leq a} = W \cap f^{-1}((-\infty, a])$ and similarly for $W_{[a,b]}$, etc. The main idea in Morse theory is to build the cohomology $H^*(W)$ from the long exact cohomology sequences of the pairs $(W_{\leq v+\epsilon}, W_{\leq v-\epsilon})$ associated to "critical values" v of $f|W$. The function f is *perfect* if the connecting homomorphisms in these long exact sequences vanish. Since W is a singular space these notions need to be made precise.

A *critical point of* $f|W$ is a point $p \in X$ in some stratum X such that $df(p)|T_pX = 0$, that is, $\lambda = df(p) \in T_X^*M$. (In particular, every zero dimensional stratum is a critical point.) The number $v = f(p)$ is said to be a critical value. It is an *isolated critical value* of $f|W$ if no other critical point $q \in W$ of $f|W$ has $v = f(q)$.

We say that f is a (stratified) Morse function for W (cf. [81]) if

- its restriction to W is proper
- $f|X$ has isolated nondegenerate critical points for each stratum X,

- at each critical point $p \in X$ the covector $\lambda = df(p) \in \Lambda_X$ is nondegenerate, that is, $df(p)(\tau) \neq 0$ for every limit of tangent spaces $\tau \subset T_pM$ from larger strata $Y > X$.

Suppose M is real analytic and $W \subset M$ is subanalytic. Then set of Morse functions is open and dense in the space of proper smooth mappings with the Whitney C^∞ topology ([100, 99]) and

The function $f : M \to \mathbb{R}$ is a stratified Morse function for W if and only if the graph $df(M)$ is transverse to (every stratum of) the conormal variety $T^*_W M$.

Proof. Suppose X is a stratum and $df(x) \in T^*_X M$. Then x is a critical point of $f|X$. The Hessian is nonsingular if and only if the graph $df(X)$ is transverse to the zero section in T^*X, which holds if and only if $df(M)$ is transverse to $T^*_X M$ in T^*M . Let $n = \dim(M)$. If the n-dimensional submanifold $df(M)$ is transverse to every stratum of $T^*_W M$ then it must miss every stratum of dimension $\leq n-1$, which is to say that $df(x) \in \Lambda_X$ is a nondegenerate covector. □

15.3. Suppose $p \in X \subset W$ is an isolated nondegenerate critical point of $f : M \to \mathbb{R}$ as above. Let $\widetilde{N} \subset M$ be a smooth submanifold intersecting X transversally at the single point $\{p\}$. Then $N = \widetilde{N} \cap W$ is a *normal slice* to the stratum X at the point p (cf. §8.3). It inherits a stratification from that of M and, at least in some neighborhood of $\{p\}$.

Suppose also that $A^\bullet$ is a complex of sheaves that is (cohomologically) constructible with respect to the stratification of W. (Or equivalently, $A^\bullet$ is a complex of sheaves on M, constructible with respect to the stratification defined by W and supported on W.) Stratified Morse theory "reduces" the problem of computing the change in cohomology $H^*(W_{\leq t}; A^\bullet)$ as we pass a critical value $t = v$ to the case of a (germ of a) stratified space $H^*(N_{\leq t}; A^\bullet)$ with a zero dimensional stratum. Theorem 15.4 is a consequence of Theorem 15.6 below which is proven in [48, 51] using stratum preserving deformations arising from various applications of Thom's first isotopy lemma 8.5.

15.4. Theorem. [51] *Let $f : M \to \mathbb{R}$ be a smooth function and suppose $f|W$ is proper. Suppose $X \subset W$ is a stratum and that $p \in X$ is an isolated nondegenerate critical point of f (in the above, stratified sense) with isolated critical value $v = f(p) \in (a, b)$. Let λ denote the Morse index of $f|X$ at p. Suppose there are no additional critical values of $f|W$ in the interval $[a, b]$. If $0 < \delta << \epsilon$ are chosen sufficiently small (cf. Lemma 15.5) then there is a natural isomorphism of* Morse groups

$$H^r(W_{\leq b}, W_{\leq a}; A^\bullet) \cong H^{r-\lambda}\left(N_{[v-\delta,v+\delta]}, N_{v-\delta}; A^\bullet|N\right). \tag{15.4.1}$$

For fixed complex $A^\bullet$ the Morse groups

$$M^t(\xi) = H^t\left(N_{[v-\delta,v+\delta]}, N_{v-\delta}; A^\bullet|N\right)$$

depend only on the nondegenerate covector $\xi = df(p) \in T^*_{X,p}M$.

For any stratum X and any given nondegenerate covector $\xi \in T^*_{X,p}M$ it is possible to find a smooth function $f : M \to \mathbb{R}$ with a nondegenerate critical point at p such that $f(p) = 0$, $df(p) = \xi$, with no other critical points of critical value 0 and so that the Hessian of $f|X$ at p is positive definite. Then $\lambda = 0$ so for such a function, and $\epsilon > 0$ sufficiently small,

$$M^t(\xi) = H^t(W_{\leq\epsilon}, W_{\leq-\epsilon}).$$

Using this formula, [73] essentially avoids discussing normal Morse data.

In the classical smooth Morse theory the Morse group is nonzero only in the single degree $r = \lambda$ where it is the group $\mathbb{Z}$. In the singular case the Morse group (15.4.1) may be nonzero in many degrees. However if W is a complex analytic variety and $A^\bullet$ is a perverse sheaf then (cf. Theorem 22.7) the Morse group again lives in a single degree $r = c + \lambda$ where $c = \mathrm{cod}_{\mathbb{C}}(\mathrm{X})$ is the complex codimension of the stratum containing the critical point p.

There are several situations in which the hypothesis "isolated nondegenerate critical point" may be relaxed. See Theorems 15.13, 15.17. The meaning of $0 < \delta << \epsilon$ is explained in the following:

15.5. Lemma. *Fix a Riemannian metric on M and let $B_\epsilon(p)$ denote the closed ball of radius ϵ centered at p. Given $p \in X \subset W \subset M \xrightarrow{f} \mathbb{R}$ and N as above, there exists ϵ_0 so that for all $\epsilon \leq \epsilon_0$ the following holds:*

(*) *the boundary sphere $\partial B_\epsilon(p)$ is transverse to every stratum of W and of $T \cap W$.*

Given such a choice ϵ there exists $\delta_0 = \delta_0(\epsilon) > 0$ such that for all $0 < \delta \leq \delta_0$ the following holds:

(**) *$f|N$ has no critical points on any stratum of $N \cap f^{-1}[v - \delta, v + \delta]$ other than $\{p\}$.*

For such a choice of ϵ, δ we write $0 < \delta << \epsilon$. The set of possible choices for ϵ, δ will be an open region in the (ϵ, δ) plane like the following shaded area:

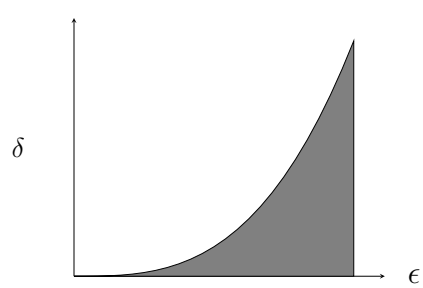


FIGURE 6. $\delta << \epsilon$ region

In this situation the main theorem of stratified Morse theory ([51]) says:

15.6. Theorem. *With $0 < \delta << \epsilon$ chosen as above, the stratified homeomorphism type of the pair $\big(N_{[v-\delta,v+\delta]}, N_{v-\delta}\big)$ is independent of ϵ, δ and there is a stratum preserving homeomorphism, smooth on each stratum, between $B_\epsilon(p) \cap W_{\leq v+\delta}$ and the adjunction space*

$$[B_\epsilon(p) \cap W_{\leq v-\delta}] \cup (D^\lambda, \partial D^\lambda) \times D^{s-\lambda} \times \big(N_{[v-\delta,v+\delta]}, N_{v-\delta}\big) \tag{15.6.1}$$

in which case we say that the *local Morse data*

$$(V_+, V_-) = B_\epsilon(p) \cap (W_{[v-\delta,v+\delta]}, W_{v-\delta}) \tag{15.6.2}$$

is homeomorphic to the product of the *tangential Morse data* $(D^\lambda, \partial D^\lambda) \times D^{s-\lambda}$ and the *normal Morse data* $(N_{[v-\delta,v+\delta]}, N_{v-\delta})$. (Here, $s = \dim(X)$.) Moreover:

If $f^{-1}([-\delta, \delta])$ contains no critical points (on any stratum) other than p then the intersection with $B_\epsilon(p)$ may be removed in the above statement.

Theorem 15.6 is illustrated in the following diagram, showing a stratified space with three 2-dimensional strata intersecting in a singular 1-dimensional stratum, like the pages of a book. The normal slice is a "Y". Normal and tangential Morse data are multiplied to give a piece of the space, however it must be glued into $W_{\leq v-\delta}$ along the red edges, which requires a delicate bending argument in order to lay the red edges flat.

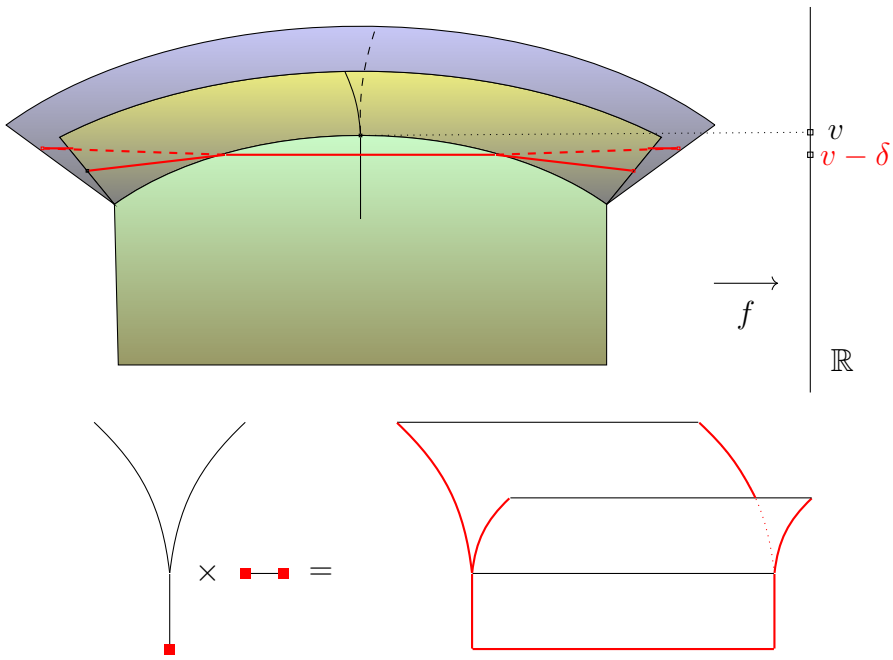


FIGURE 7. Normal Morse data $\times$ Tangntial Morse data $=$ Total Morse data
Glue along the red subspace

15.7. Homotopy type of the normal Morse data. Let $X \subset W \subset M \xrightarrow{f} \mathbb{R}$ as in §15.5 with X a stratum of a Whitney stratified set W contained in a smooth manifold M and a funtion f with a nondegenerate critical point (in the stratified sense) $x \in X$ and $v = f(x)$. Choose $0 \leq \delta << \epsilon$ as in §15.5 and let $N = \tilde{N} \cap W$ be a normal slice to the stratum X with $N \cap X = \{x\}$ and $N_{v-\delta} = N \cap f^{-1}(v-\delta)$. In the definition of the normal Morse data the (negative) *halflink and its boundary* is the pair

$$(\ell^-, \partial\ell^-) = (B_\epsilon(x) \cap N_{v-\delta}, \partial B_\epsilon(x) \cap N_{v-\delta})$$

with a similarly defined $(\ell^+, \partial\ell^+)$ and a stratum preserving homeomorphism $h : \partial\ell^+ \cong \partial\ell^-$. There is a stratum preserving homeomorphism of the link $L_x \cong \ell^+ \cup_h \ell^-$ with the two halflinks, glued along their common boundary. By [51] §3.11,

There is a stratum preserving deformation retraction of the normal Morse data $(N_{[v-\delta,v+\delta]}, N_{v-\delta})$ to the pair $(\mathrm{cone}(\ell^-), \ell^-)$.

If $A^\bullet \in D^b_c(W)$ is a constructible sheaf on W and if $M^t = M^t(df(x), A^\bullet)$ denotes the Morse group for f at the critical point x then the long exact cohomology sequence for the pair $(\mathrm{cone}(\ell^-), \ell^-)$

becomes:

$$\cdots \to H^{t-1}(\ell^-, A^\bullet) \to M^t \to H^t_x(A^\bullet) \to H^t(\ell^-, A^\bullet) \to \cdots$$

which expresses the Morse group in terms of the stalk cohomology and the cohomology of the halflink. In the complex analytic case this sequence becomes the braid diagram of §16.5.

15.8. The stratified homeomorphism type of each of these stratified spaces is indepenent of the choice of Riemannian metric, ϵ and δ (provided $0 < \delta << \epsilon$). To some degree it is even independent of the Morse function f. As above assume $x \in X \subset W$ is a nondegenerate critical point (in the stratified sense) of $f : M \to \mathbb{R}$ and let $\xi = df(x) \in \Lambda_X$. Suppose $g : M \to \mathbb{R}$ is a another smooth function with a nondegenerate critical point $y \in X$ and let $\eta = dg(y) \in \Lambda_X$. In [51] Theorem 7.5.1 Thom's isotopy lemma is used to prove

> If ξ and η lie in the same connected component of the set Λ_X of nondegenerate covectors then there is a stratum preserving homeomorphism between the normal Morse data (resp. half link and boundary) for ξ and for η.

(This result says nothing about the tangential Morse data and in fact the Morse index of $f|X$ at x may differ from that of $g|X$ at y.) The homeomorphism is far from unique due to the existence of nontrival monodromy automorphisms of the normal Morse data, see §16.4.

If $df(x)$ is a degenerate covector then Morse data for g may differ from Morse data for f even if $x = y$ and $df(x) = dg(x)$.

15.9. Sheaf theoretic expression. Kashiwara and Schapira [74] §5.1, §5.4 and Schúermann [107] prefer a sheaf-theoretic expression for the Morse group. Let

$$x_0 \in X \subset W \subset M \xrightarrow{f} \mathbb{R}$$

as in Theorem 15.4 above and suppose $f(x_0) = 0$ is an isolated critical value of $f|W$. Let $A^\bullet \in D^b_c(W)$ be a constructible complex of sheaves. Set

$$Z = \{x \in W | f(x) \geq 0\}$$

with inclusion $i : Z \to W$ and let $S^\bullet_Z = i^!_Z A^\bullet = R\Gamma_Z A^\bullet$ denote the sheaf obtained from $A^\bullet$ with sections supported in Z, cf. §**??**. Let $U = B_\epsilon(x_0) \cap W$ be a basic neighborhood of the critical point x_0. If $a < 0 < b$ and $[a, b]$ contains no critical values other than 0 then for $0 < \delta << \epsilon$ Thom's first isotopy lemma (Lemma 8.5) gives isomorphisms of the Morse groups:

$$\begin{aligned} H^r(W_{\leq b}, W_{\leq a}; A^\bullet) &\cong H^r(U_{\leq \delta}, U_{\leq -\delta}; A^\bullet) \\ &\cong H^r(U_{\leq \delta}, U_{<0}; A^\bullet) \\ &\cong H^r(U_{\leq \delta}; i^!_Z A^\bullet) \\ &\cong \underline{\underline{H}}^{\mathbf{r}}_{x_0}(i^!_Z A^\bullet) = \underline{\underline{\mathbf{H}}}^r_{x_0}(R\Gamma_Z A^\bullet) \end{aligned}$$

since the stalk cohomology is the limit as $\epsilon, \delta \to 0$ provided they satisfy (*) and (**) of Lemma 15.5. If we apply the main theorem in stratified Morse theory, this Morse group is identified with

$$M^{r-\lambda}(\xi; A^\bullet) = \underline{\underline{\mathbf{H}}}^{r-\lambda}_{x_0}(i^!_{Z\cap N}(A^\bullet|N))$$

where $\xi = df(x_0) \in \Lambda_X \subset T^*_X M$ and where $N = T \cap W \cap B_\epsilon(p)$ denotes the normal slice to the stratum X. Except for the shift λ (which comes from the tangential Morse data), this expression depends only on the nondegenerate covector $\xi = df(x_0)$, cf. §15.8. (If a covector $\xi \in T^*_X M$ is degenerate then the Morse group $M^t(\xi; A^\bullet)$ is not necessarily well defined.) If we choose the function f so that the Hessian of $f|X$ is positive definite at x_0 then $\lambda = 0$ and so the Morse group is canonically identified with the derived functor of sections with support:

$$M^t(\xi; A^\bullet) = \underline{\underline{\mathbf{H}}}^t_{x_0}(i^!_Z A^\bullet) = H^t(R\Gamma_Z A^\bullet_{x_0}).$$

15.10. Singular support. Suppose $W \subset M$ as above, and $A^\bullet \in D^b_c(W)$ is constructible with respect to a given stratification of W. From the preceding paragraph we see that Morse groups $M^*(\xi; A^\bullet)$ for $A^\bullet$ are assigned to every nondegenerate covector $\xi \in \bigcup_X \Lambda_X$ (union over strata X of W). Such a covector is not in the singular support $SS(A^\bullet)$ if all the Morse groups vanish: $M^t(\xi; A^\bullet) = 0, \forall t$. *Thus, $SS(A^\bullet)$ is the closure of the set of nondegenerate covectors $\xi \in \bigcup_X T_X M$ for which some Morse group is nonzero.*

If $\xi \in T^*_p M$ is a nondegenerate covector for W, the Morse groups $M^*(\xi, A^\bullet)$ form the stalk cohomology of a complex of sheaves $\mu_M(A^\bullet)$, the *microlocalization* of $A^\bullet$, on T^*M, which is supported on $SS(A^\bullet)$ [74].

If a complex of sheaves $A^\bullet$ is not necessarily constructible then the singular support may be defined in the same manner (cf. [74]) but it does not necessarily have the simple interpretation described here.

15.11. Characteristic cycle. Suppose that M^n is an oriented analytic manifold and $W \subset M$ is a subanalytically stratified subanalytic subset. Let $A^\bullet \in D^b_c(W)$. In [71] M. Kashiwara associates a Borel-Moore chain $CC(A^\bullet)$ in $C^{BM}_n T^*M$ such that $\partial CC(A^\bullet) = 0$. It is called the characteristic cycle of $A^\bullet$. It can be described Morse-theoretically (see [106]) as follows. Let X be a stratum of W and let $\Lambda_X \subset T^*_X M$ be the collection of nondegenerate covectors for X. It is a finite union $\Lambda_X = \cup_\alpha \Lambda_{X,\alpha}$ of connected components. Each component has dimension n and a canonical orientation, hence a fundamental class in Borel Moore homology:

$$[\Lambda_{X,\alpha}] \in C^{BM}_n(\Lambda_{X,\alpha}) \to C^{BM}_n T^*M.$$

For each α the Morse groups $M^t(\xi; A^\bullet)$ are independent of the choice of $\xi \in \Lambda_{X,\alpha}$ so the Euler characteristic $\chi_{X,\alpha} = \sum(-1)^t \dim M^t(\xi; A^\bullet)$ is well defined. Then the characteristic cycle is the sum over strata X with multiplicity $\chi_{X,\alpha}$ of these fundamental classes:

$$CC(A^\bullet) = \sum_X \sum_\alpha \chi_{X,\alpha} [\Lambda_{X,\alpha}].$$

Kashiwara proves [71] that $\partial CC(A^\bullet) = 0$ and the *Kashiwara index theorem* [71, 74, 106] says if W is compact then the Euler characteristic of $A^\bullet$ is the intersection number

$$\chi(W; A^\bullet) = T^*_M M \cap CC(A^\bullet)$$

of the characteristic cycle with the zero section.

15.12. Hyperbolic Lefschetz numbers. Let $f : W \to W$ be a subanalytic self map defined on a subanalytically stratified subanalytic set W. A fixed point x of f is *contracting* if all nearby points are mapped closer to x, and it is expanding if nearby points are mapped farther away from x. In some cases it is possible to identify contracting and expanding directions:

A connected component V of the fixed point set of f is said to be *hyperbolic* if there is a neighborhood $U \subset W$ of V and a subanalytic mapping $r = (r_1, r_2) : U \to \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ so that $r^{-1}(0) = V$ and so that $r_1(f(x)) \geq r_1(x)$ and $r_2(f(x)) \leq r_2(x)$ for all $x \in U$.

(There is an unfortunate misprint in [52] and equation (3.1)(c)(ii) should read $\geq$.) The self map $\mathbb{C} \to \mathbb{C}$ given by $z \mapsto z/(1+z)$ $(z \neq -1)$ has a non-hyperbolic isolated fixed point at $z = 0$. Hyperbolic behavior of $f : W \to W$ is illustrated in the following diagram. (Flow lines connecting $r(x)$ and $r(f(x))$ do not exist in general).

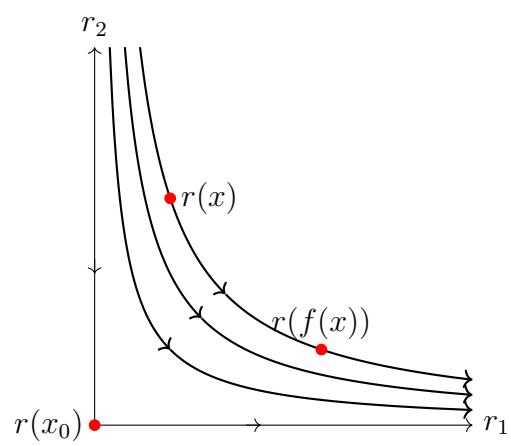


FIGURE 8. Behavior near a hyperbolic fixed point

Suppose V is a hyperbolic fixed point component. Let $V^+ = r^{-1}(Y)$ and $V^- = r^{-1}(X)$ where X, Y denote the X and Y axes in $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ with inclusions $j^\pm, h^\pm$:

$$V \xrightarrow{j^\pm} V^\pm \xrightarrow{h^\pm} W. \tag{15.12.1}$$

So V^+ consists of points that converge into the fixed component V, while points in V^- move away from V. Let $A^\bullet \in D^b_c(W)$ be a constructible complex of sheaves. A morphism $\Phi : f^*(A^\bullet) \to A^\bullet$ is

called a *lift* of f to $A^\bullet \in D^b_c(W)$. Such a lift induces a homomorphism $\Phi_* : H^i(W; A^\bullet) \to H^i(W; A^\bullet)$ and defines the Lefschetz number

$$\mathrm{Lef}(f, A^\bullet) = \sum_{i\in\mathbb{Z}} (-1)^i \,\mathrm{Trace}\left(\Phi_* : H^i \to H^i\right).$$

If V is a hyperbolic connected component of the fixed point set define the following *hyperbolic localizations* or restrictions of $A^\bullet$ to V:

$$A_V^{!*} = (j^-)^!(h^-)^* A^\bullet \quad \text{and} \quad A_V^{*!} = (j^+)^*(h^+)^! A^\bullet.$$

Then $H^*(V; A_V^{!*})$ is the cohomology of a neighborhood of V with closed supports in the directions "flowing" into V and with compact supports in the directions "flowing" away from V.

In [52] it is shown that the self map Φ also induces self maps $\Phi_V^{!*}$ on the sheaf $A_V^{!*}$ and $\Phi_V^{*!}$ on $A_V^{*!}$ and that associated *local Lefschetz numbers* $\mathrm{Lef}(\Phi_V^{!*}; A_V^{!*})$ and $\mathrm{Lef}(\Phi_V^{*!}; A_V^{*!})$ are equal[6].

Hyperbolic fixed points arise naturally as fixed points of Hecke correspondences on compactifications of locally symmetric spaces. In these cases the *weighted cohomology* sheaf has a natural lift to the correspondence, and the resulting Lefschetz numbers have been calculated ([45, 53, 44]).

15.13. Theorem. [52] *Given a self map $f : W \to W$, a complex of sheaves $A^\bullet \in D^b_c(W)$ and a lift $\Phi : f^*(A^\bullet) \to A^\bullet$, suppose that W is compact and that all connected components of the fixed point set are hyperbolic. Then the global* $\mathrm{Lef}(f, A^\bullet)$ *is the sum over connected components of the fixed point set of the local Lefschetz numbers:*

$$\mathrm{Lef}(f, A^\bullet) = \sum_V \mathrm{Lef}(\Phi_V^{!*}; A_V^{!*}) = \sum_V \mathrm{Lef}(\Phi_V^{*!}; A_V^{*!}).$$

Moreover, each local Lefschetz number $\mathrm{Lef}(\Phi_V^{!*})$ is the Euler characteristic of a constructible function $\mathrm{Lef}(\Phi_x, A_V^{!*})$ for $x \in V$ (see §13.1 above). Let $V = \coprod V_r$ be a stratification of the fixed point component V so that the pointwise Lefschetz number $\mathrm{Lef}(\Phi_x, A_V^{!*})$ is constant on each stratum V_r, and call it $L_r(\Phi; A_V^{!*})$. If V is compact then Lemma 13.2 (cf. [52] §11.1) gives

$$\mathrm{Lef}(\Phi_V^{!*}; A_V^{!*}) = \sum_r \chi_c(V_r) L_r(\Phi; A_V^{!*}).$$

The parallel statement also holds for $\mathrm{Lef}(\Phi_V^{*!}; A_V^{*!})$.

15.14. Outline of proof. In 1965 Grothendieck and Illusie considered this situation and even the case when $f : W \to W$ is replaced by a correspondence. They proved [57] that the Lefschetz number $L(f, A^\bullet)$ is equal to a sum of contributions $\ell.c.(V)$ from each conneted component V of the fixed point set and that this contribution was locally defined by the restriction of $A^\bullet$ and f to an arbitrary neighborhood of V. Their theorem may be considered an existence proof of a Lefschetz formula, but it is not very illuminating when it comes to determining the individual contributions.

[6]In many cases the hyperbolic localizations $A_V^{!*}$ and $A_V^{*!}$ are canonically isomorphic, cf. [52, 11]

Let $U \subset X$ be a hyperbolic neighborhood of a single connected component V of the fixed point set. It turns out that if we "cut off' the sheaf $A^\bullet$ by setting it equal to 0 outside U, and using compact support boundary conditions on the parts of U for which f is expanding, then the self map f will induce a morphism on the resulting sheaf, call it $A^\bullet_{cut}$. Geometrically, thinking about the dual situation of chains rather than cochains, (hence interchanging compact supports with closed supports) then a chain ξ with closed support (that is, a Borel-Moore chain) near the expanding part of the boundary, and compact support near the contracting part of the boundary will be again taken into such a chain by f, that is, $f(\xi)$ will again have closed support near the expanding boundary and compact support near the contracting boundary. So f induces

$$f^*_{cut} : H^*(U; A^\bullet_{cut}) \to H^*(U; A^\bullet_{cut}).$$

Now applying the Grothendieck-Illusie theorem to this sheaf gives $L(f^*_{cut}) = \ell.c.(V)$. However we have not modified the sheaf or the self map f in a (smaller) neighborhood of V so again the Grothendieck-Illuse theorem says that the local contribution arising from the pair $(A^\bullet_{cut}, f^*_{cut})$ is the same as the local contribution arising from $(A^\bullet, f)$.

Apply this idea to a neighborhood $U = r^{-1}(\text{box})$ as shown in Figure 8 where the "box" is a small open region $[0, a) \times [0, b) \subset \mathbb{R}^2$. Assume the mapping $r : U \to \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ has been Whitney stratified and the box has been chosen so that its top and right edges are transverse to the strata (depicted in red in Figure 9) and so that the top right corner does not intersect any strata. Cut off the sheaf $A^\bullet$ with compact supports along the right side of the box, and closed supports along the top (and either condition at the top right corner). As we imagine shrinking the box a tricky but interesting point arises.

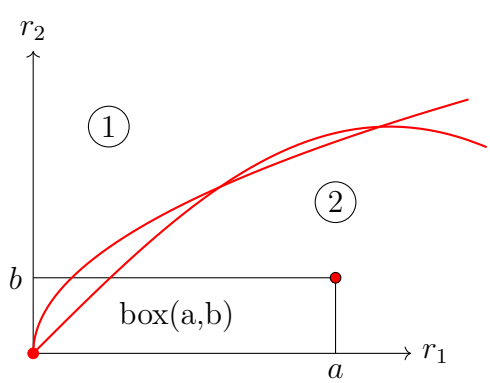


FIGURE 9. Shrinking the box

By the isotopy lemma, the cohomology $H^*(r^{-1}(\text{box}); A^\bullet_{cut})$ will not change as long as the top right corner (a, b) in Figure 9 does not cross any strata and the top and right side of the box remain transverse to the strata. So we may let $b \to 0$ first, then followed by $a \to 0$, obtaining $H^*(r^{-1}(\text{box}); A^\bullet_{cut}) \cong H^*(V; A^{!*}_V)$. On the other hand, if the top right corner (a, b) of the box lies

in the region 1 then we may let $a \to 0$ first, followed by $b \to 0$ obtaining $H^*(r^{-1}(\text{box}; A^\bullet_{cut}) \cong H^*(V; A^{*!}_V)$. Although these cohomology groups are not necessarily isomorphic, they will have the same Lefschetz number as required by Grothendieck's theorem. □

What happens if the self map is "neutral" near the fixed point component, neither contracting nor expanding? In this case, the self map preserves the boundary of some neighborhood U of the fixed point component and it acts on U without fixed points. This implies that the Lefschetz number of the map is zero on the cohomology of U. Therefore the compact support Lefschetz number on U coincides with the closed support Lefschetz number on U. In other words we may consider the map to be either "expanding" or "contracting" and we will get the same answer.

15.15. Torus actions. The Morse-Bott theory of critical points for smooth manifolds also has various extensions to singular spaces. Suppose the torus $T = \mathbb{C}^*$ acts algebraically on a (possibly singular) normal projective algebraic variety W and suppose the action extends to some projective space $M = \mathbb{P}^m$, containing W. The imaginary part of the K$\acute{}$ahler form on M is a nondegenerate symplectic 2-form whose restriction to any complex submanifold of M is also nondegenerate. The action of the circle $S^1 = \{e^{i\theta}\} \subset \mathbb{C}^*$ defines a Hamiltonian vector field on M with resulting (cf. §23.2) moment map $\mu : M \to \mathbb{R}$ which may be thought of as a Morse function that increases along the real directions of the $\mathbb{C}^*$ action. The critical points of μ are exactly the fixed points of the torus action.

Let $W^T = \coprod_r V_r$ denote the fixed point components of the torus action and define

$$\begin{aligned} V_r^- &= \left\{ x \in W \mid \lim_{t\to\infty} t.x \in V_r \right\} \\ V_r^+ &= \left\{ x \in W \mid \lim_{t\to 0} t.x \in V_r \right\}. \end{aligned} \tag{15.15.1}$$

with inclusions as in equation (15.12.1):

$$V_r \xrightarrow{j_r^\pm} V_r^\pm \xrightarrow{h_r^\pm} W \tag{15.15.2}$$

In [15] Bialynicki-Birula proves that if W is nonsingular then each $V_r^+ \to V$ has the natural structure of an algebraic bundle of affine spaces (of some dimension d_r) and the moment map $\mu : W \to \mathbb{R}$ is a $\mathbb{Q}$-perfect Morse-Bott[7] function.

15.16. Theorem. *If W is nonsingular then for each $j \geq 0$ there is an isomorphism*

$$H^j(W;\mathbb{Q}) \cong \bigoplus_r H^{j-2d_r}(V_r;\mathbb{Q}) \quad \textit{and} \quad H_j(W;\mathbb{Q}) \cong \bigoplus_r H_{j-2d_r}(V_r;\mathbb{Q}). \tag{15.16.1}$$

[7]meaning that for each component V of the fixed point set, the Hessian $\partial^2 f/\partial x_i \partial x_j$ is nondegenerate on each normal space T_pM/T_pV and the connecting homomorphisms vanish in the long exact sequences of the pairs $H^i(W_{\leq v+\epsilon}, W_{\leq v-\epsilon}; \mathbb{Q})$ where $v = \mu(p)$

The original proof involves the Weil conjectures and counting points mod p. In many cases these isomorphisms can be described geometrically. Suppose $Z \subset V_r$ is an compact d-dimensional algebraic subvariety with fundamental class $[Z] \in H_{2d}(Z) \to H_{2d}(V_r)$. Let $\pi : V_r^+ \to V_r$ be the bundle projection and let $Z^+ = \pi^{-1}(Z)$. Then the closure $\overline{Z^+}$ is a $d + d_r$ dimensional variety whose fundamental class $[\overline{Z^+}] \in H_{2d+2d_r}(W)$ is the image of $[Z]$ under the second isomorphism in (15.16.1), cf. [26].

Returning to the general case, suppose W is possibly singular. Each fixed point $p \in W \subset M$ of the torus action is a critical point for the moment map $\mu : M \to \mathbb{R}$ but if p is in the singular set of W then the restriction $\mu|W$ will fail to be nondegenerate (at p) in the sense of §15.2 because $d\mu(p)$ kills all limits of tangent vectors. So $\mu : W \to \mathbb{R}$ is not a Morse function (or a Morse-Bott function) in the sense of §15.2. However some aspects of Morse theory continue to work in this case.

Each of the sets $V_r^\pm$ is algebraic and the projection $V_r^\pm \to V_r$ is algebraic. The time $t = 1$ map of the torus action is hyperbolic (in the sense of §15.12) at each fixed point component. Suppose $A^\bullet \in D_c^b(W)$ is a complex of sheaves on W. As in §15.12 define

$$A_r^{!*} = (j_r^+)^!(h_r^+)^* A^\bullet \quad \text{and} \quad A_r^{*!} = (j_r^-)^*(h_r^-)^! A^\bullet.$$

If the sheaf $A^\bullet$ is *weakly hyperbolic*[8] or if $A^\bullet$ has a lift to the equivariant derived category of [14] then ([52, 11]) there is a canonical isomorphism $A_r^{*!} \cong A_r^{!*}$ so (cf. Theorem 15.13):

If $A^\bullet$ is weakly hyperbolic and$W$ is compact then $\chi(W; A^\bullet) = \sum_r \chi(V_r; A_r^{*!})$.

The intersection complex $I^{\mathbf{p}}C^\bullet$ with arbitrary perversity $\mathbf{p}$ on W (see §?? and §?? below) is weakly hyperbolic in a canonical way. In [78] F. Kirwan generalized the Bialynick-Birula theorem (15.16.1) to the case of singular varieties. She uses the decomposition theorem (Theorem 22.8) to prove the following for the *middle* perversity intersection complex $\underline{\underline{IC}}^\bullet$:

15.17. Theorem. *([78]) The moment map μ is a perfect Morse Bott function and it induces a decomposition for all i, expressing the intersection cohomology of W as a sum of locally defined cohomology groups of the fixed point components:*

$$IH^i(W) \cong \bigoplus_r H^i(V_r; \underline{\underline{IC}}_r^{!*}). \tag{15.17.1}$$

The result also generalizes to actions of a torus $T = (\mathbb{C}^*)^m$. In [11] the sheaf $\underline{\underline{IC}}_r^{!*}$ is shown to be a direct sum of $\underline{\underline{IC}}^\bullet$ sheaves of subvarieties of V_r.

[8]meaning that there exists a local system $\mathcal{L}$ on T and a quasi-isomorphism $\mu^*(A^\bullet) \cong \mathcal{L} \boxtimes A^\bullet$ where $\mu : T \times W \to W$ is the torus action

16. Complex Morse theory of sheaves

16.1. Three magical phenomena arise when Morse theory is applied to complex analytic (or algebraic) spaces. First, the index of certain Morse functions on a complex manifold can often be estimated, due to a relation between the eigenvalues of a complex symmetric or Hermitian matrix (the Levi form) and an associated symmetric matrix (the Hessian), see §16.2 below. The relation may be expressed in many different ways, beginning with R. Thom (unpublished lecture), Andreotti and Frankel [1] (1958), Milnor [98] §7 (1963), see also [55] p. 158, [51] II.A.4, [7] p. 311, or [80] 3.1.7.

The second piece of magic concerns the topology of singular algebraic and analytic varieties, as described by Lefschetz [82] and later developed using stratification theory.

Finally, if X is a connected complex analytic stratum in a complex analytic subvariety $W \subset M$ of a complex manifold M then the set Λ_X of nondegenerate covectors is connected, since its intersection with each conormal space is the complement of a proper complex analytic subvariety.

16.2. Levi form. Let $f : \mathbb{C}^n \to \mathbb{R}$ be a smooth function. The *Levi form* of f is the $k \times k$ Hermitian matrix $L(f) = (\partial^2 f/\partial z_i \partial \bar{z}_j)$. For $x \in \mathbb{C}^n$ let $\sigma_x(L(f))$ denote the maximal (complex) dimension of a (complex) subspace on which $L(f)(x)$ is negative definite and let $\sigma_x(H(f))$ denote the maximal (real) dimension of a (real) subspace on which the Hessian $H(f)(x)$ is negative definite. The following important estimate holds:

If the Hessian $H(f)(x)$ is nondegenerate then
$$n + \sigma_x(L(f)) \geq \sigma_x(H(f)) \geq n - \sigma_x(L(-f))$$

For $n = 1$ and $z = x + iy$, the Levi form $L(f) = \partial^2 f/dx^2 + \partial^2 f/dy^2$ is the trace of the Hessian $H(f)$ so the result is clear. For the general case let $V = H_p \cap L_n$ where H_p is a maximal real subspace of $\mathbb{C}^n$ on which H is positive definite, and L_n is a maximal complex subspace of $\mathbb{C}^n$ on which L is negative semidefinite. By the one dimensional case, V contains no complex line, so $V \cap iV = \{0\}$. Therefore $V \oplus iV$ is a complex subspace of L_n so
$$\dim_{\mathbb{C}}(L_n) \geq \dim_{\mathbb{R}}(V) \geq \dim_{\mathbb{R}}(H_p) + 2\dim_{\mathbb{C}}(L_n) - 2n.$$
Therefore $\sigma_x(H) = 2n - \dim(H_p) \geq \dim(L_n) = n - \sigma_x(-L)$ which proves the second inequality. The first follows from this by replacing f with $-f$. □

If $A \subset \mathbb{C}^n$ is a complex submanifold and $x \in A$ then the Levi form of $f|A$ at x is equal to the restriction of the $L(f)$ to T_xA, a fact which is not generally true for the Hessian. Fix a point $p \in \mathbb{C}^n$ so that the function $f(x) = \mathrm{dist}^2(x, p)$ has a nondegenerate critical points when restricted to A. At such a critical point $x \in A$ it follows that

$$\sigma_x(H(f|A)) \leq \dim_{\mathbb{C}} A \text{ and } \sigma_x(H(-f|A)) \geq \dim_{\mathbb{C}} A. \tag{16.2.1}$$

16.3. The complex link. Details for this section may be found in [51] part II §2. (See also [107] chapt. 5.) Throughout this section W denotes a complex analytically stratified complex analytic

variety of complex dimenion n contained in some complex manifold M, say, of dimension m. Let X be a stratum of W, let $p \in X$ and let $N = \widetilde{N} \cap W$ be a normal slice at p, that is, the intersection with a complex submanifold $\widetilde{N}$ that is transverse (and complementary) to X with $\widetilde{\cap} X = \{p\}$.

Suppose $\xi \in T^*_{X,p}M$ is a nondegenerate covector. Since

$$T^*_p M = \mathrm{Hom}_{\mathbb{C}}(T_pM, \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{R}}(T_pM, \mathbb{R})$$

we may choose local coordinates near p that identify ξ with the linear projection $\mathbb{C}^m \to \mathbb{C}$ with $p = 0 \in \mathbb{C}^m$ and $\pi(p) = 0$. The restriction $\pi|N : N \to \mathbb{C}$ takes the zero dimensional singular point to the origin. Every other stratum of N is taken sumersively to $\mathbb{C}$ (over a neighborhood of 0), so $N \to \mathbb{C}$ is a stratified fiber bundle except at the point p. Let $r(z)$ denote the square of the distance from p in $\mathbb{C}^m$. As in Lemma 15.5 there is a region $0 < \delta << \epsilon \subset \mathbb{R}^2$ such that for any pair (δ, ϵ) in this region the following holds:

(1) $\partial B_\epsilon(p)$ is transverse to every stratum of N (where $B_\epsilon(p)$ denotes the ball of radius ϵ).
(2) For each stratum Y of N (except $Y = \{p\}$) the restriction $\pi|Y$ has no critical values in the disk $D_\delta(0) \subset \mathbb{C}$
(3) For each stratum Y of N with $\dim_{\mathbb{C}}(Y) \geq 2$, and for each point $z \in Y \cap \partial B_\epsilon(p)$ if $|\pi(z)| \leq \delta$ then the complex linear map

$$(dr(z), d\pi(z)) : T_zY \to \mathbb{C}^2$$

has rank 2.

Identify $\delta = \delta + 0i \in \mathbb{C}$ and define the *complex link* ([47], [48], [51] II §2):

$$\mathcal{L} = \pi^{-1}(\delta) \cap N \cap B_\epsilon(0),\ \partial\mathcal{L} = \pi^{-1}(\delta) \cap \partial B_\epsilon(0).$$

It is a single fiber of the (stratified) fiber bundle over the circle $S^1 = \partial D_\delta \subset \mathbb{C}$,

$$\mathcal{E} = \pi^{-1}(\delta e^{i\theta}) \cap N \cap B_\epsilon(0),\ \partial\mathcal{E} = \pi^{-1}(\delta e^{i\theta}) \cap N \cap \partial B_\epsilon(0)$$

$$(\mathcal{E}, \partial\mathcal{E}) \to S^1 = \partial D_\delta = \{\delta e^{i\theta} |\ 0 \leq \theta \leq 2\pi\}$$

and the boundary $\partial\mathcal{E}$ is a trivial bundle over $S^1 = \partial D_\delta$. See Figure 10. The stratified homeomorphism type of the spaces $\mathcal{L}, \partial\mathcal{L}$ is independent of the choice of ϵ, δ.

16.4. Theorem. *([47],[48] ,[51] §II.2.4) The bundle $\mathcal{E}$ is stratified-homeomorphic to the mapping cylinder of a (stratified) monodromy homeomorphism*

$$\mu : (\mathcal{L}, \partial\mathcal{L}) \to (\mathcal{L}, \partial\mathcal{L})$$

that is the identity on $\partial\mathcal{L}$ and is well defined up to stratum preserving isotopy. The link $L_X(p)$ is homeomorphic to the "cylinder with caps",

$$L_X(p) \cong \mathcal{E} \cup_{\partial\mathcal{E}} (\partial\mathcal{L} \times D_\delta)\,. \tag{16.4.1}$$

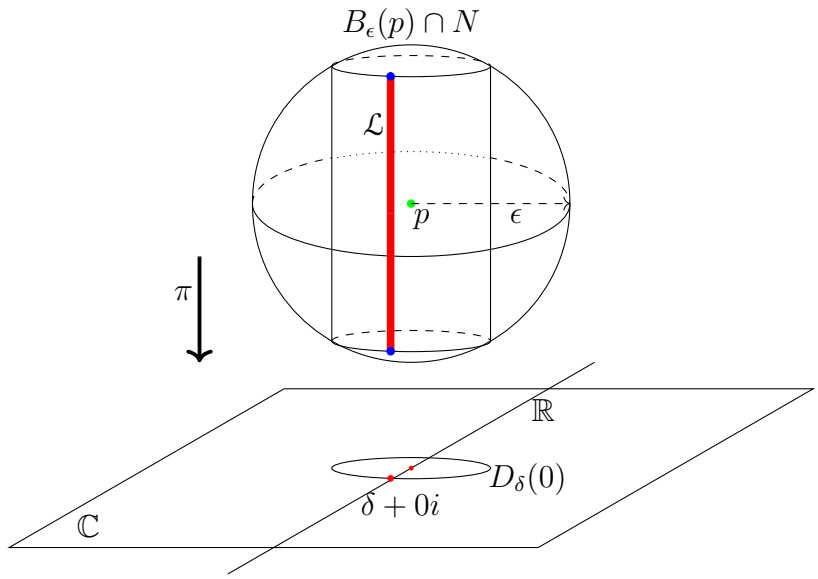


FIGURE 10. Normal slice with complex link, ϵ-ball and δ-disk

16.5. The braid diagram. All these facts fit together into a single commutative diagram that explains their cohomological interrelationships. Throughout this section we fix a constructible complex of sheaves $A^\bullet \in D^b_c(W)$ of K-vector spaces, where K is a (coefficient) field. Fix a point $x \in W$ in some stratum X with normal slice N. Fix a nondegenerate covector $\xi \in T^*_{X,x}M$ with resulting complex link $\mathcal{L}$. As in Theorem 15.4 we may identify $\xi = d\phi$ with the differential of a locally defined (near x) smooth function $\phi : M \to \mathbb{R}$ such that $\phi(x) = 0$ and $d\phi|T_xX = 0$. Let

$$H^r(\xi, A^\bullet) = H^r(N_{\leq\delta}, N_{<0}; A^\bullet) = H^r(R\Gamma_Z A^\bullet)$$

denote the Morse group for ξ in degree r for $\delta > 0$ sufficiently small. Let $L = L_X(x)$ denote the link of the stratum X at x.

The homeomorphisms described in the Theorem 16.4 are stratum preserving so they induce isomorphisms on cohomology with coefficients in $A^\bullet$ and they allow us to interpret these cohomology groups,

$$\begin{aligned} H^r(N - x; A^\bullet) &\cong H^r(L_X(x); A^\bullet) & H^r(N, N - x; A^\bullet) &\cong H^r(i^!_x A^\bullet) \\ H^r(N; A^\bullet) &\cong H^r(i^*_x A^\bullet) & H^r(N, N_{<0}; A^\bullet) &\cong H^r(\xi; A^\bullet) \\ H^r(N_{<0}; A^\bullet) &\cong H^r(\mathcal{L}; A^\bullet) & H^{r+1}(N - x, N_{<0}; A^\bullet) &\cong H^r(\mathcal{L}, \partial\mathcal{L}; A^\bullet) \end{aligned}$$

By (16.4.1) the "variation" map $I - \mu : H^r(\mathcal{L}; A^\bullet) \to H^r(\mathcal{L}, \partial\mathcal{L}; A^\bullet)$ may be identified with the connecting homomorphism in the long exact sequence for the pair $(N - x, N_{<0})$, cf. [47]. As in [51] p. 215 the three long exact sequences for the triple of spaces

$$N_{<0} \subset N - \{x\} \subset N$$

may be assembled into a braid diagram with exact sinusoidal rows, capturing the Morse group in the middle. (cf. [107] §6.1 where the same sequences are considered separately):

$$\begin{tikzcd}
H^{r-1}(i_x^* A^\bullet) \arrow[rr, bend left] \arrow[dr] & & H^{r-1}(\mathcal{L}, A^\bullet) \arrow[rr, bend left, "var"] \arrow[dr] & & H^{r-1}(\mathcal{L}, \partial\mathcal{L}; A^\bullet) \arrow[rr, bend left] \arrow[dr] & & H^{r+1}(i_x^! A^\bullet) \\
& H^{r-1}(L; A^\bullet) \arrow[ur] \arrow[dr] & & H^r(\xi; A^\bullet) \arrow[ur] \arrow[dr] & & H^r(L; A^\bullet) \arrow[ur] \arrow[dr] & \\
H^{r-2}(\mathcal{L}, \partial\mathcal{L}; A^\bullet) \arrow[ur] \arrow[rr, bend right] & & H^r(i_x^! A^\bullet) \arrow[ur] \arrow[rr, bend right] & & H^r(i_x^* A^\bullet) \arrow[ur] \arrow[rr, bend right] & & H^r(\mathcal{L}; A^\bullet)
\end{tikzcd}$$

FIGURE 11. Braid diagram

In order to use the braid diagram to determine the Morse group it is first necessary to understand the cohomology of the complex link.

16.6. Theorem. *Suppose $A^\bullet \in D_c^b(W)$ is a complex of sheaves of K-vector spaces on a complex analytic set W such that for each stratum X and for each point $x \in X$, with $i_x : \{x\} \to W$, the stalk cohomology vanishes:*

$$H^r(i_x^* A^\bullet) = 0 \text{ whenever } r > \operatorname{codim}_{\mathbb{C}}(X). \tag{16.6.1}$$

Then $H^r(\mathcal{L}; A^\bullet) = 0$ for all $r > \ell = \dim_{\mathbb{C}}(\mathcal{L})$. If the Verdier dual $\mathbb{D}_W(A^\bullet)$ satisfies (16.6.1), or equivalently, (if W has pure (complex) dimension n and)

$$H^r(i_x^! A^\bullet) = 0 \text{ whenever } r < n + \dim_{\mathbb{C}}(X), \tag{16.6.2}$$

then $H^r(\mathcal{L}, \partial\mathcal{L}; A^\bullet) = 0$ for all $r < \ell$.

Proof. The proof uses three ingredients: the Levi form §16.2, the normal $\times$ tangential theorem §15.6, and the braid diagram. It is the prototype for all Morse-theoretic results for singular complex varieties so we give the complete proof. The second part if the theorem follows from the first part by Verdier duality. We prove the first part by induction on the complex codimension c of the stratum X, with the case $c = 0$ trivial. Fix $x \in X \subset W$. As in Figure 10 the complex link

$$\mathcal{L} = \pi^{-1}(\delta) \cap N \cap B_\epsilon(x)$$

is stratified in a codimension preserving way, by its intersection with the strata of W; the boundary $\partial\mathcal{L}$ is collared in $\mathcal{L}$; and the function $f(y) = \operatorname{dist}(x, y)^2$ has Morse critical points in $\mathcal{L} - \partial\mathcal{L}$. So $H^*(\mathcal{L}, A^\bullet) \cong H^*(\mathcal{L} - \partial L, A^\bullet)$. Build $\mathcal{L} - \partial\mathcal{L}$ by Morse theory, starting with the empty set.

Suppose $x' \in \mathcal{L}$ is a critical point of the distance function $f|\mathcal{L}$. It lives in some stratum Y of $\mathcal{L}$, say $Y = X' \cap \mathcal{L}$ where X' is a stratum of W with codimension $\mathrm{cod_W X' = cod_{\mathcal{L}} Y = c' < c}$. Then $\dim(Y') = \dim(\mathcal{L}) - c' = c - 1 - c'$. By equation (16.2.1) the index λ of $f|Y$ at x is $\lambda \leq c - 1 - c'$. Let $\mathcal{L}'$ denote the complex link of the stratum Y in $\mathcal{L}$ at the point x'; it also equals the complex link of the stratum X' in W and its dimension is $c' - 1$. By induction, $H^r(\mathcal{L}', A^\bullet) = 0$ for all $r > c' - 1$. By assumption $H^s(i^*_{x'} A^\bullet) = 0$ for all $s > \mathrm{cod_W(X') = cod_{\mathcal{L}} Y = c'}$. So the braid diagram implies the Morse group vanishes, $H^r(\xi, A^\bullet) = 0$ for all $r > c'$.

If $v = f(x')$ denotes the critical value of $f|\mathcal{L}$ (which we may assume to be isolated) then the main theorem of stratified Morse theory, equation (15.4.1) gives, for sufficiently small $\epsilon > 0$, that

$$H^t(\mathcal{L}_{\leq v+\epsilon}, \mathcal{L}_{\leq v-\epsilon}, A^\bullet) = H^{t-\lambda}(\xi, A^\bullet).$$

If this is nonzero then $t - \lambda \leq c'$ so $t \leq c' + \lambda \leq c - 1$. So the cohomology of $\mathcal{L}'$ cannot change in degrees that are greater than $c - 1$. Applying this argument to every critical point implies $H^t(\mathcal{L}; A^\bullet) = 0$ for all $t > c - 1 = \dim(\mathcal{L})$ which concludes the inductive step. □

16.7. Vanishing theorems. The same proof as above, using the braid diagram, together with induction and the estimates in §16.2 for the distance[2] from a point, may be used to prove many vanishing theorems and Lefschetz-type theorems in sheaf cohomology, see [3, 56, 47, 51, 107, 72, 60, 59]. Since Theorem 15.6 and Theorem 16.4 are statements about homeomorphism type, this method also gives homotopy-theoretic results, for example ([47, 48, 51]):

16.8. Theorem. *Let W be a Stein space or an affine complex algebraic variety of dimension n and let $A^\bullet \in D^b_c(W)$ be a complex of sheaves on W that satisfies (16.6.1). Then $H^r(W; A^\bullet) = 0$ for all $r > n$ and in fact W has the homotopy type of a CW complex with cells of dimension $\leq n$. Let W be a projective variety and let H be a hyperplane that is transverse to each stratum of some Whitney stratification of W. Let $A^\bullet$ be a complex of sheaves on W that satisfies (16.6.2). Then $H^r(W, W \cap H; A^\bullet) = 0$ for all $r < n$ and in fact W is obtained from $W \cap H$ up to homotopy by attaching cells of diimension $\geq n$.*

16.9. Let $A^\bullet$ be a constructible complex of sheaves on a complex analytic variety W. By examining the braid diagram again it is easy to see that vanishing of the Morse group $H^r(\xi, A^\bullet)$ at a critical point $x \in W$ and induction, as in the proof of Theorem 16.6, implies vanishing of the stalk cohomology $H^r(i^*_x A^\bullet)$.

16.10. Corollary. [89] *The complex $A^\bullet$ satisfies (16.6.1) (resp. (16.6.2)) if and only if for every stratum X of W and every point $x \in X$ and every nondegenerate covector $\xi \in T^*_{X,x}M$ the Morse group $M^j(\xi; A^\bullet)$ vanishes for all $j > \mathrm{cod_{\mathbb{C}}(X)}$ (resp. for all $j < \mathrm{cod_{\mathbb{C}}(X)}$).*

Let us say that the complex $A^\bullet$ is *Morse-pure* if, for every point $x \in W$ and for every nondegenerate covector $\xi \in T_{X,x}$, the Morse group $H^j(\xi, A^\bullet)$ vanishes in all degrees except for one, $j = \mathrm{cod}_{\mathbb{C}}(\mathrm{X})$, where X is the stratum containing x.

This recalls the classical situation of Morse theory on smooth manifolds, where the Morse group is the integers and it lives in a single degree, the Morse index. In the language of §19.2, a *perverse sheaf* is a complex satisfying (16.6.1) and (16.6.2). So Corollary 16.10 says

The complex of sheaves $A^\bullet$ is Morse-pure if and only if it is perverse.

17. Intersection Homology

17.1. A motivating example. Consider $W = \Sigma T^3$, the suspension of the 3-torus with singular points denoted $\{N\}, \{S\}$. We have natural cycles, point, T^1, ΣT^1, T^2, ΣT^2, T^3, ΣT^3. Some of these hit the singular points, some do not. The ones that do not are homologous to zero by a "homology" or bounding chain that hits the singular points. If we restrict cycles and homologies by not allowing them to hit the singular points, this will change the resulting homology groups. For $p = 0, 1, 2$ define $C_i^p(W) = \{\xi \in C_i(W) |\ \xi \cap \{N, S\} = \phi$ unless $i \geq 4 - p\}$ Here are the resulting homology groups.

	$p=0$	$p=1$	$p=2$
$i=4$	ΣT^3	ΣT^3	ΣT^3
$i=3$	0	ΣT^2	ΣT^2
$i=2$	T^2	0	ΣT^1
$i=1$	T^1	T^1	0
$i=0$	$\{pt\}$	$\{pt\}$	$\{pt\}$

	$p=0$	$p=1$	$p=2$
$i=4$	T^3	T^3	T^3
$i=3$	0	T^2	T^2
$i=2$	0	0	T^1
$i=1$	0	0	0
$i=0$	0	0	0

Figure 12. Intersection homology and stalk homology of ΣT^3

The larger the number p the more cycles are allowed into the singular points. If there are more strata we can assign such numbers to each stratum separately. Cycles with a large value of p may get "locked in" to the singular set. Their obstinate and perverse refusal to move away from the singular set led to the notion of a "perversity" vector.

17.2. Digression on transversality. Let $K \subset \mathbb{R}$ be the Cantor set. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth (C^∞) function that vanishes precisely on K. Let $A \subset \mathbb{R}^2$ denote the graph of f and let B denote the x-axis. Then A, B are smooth submanifolds of $\mathbb{R}^2$ but their intersection is the Cantor set. This sort of unruly behavior can be avoided using transversality.

Two submanifolds $A, B \subset M$ of a smooth manifold are said to be transverse at a point of their intersection $x \in A \cap B$ if $T_xA + T_xB = T_xM$. If A and B are transverse at every point of their intersection then $A \cap B$ is a smooth submanifold of M of dimension $\dim(A) + \dim(B) - \dim(M)$. Arbitrary submanifolds $A, B, \subset M$ can be made to be transverse by moving either one of them, say $A' = \phi_\epsilon(A)$ by the flow, for an arbitrarily small time, of a smooth vector field on M. If V is a finite dimensional vector space of vector fields on M which span the tangent space T_xM at every point $x \in M$ then there is an open and dense subset of V consisting of vector fields v such that the time $= 1$ flow ϕ_1 of v takes A to a submanifold $A' = \phi_1(A)$ that is transverse to B.

This is a very powerful result. It says, for example, that two submanifolds of Euclidean space can be made transverse by an arbitrary small translation. The proof, due to Marston Morse, is so elegant, that I decided to include it in the Appendix §B.

Two Whitney stratified subsets $W_1, W_2 \subset M$ are said to be transverse if each stratum of W_1 is transverse to each stratum of W_2, in which case the intersection $W_1 \cap W_2$ is canonically Whitney stratified. As in the preceding paragraph, Whitney stratified sets can be made to be transverse by the application of the flow, for an arbitrarily small time, of a similarly chosen smooth vector field on M.

17.3. Intersection homology. Let W be a compact n-dimensional Whitney stratified pseudomanifold with strata S_α (α in some index set I, partially ordered by the closure relations between strata with S_0 being the stratum of dimensioin n) and let $0 \le p_\alpha \le \mathrm{cod}(\mathrm{S}_\alpha) - 2$ be a collection of integers which we refer to as a perversity $\bar{p}$. Define the intersection chains,

17.4. Definition.

$$IC_i^{\bar{p}}(W) = \left\{ \xi \in C_i(W) \;\middle|\; \begin{array}{l} \dim(\xi \cap S_\alpha) \le i - \mathrm{cod}(\mathrm{S}_\alpha) + \mathrm{p}_\alpha \\ \dim(\partial\xi \cap S_\alpha) \le i - 1 - \mathrm{cod}(\mathrm{S}_\alpha) + \mathrm{p}_\alpha \end{array} \text{ for } \alpha > 0 \right\} \tag{17.4.1}$$

Having placed the same restrictions on the chains as on their boundaries, we obtain a chain complex, in fact a complex of (soft) sheaves $\underline{\underline{IC}}^{\bar{p}}$ with resulting cohomology groups $IH_i^{\bar{p}}(W)$ or $I^{\bar{p}}H_i(W)$. (As usual, "chains" could refer to PL chains, singular chains, subanalytic chains, etc.) Because W is a pseudomanifold the singular strata have codimension at least 2. The condition $p_\alpha \le \mathrm{cod}(\mathrm{S}_\alpha) - 2$ implies that most of the chain, and most of its boundary are completely contained within the top stratum S_0. So a cycle ($\partial\xi = 0$) in $IC_i^{\bar{p}}$ is also a cycle for ordinary homology and similarly a bounding chain in $IC_i^{\bar{p}}$ is also a bounding chain for ordinary homology. So we have a homomorphism $IH_i^{\bar{p}}(W) \to H_i(W)$. Moreover, if $\xi \in IC_i^{\bar{p}}(W)$ and if $\eta \in IC_j^{\bar{q}}(W)$ and if we can arrange that $\xi \cap S_\alpha$ and $\eta \cap S_\alpha$ are transverse within each stratum S_α then we will have an

intersection

$$\xi \cap \eta \in IC^{\bar{p}+\bar{q}}_{i+j-n}(W)$$

which is well defined provided that $p_\alpha + q_\alpha \leq \mathrm{cod}(\mathrm{S}_\alpha) - 2$ for all $\alpha > 0$.

The first problem with this construction is that it is obviously dependent on the stratification. Moreover, if we are not careful, large values of p_α for small strata $S_\alpha < S_\beta$ will have the effect of allowing chains into S_α but not into S_β thereby "locking" the chain into passing through a small stratum, resulting in a stratification-dependent homology theory. This issue can be avoided by requiring that p_α depends only on $\mathrm{cod}(\mathrm{S}_\alpha)$ and that $\beta > \alpha \implies p_\beta \geq p_\alpha$.

The second problem involves the effect of refining the stratification. For a simple case, suppose W consists only of two strata, S_0 and S_c, the singular stratum having codimension $c \geq 3$, to which we assign a perversity p_c. Now suppose we refine this stratum by introducing a "fake" stratum, S_r of codimension $r > c$. Chains in $IC^p_i(W)$ may intersect S_c in dimension $\leq i - c + p_c$ and for all we know, they may lie completely in S_r, meaning that the chain will have "perversity" $p_r = p_c + c - r$. On the other hand if we assume, as before, that we can arrange for this chain to be transverse to the fake stratum S_r within the stratum S_c then its intersection with S_r will have dimension $\leq i - c + p_c - (r - c) = i - r + p_c$ which is to say that it has "perversity" $p_r = p_c$. This argument shows (or suggests) that in this case we have natural isomorphisms between the intersection homology $IH^{p_c}_i(W)$ as computed before the refinement, and the intersection homology IH^{p_c,p_r}_i after refinement, for any p_r with $p_c \leq p_r \leq p_c + r - c$, that is,

$$IH^{p_c,p_c}_i(W) \cong IH^{p_c,p_c+1}_i(W) \cong \cdots \cong IH^{p_c,p_c+r-c}_i(W)$$

In summary, assuming that $p_c \leq p_r \leq p_c + r - c$ the resulting homology group IH^{p_c,p_r}_i is unchanged after refinement. This leads to the formal definition of intersection homology.

17.5. Definition. A *perversity* is a function $\bar{p} = (p_2, p_3, \cdots)$ with $p_2 = 0$ and with $p_c \leq p_{c+1} \leq p_c + 1$. The complex of sheaves of intersection chains is the complex with sections

$$\Gamma(U, \underline{\underline{IC}}^{-i}_{\bar{p}}) = \left\{ \xi \in C_i(U) \;\middle|\; \begin{array}{l} \dim(\xi \cap S_c) \leq i - c + p_c \\ \dim(\partial\xi \cap S_c) \leq i - 1 - c + p_c \end{array} \text{ for } c \geq 2 \right\} \tag{17.5.1}$$

where S_c denotes the union of all strata of codimension $c \geq 2$.

Intersection homology with coefficients in a local system is defined similarly, however something special happens in this case. For any triangulation of a chain $\xi \in IC^{-i}_{\bar{p}}$ all of its i-dimensional simplices and all of its $i-1$ dimensional simplices will be completely contained within the top stratum (or "nonsingular part") of W. So if $\mathcal{L}$ is a local coefficient system *defined only on the top stratum of* W, we can still construct the sheaf of intersection chains $\underline{\underline{IC}}^{\bullet}_{\bar{p}}(\mathcal{L})$ exactly as above.

Let $\bar{0}$ be the perversity $0_c = 0$ and let $\bar{t}$ be the perversity $t_c = c - 2$.

17.6. Theorem. *Let W be an oriented stratified pseudomanifold of dimension n. For any choice of perversity $\bar{p}$ equation (17.5.1) defines a complex of soft sheaves $\underline{\underline{IC}}^{\bullet}_{\bar{p}}$ on W and the following holds.*

(1) The cohomology sheaves $\underline{\underline{IH}}^{-m}_{\bar{p}}$ and the hypercohomology groups $IH^{\bar{p}}_i(W)$ are well defined and are independent of the stratification;

(2) in fact they are topological invariants.

(3) There are canonical maps

$$H^{n-i}(W) \to IH^{\bar{p}}_i(W) \to H_i(W)$$

that factor the Poincaré duality map,

(4) if $\bar{p} \leq \bar{q}$ then there are also compatible mappings $IH^{\bar{p}} \to IH^{\bar{q}}$. In sheaf language we have natural maps

$$\underline{\underline{\mathbb{Z}}}_W[n] \to \underline{\underline{IC}}^{\bar{p}}_W \to \underline{\underline{IC}}^{\bar{q}}_W \to \underline{\underline{C}}^{BM}_W.$$

(5) If the link L_X of each stratum X is connected then for $\bar{p} = \bar{0}$ the first of these maps is a quasi-isomorphism, and for $\bar{q} = \bar{t}$ the second map is a quasi-isomorphism.

(6) If $p_c + q_c \leq t_c = c - 2$ for all c then the intersection of transversal *chains determines a pairing*

$$IH^{\bar{p}}_i(W) \times IH^{\bar{q}}_j(W) \to IH^{\bar{p}+\bar{q}}_{i+j-n}(W)$$

(7) If $\bar{p} + \bar{q} = \bar{t}$ then the resulting pairing

$$IH^{\bar{p}}_i(W) \times IH^{\bar{q}}_{n-i}(W) \to H_0(W) \to \mathbb{Z}$$

is nondegenerate over $\mathbb{Q}$ (or over any field).

The last statement in Theorem 17.6, Poincaré duality, was the big surprise when intersection homology was discovered for it is a duality statement that applies to singular spaces. Especially, if the stratification of W consists only of even codimension strata then there is a "middle" choice for p, that is, $p_c = (c-2)/2$ for which $IH^{\bar{p}}(W;k)$ is self-dual for any field k.

There is a technical problem with moving chains within a Whitney stratified set W, so as to be transverse within each stratum of W. This can be accomplished with piecewise-linear chains within a piecewise-linear stratified set W, and has recently been accomplished using semi-analytic chains within a semi-analytic stratified set, but to my knowedge, it has not been accomplished in any other setting. This is one of the many problems that is avoided with the use of sheaf theory. The proof of topological invariance depends entirely on sheaf theory. Other results such as the proof of Poincaré duality, that can be established using chain manipulations, are incredibly awkward, requiring a choice of model for the chains, and delicate manipulations with individual chains. These constructions are easier, but less geometric, if they are all made using sheaf theory. For this purpose we need to identify the quasi-isomorphism class of the complex of sheaves $\underline{\underline{IC}}^{\bar{p}}$.

17.7. Proposition. *Let W be a Whitney stratified pseudomanifold and let $\mathcal{E}$ be a local coefficient system defined on the top stratum. Fix a perversity $\bar{p}$, and let $x \in X$ be a point in a stratum X of codimension $c \geq 2$. Then the stalk of the intersection homology sheaf at x is*

$$\underline{\underline{H}}^{-i}(\underline{\underline{IC}}^{\bar{p}}(\mathcal{E}))_X = IH_i^{\bar{p}}(W, W-x; \mathcal{E}) = \begin{cases} 0 & \text{if } i < n - p_c \\ IH_{i-n+c-1}(L_X; \mathcal{E}) & \text{if } i \geq n - p_c \end{cases}$$

and the stalk cohomology with compact supports is

$$H_c^{-i}(U_x; \underline{\underline{IC}}^{\bar{p}}(\mathcal{E})) = IH_i^{\bar{p}}(U_x) = \begin{cases} H_i(L_x; \mathcal{E}) & \text{if } i \leq c - p_c \\ 0 & \text{if } i > c - p_c \end{cases}$$

Proof. Use the local product structure of a neighborhood $U_x \cong c^o(L_x) \times B_\epsilon^{n-c}$ and the Künneth formula[9]

$$IH_i^{\bar{p}}(U, \partial U; \mathcal{E}) \cong IH_{i-(n-c)}^{\bar{p}}(c(L_X), L_X; \mathcal{E}).$$

If $\xi \in IC_i^{\bar{p}}$ and if $(i-n+c)-c+p_c \geq 0$ then the chain ξ is allowed to hit the cone point, otherwise it is not. When it is allowed to hit the cone point, we may assume (using a homotopy argument) that it locally coincides with the cone over a chain in L_X which satisfies the same allowability conditions. Similar remarks apply to $\partial\xi$. On the other hand, suppose ξ is a compact i-dimensional chain in the link L_X. It is the boundary of the cone $c(\xi)$ so the homology class $[\xi]$ vanishes in the neighborhood U_x provided that cone is allowable, which occurs if

$$\dim(c(\xi) \cap X) = 0 \leq (i+1) - c + p_c \quad \text{that is, if} \quad i > c - p_c. \qquad \square$$

Comparing this to the calculation (9.4.1) of $Rj_*j^*(\underline{\underline{IC}}^{\bar{p}})$ where $j : U = W - \overline{S_c} \to W$ is the inclusion of the open complement of the closure of S_c we see that the intersection homology sheaf on S_c is the *truncation* (§18.1) of the sheaf $Rj_*(\underline{\underline{IC}}^{\bar{p}}|U)$. For example, suppose $\dim(W) = 8$ has strata of dimension $0, 2, 4, 6, 8$ and the perversity is the middle one, $p_c = (c-2)/2$. Then the stalk cohomology $H^*(\underline{\underline{IC}}_x^{\bar{p}})$ of the sheaf $\underline{\underline{IC}}^{\bar{p}}$ is described in Figure 13, where L^r means the r-dimensional link of the codimension $r+1$ stratum and the red zeroes represent homology groups that have been killed by the perversity condition.

This gives an inductive way to construct intersection homology using purely sheaf-theoretic operations, to be described in the next section.

[9]The homology $H_*(B_\epsilon, \partial B_\epsilon)$ is torsion-free so the Tor terms vanish.

i	cod0	cod2	cod4	cod6	cod8
0					0
-1					0
-2					0
-3				0	0
-4				0	0
-5			0	0	$IH_4(L^7)$
-6			0	$IH_3(L^5)$	$IH_5(L^7)$
-7		0	$IH_2(L^3)$	$IH_4(L^5)$	$IH_6(L^7)$
-8	$\mathbb{Z}$	$IH_1(L^1)$	$IH_3(L^3)$	$IH_5(L^5)$	$IH_7(L^7)$

cod0	cod2	cod4	cod6	cod8
$\mathbb{Z}$	$IH_0(L^1)$	$IH_0(L^3)$	$IH_0(L^5)$	$IH_0(L^7)$
	0	$IH_1(L^3)$	$IH_1(L^5)$	$IH_1(L^7)$
		0	$IH_2(L^5)$	$IH_2(L^7)$
		0	0	$IH_3(L^7)$
			0	0
			0	0
				0
				0
				0

FIGURE 13. Stalk cohomology and compact support cohomology of $\underline{\underline{IC}}^\bullet$

18. TRUNCATION

18.1. Truncation. If $A^\bullet$ is a complex of sheaves define

$$(\tau_{\leq r}A^\bullet)^i = \begin{cases} 0 & \text{if } i > r \\ \ker(d^r) & \text{if } i = r \\ A^i & \text{if } i < r \end{cases}$$

Then $\underline{\underline{H}}^i(\tau_{\leq r}A^\bullet) = \underline{\underline{H}}^i(A^\bullet)$ for $i \leq r$ and is zero for $i > r$. This gives a distinguished triangle

$$\begin{tikzcd} \tau_{\leq r}A^\bullet \arrow[rr] & & A^\bullet \arrow[dl] \\ & \tau^{\geq r+1}A^\bullet \arrow[ul, "{[1]}"] & \end{tikzcd}$$

where

$$\left(\tau^{\geq r}A^\bullet\right)^i = \begin{cases} A^i & \text{if } i > r \\ A^r/d(A^{r-1}) & \text{if } i = r \\ 0 & \text{if } i < r. \end{cases}$$

During a conversation in October 1976 Pierre Deligne suggested that the following construction might generate the intersection homology sheaf.

18.2. Definition. Let W be a purely n dimensional oriented Whitney stratified pseudomanifold and let W_r denote the union of the strata of dimension $\leq r$. Set $U_k = W - W_{n-k}$ with inclusions

$$U_2 \xrightarrow[j_2]{} U_3 \xrightarrow[j_3]{} \cdots \xrightarrow[j_{n-1}]{} U_n \xrightarrow[j_n]{} U_{n+1} = W$$

Let $\mathcal{E}$ be a local coefficient system defined on the top stratum $U = U_2$. Set

$$\underline{\underline{P}}^\bullet_{\bar{p}}(\mathcal{E}) = \tau_{\leq p(n)}Rj_{n*}\cdots\tau_{\leq p(3)}Rj_{3*}\tau_{\leq p(2)}Rj_{2*}\mathcal{E}.$$

The resulting complex of sheaves will have stalk cohomology that is illustrated in Figure 14 (in the case of middle perversity, with $\mathcal{E} = \mathbb{Z}$, for a stratified psuedomanifold W with $\dim(W) = 8$ that is stratified with strata of codimension 0, 2, 4, 6, 8). In this figure we suppress the $\bar{p}$ on $\underline{\underline{P}}^\bullet_{\bar{p}}$.

i	cod0	cod2	cod4	cod6	cod8
8					
7					0
6					0
5				0	0
4				0	0
3			0	0	$H^3(L^7, \underline{\underline{P}}^\bullet)$
2			0	$H^2(L^5, \underline{\underline{P}}^\bullet)$	$H^2(L^7, \underline{\underline{P}}^\bullet)$
1		0	$H^1(L^3, \underline{\underline{P}}^\bullet)$	$H^1(L^5, \underline{\underline{P}}^\bullet)$	$H^1(L^7, \underline{\underline{P}}^\bullet)$
0	$\mathbb{Z}$	$H^0(L^1, \underline{\underline{P}}^\bullet)$	$H^0(L^3, \underline{\underline{P}}^\bullet)$	$H^0(L^5, \underline{\underline{P}}^\bullet)$	$H^0(L^7, \underline{\underline{P}}^\bullet)$

FIGURE 14. Stalk cohomology of $\underline{\underline{P}}^\bullet$

We remark, for example, at a point $x \in W$ that lies in a stratum X of codimension 6, the stalk cohomology at x equals the cohomology of the link L^5 of X with coefficients in the part of the sheaf $\underline{\underline{P}}^\bullet|U_6$ that has been previously constructed over the strictly *larger* strata $Y > X$.

18.3. Theorem. [50] *Let W be an oriented n-dimensional stratified pseudomanifold and let $\mathcal{E}$ be a local coefficient system on the top stratum. The orientation map $\mathcal{E}[n] \to \underline{\underline{C}}^{-n}_U(\mathcal{E})$ induces an isomorphism $\underline{\underline{P}}^\bullet_{\bar{p}}(\mathcal{E})[n] \cong \underline{\underline{IC}}^\bullet_{\bar{p}}(\mathcal{E})$ where $U = U_2$ denotes the largest stratum.*
If the link L_x of every stratum is connected then $\underline{\underline{P}}^\bullet_{\bar{t}}(\mathbb{Z})[n] \to \underline{\underline{C}}^\bullet(\underline{\underline{\mathbb{Z}}}_U)$ is a quasi-isomorphism (so that $IH^i_{\bar{t}}(W) = H_{n-i}(W)$) and $\underline{\underline{P}}^\bullet_{\bar{0}}(\mathbb{Z}) \to \underline{\underline{\mathbb{Z}}}$ is a quasi-isomorphism (so that $IH^i_{\bar{0}}(W) = H^i(W)$), where $\bar{0}_c = 0$ and where $\bar{t}_c = c - 2$ are the "bottom" and "top" perversities respectively. If $\bar{p} + \bar{q} \leq \bar{t}$ (where $t_c = c - 2$) and if $\mathcal{E}_1 \otimes \mathcal{E}_2 \to \mathcal{E}_3$ is a morphism of local systems on U then it extends canonically to a product

$$\underline{\underline{P}}^\bullet_{\bar{p}}(\mathcal{E}_1) \otimes \underline{\underline{P}}^\bullet_{\bar{q}}(\mathcal{E}_2) \to \underline{\underline{P}}^\bullet_{\bar{p}+\bar{q}}(\mathcal{E}_3).$$

Proof. For simplicity we discuss the case of constant coefficients. There are two problems (a) to show that the orientation map $\underline{\underline{\mathbb{Z}}}_U \to \underline{\underline{C}}^{-n}_U$ extends to a (uniquely defined) map in the derived category $\underline{\underline{P}}^\bullet[n] \to \underline{\underline{IC}}^\bullet$ (for a fixed perversity, which we suppress in the notation) and (b) to show that this map is a quasi-isomorphism. These are proven by induction, adding one stratum at a time. Consider the diagram

$$U_k \xrightarrow[j_k]{} U_{k+1} \xleftarrow[i_k]{} X^{n-k}$$

where X^{n-k} is the union of the codimension k strata. Suppose by induction that we have constructed a quasi-isomorphism $\underline{\underline{P}}^\bullet_k \to \underline{\underline{IC}}^\bullet_k$ of sheaves over U_k (where the subscript k denotes the restriction to U_k). Now compare the two distinguished triangles (writing $i = i_k$ and $j = j_k$ to simplify notation),

$$\begin{tikzcd} Ri_*i^!(\underline{\underline{P}}^\bullet_{k+1}) \arrow[rr] & & \underline{\underline{P}}^\bullet_{k+1} \arrow[dl] \\ & Rj_*j^*(\underline{\underline{P}}^\bullet_{k+1}) = Rj_*(\underline{\underline{P}}^\bullet_k) \arrow[ul] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Ri_*i^!(\underline{\underline{IC}}^\bullet_{k+1}) \arrow[rr] & & \underline{\underline{IC}}^\bullet_{k+1} \arrow[dl] \\ & Rj_*j^*(\underline{\underline{IC}}^\bullet_{k+1}) = Rj_*(\underline{\underline{IC}}^\bullet_k) \arrow[ul] & \end{tikzcd}$$

We are actually concerned with the right side of these triangles. By induction we have an isomorphism on the bottom row, so we get an isomorphism of the truncations:

$$\underline{\underline{P}}^\bullet_{k+1} = \tau_{\le p(k)} Rj_* \underline{\underline{P}}^\bullet_k \to \tau_{\le p(k)} Rj_*(\underline{\underline{IC}}^\bullet_k)$$

This is the upper right corner of the first triangle and we wish to identify it with the upper right corner of the second triangle. So it suffices to show that we have an isomorphism (in the derived category),

$$\underline{\underline{IC}}^\bullet_{k+1} \cong \tau_{\le p(k)} Rj_*(\underline{\underline{IC}}^\bullet_k).$$

But this is exactly what the local calculation says: the stalk of the intersection cohomology is the truncation of the intersection cohomology of the link.

In fact, the formula $\underline{\underline{P}}^\bullet_{k+1} = \tau_{\le p(k)} Rj_* \underline{\underline{P}}^\bullet_k$ implies that the attaching morphism $\underline{\underline{P}}^\bullet_{k+1} \to Rj_*j^* \underline{\underline{P}}^\bullet_{k+1}$ is an isomorphism in degrees $r \le p(k)$, or equivalently, that $H^r(i^! \underline{\underline{P}}^\bullet) = 0$ for $r \le p(k)+1$. This is the same as saying that for any $x \in X^{n-k}$,

$$H^r_c(U_x; \underline{\underline{P}}^\bullet) = H^r(i_x^! \underline{\underline{P}}^\bullet) = 0 \text{ for } r < p(k) + 2 + (n-k) = n - q(k)$$

where $q(k) = k - 2 - p(k)$ is the complementary perversity, $i_x : \{x\} \to W$ is the inclusion and U_x is a basic open neighborhood of x in W. (See also Proposition C.3.)

The construction of the pairing is similar. Start with the multiplication

$$\underline{\underline{\mathbb{Z}}}_{U_2} \otimes \underline{\underline{\mathbb{Z}}}_{U_2} \to \underline{\underline{\mathbb{Z}}}_{U_2}.$$

Now apply $\tau_{\le p(2)} Rj_*$. The truncation of a tensor product is not simply the tensor product of the truncations, there are a lot of cross terms. By examining the effect on the stalk cohomology one eventually arrives at a pairing

$$\underline{\underline{P}}^\bullet_{\bar p} \otimes \underline{\underline{P}}^\bullet_{\bar q} \to \underline{\underline{P}}^\bullet_{\bar p + \bar q}. \qquad \square$$

The Poincaré duality theorem for intersection cohomology, translated into cohomology indexing says the following: if $\bar p + \bar q = \bar t$ then the resulting pairing

$$IH^i_{\bar p}(W) \times IH^{n-i}_{\bar q}(W) \to IH^{\bar t}_n(W) \to \mathbb{Z}$$

is nondegenerate when tensored with any field. In the next section this will be expressed in a sheaf theoretic way.

Finally we can state the sheaf theoretic statement of duality for intersection cohomology. Let K be a field.

18.4. Theorem. [49, 50] *Let X be a Whitney stratified space and let $\bar{p}+\bar{q}=\bar{t}$ be complementary perversities. Let $\mathcal{E}_1\otimes\mathcal{E}_2\to\underline{\underline{K}}$ be a dual pairing of local systems (of K-vector spaces) defined over the top stratum of X. Then the resulting pairing*

$$\underline{\underline{IC}}^\bullet_{\bar{p}}(\mathcal{E}_1)[n]\otimes\underline{\underline{IC}}^\bullet_{\bar{q}}(\mathcal{E}_2)\to\underline{\underline{IC}}^\bullet_{\bar{t}}(K)\to\mathbf{D}^\bullet_X$$

is a Verdier dual pairing.

The proof of Theorem 18.4 is by induction on the strata, as before, adding one stratum at a time, using the long exact sequences, duality, and the above formal properties.

18.5. The middle perversity. Suppose the codimension of every stratum X of the stratified space W is even. Then there is a middle perversity $\bar{m}(c)=(c-2)/2$ and $\underline{\underline{IC}}^\bullet_{\bar{m}}(K)$ is self dual for any field K. If odd codimension strata are present there is an upper middle $\bar{m}^+(c)=\lceil\frac{c-2}{2}\rceil$ and lower middle $\bar{m}^-(c)=\lfloor\frac{c-2}{2}\rfloor$ perversity with a canonical morphism

$$\Phi:\underline{\underline{IC}}^\bullet_{\bar{m}^-}(K)\to\underline{\underline{IC}}^\bullet_{\bar{m}^+}(K)$$

between the dual sheaves $\underline{\underline{IC}}^\bullet_{\bar{m}^+}(K)$ and $\underline{\underline{IC}}^\bullet_{\bar{m}^-}(K)$. The morphism Φ is a quasi-isomorphism if and only if W is a *K-Witt space* ([111, 54]), meaning that for each stratum X of odd codimension c the stalk cohomology vanishes in the middle dimension:

$$IH^{(c-1)/2}_{\bar{m}^-}(L_X;K)=0$$

where L_X denotes the link of the stratum X. For such a space

$$\underline{\underline{IC}}^\bullet_{\bar{m}^-}(K)\cong\underline{\underline{IC}}^\bullet_{\bar{m}^+}(K)$$

is self dual.

18.6. Orientation sheaf. (See also §5.) Suppose X is Whitney stratified and n-dimensional, with largest stratum X^0. The orientation sheaf $\mathcal{O}$ on X^0 is the local system whose stalk at each point $x\in X^0$ is the local homology $H_n(X,X-x;\mathbb{Z})=H_n(X^0,X^0-x;\mathbb{Z})$. If it is placed in degree $-n$ then it becomes (quasi) isomorphic to the dualizing sheaf $\omega=\mathbb{D}(X^0)$, that is $\omega=\mathcal{O}[n]$. Then $\mathbb{P}(\omega)$ is the intersection *homology* sheaf; it's cohomology is:

$$H^{-r}(\mathbb{P}(\omega))=IH_r(X;\mathbb{Z})$$

for any $r\geq 0$. Let $\underline{\underline{\mathbb{Z}}}$ be the constant sheaf on X^0, placed in degree 0. Then

$$H^s(\mathbb{P}(\underline{\underline{\mathbb{Z}}}))=IH^s(X;\mathbb{Z})$$

is the intersection *cohomology*. (Truncations also need to be shifted by n which accounts for the difference between [16] and [90].) The canonical pairings of sheaves on X^0,

$$\underline{\underline{\mathbb{Z}}}\otimes\underline{\underline{\mathbb{Z}}}\to\underline{\underline{\mathbb{Z}}}\quad\text{and}\quad\underline{\underline{\mathbb{Z}}}\otimes\omega\to\omega$$

induce (with appropriately chosen perversities) cup products $IH^* \otimes IH^* \to IH^*$ in intersection cohomology and cap products $IH^* \otimes H_* \to H_*$. These constructions work over the integers or more generally over any commutative ring of finite cohomological dimension.

There is always an isomorphism $\underline{\underline{\mathbb{Z}}}[n] \otimes \mathbb{Z}/(2) \to \omega \otimes \mathbb{Z}/(2)$ of sheaves on X^0. An *orientation* of X^0, if one exists, is an isomorphism $\underline{\underline{\mathbb{Z}}}[n] \to \omega$. So an orientation induces an isomorphism $\mathbb{P}(\underline{\underline{\mathbb{Z}}})[n] \to \mathbb{P}(\omega)$ hence a Poincaré duality isomorphism $IH^*(X) \to IH_*(X)$. In this case the product $\underline{\underline{\mathbb{Z}}} \otimes \underline{\underline{\mathbb{Z}}} \to \underline{\underline{\mathbb{Z}}}$ on X^0 becomes $\omega \otimes \omega \to \omega[n]$ which induces (for appropriate choice of perversity) the (geometrically defined) original intersection product in intersection *homology*. The proof of the following is in §C.2.

18.7. Theorem. [50] *Let W be an n-dimensional Whitney stratified set with biggest stratum $U = U_2$ and let $\bar{p}$ be a pervesity. Then Deligne's construction*

$$\mathcal{E} \mapsto \underline{\underline{P}}^\bullet(\mathcal{E}) = \tau_{\leq p(n)} Rj_{n*} \cdots \tau_{\leq p(3)} Rj_{3*} \tau_{\leq p(2)} Rj_{2*} \mathcal{E}$$

defines an equivalence of categories between the category of local systems of K-vector spaces (K a field) on the nonsingular part $U = U_2$, and the full subcategory of $D^b_c(W)$ consisting of "IC sheaves", that is, complexes of sheaves $A^\bullet$, constructible with respect to the given stratification, such that the following conditions hold

(1) $A^\bullet|U_2 \cong \mathcal{E}$ is isomorphic to a local coefficient system
(2) $\underline{\underline{H}}^r(A^\bullet) = 0$ for $r < 0$
(3) $H^r(i_x^* A^\bullet) = 0$ for $r > p(c)$ ("support condition")
(4) $H^r(i_x^! A^\bullet) = 0$ for $r < n - q(c)$ ("cosupport condition")

for all points $x \in W - U_2$, where $i_x : \{x\} \to W$ is the inclusion of the point and c denotes the (real) codimension of the stratum containing x and where $q(c) = c - 2 - p(c)$ is the complementary perversity.

If $\mathcal{E}$ is a local system on U_2 and if $A^\bullet$ is a constructible complex of sheaves that satisfies the conditions (1)-(4), Theorem 18.7 says that there is a canonical isomorphism $A^\bullet \cong \underline{\underline{IC}}^\bullet_{\bar{p}}(\mathcal{E})$. Since the category of IC sheaves is a full subcategory of the derived category, the theorem also says that

$$\mathrm{RHom}(\underline{\underline{IC}}^\bullet(\mathcal{E}_1), \underline{\underline{IC}}^\bullet(\mathcal{E}_2)) \cong \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2).$$

whenever $\mathcal{E}_1, \mathcal{E}_2$ are local systems on U_2. If $\mathcal{E}$ is an indecomposable local system (which is not isomorphic to a direct sum of two nontrivial local systems) then $\underline{\underline{IC}}^\bullet_{\bar{p}}(\mathcal{E})$ is an indecomposable complex of sheaves (and is not isomorphic to a direct sum of two nontrivial complexes of sheaves).

For a perversity $\bar{p}$ let $p^{-1}(t) = \min\{c|\ p(c) \geq t\}$ and $p^{-1}(t) = \infty$ if $t > p(n)$. We can reformulate these conditions (2,3,4) in a way that does not refer to a particular stratification as follows:

(S1) $\dim\{x \in W|\ H^r(i_x^* A^\bullet) \neq 0\} \leq n - p^{-1}(t)$ for all $t > 0$

(S2) $\dim\{x \in W|\ H^r(i_x^! A^\bullet) \neq 0\} \leq n - q^{-1}(n - t)$ for all $t < n$.

As above, the condition (S2) is the Verdier dual of condition (S1) and may be expressed as

(S2') $\dim\{x \in W |\ H^r(i_x^* \mathbf{D}(A^\bullet)) \neq 0\} \leq n - q^{-1}(t)$ for all $t > 0$.

Part 2. Perverse sheaves

19. Perverse sheaves

19.1. Let W be a Whitney stratified space with a given stratification. We have two notions of the constructible derived category: (1) as complexes of sheaves that are cohomologically constructible with respect to the given stratification (2) as complexes of sheaves that are cohomologically constructible with respect to some stratification. In order to reduce the total number of words in these notes, we shall simply refer to "the constructible derived category", meaning either one of these two possibilities.

The category of perverse sheaves is defined by relaxing the support and cosupport conditions (§18.7) for the IC sheaf by one. Here is the precise definition in the case of middle perversity:

19.2. Definition. Let W be a n-dimensional Whitney stratified (or stratifiable) space that can be stratified with strata of even dimension. Let K be a field.

A *middle perversity* perverse sheaf on W is a complex of sheaves $A^\bullet$ in the bounded constructible derived category $D_c^b(W)$ of K-vector spaces, with the following property. If S is a stratum of dimension d let $j_S : S \to W$ and $j_x : \{x\} \to W$. Then

$$H^i(j_x^* A^\bullet) = 0 \text{ for all } i > (n-d)/2$$
$$H^i(j_x^! A^\bullet) = 0 \text{ for all } i < (n+d)/2$$

Since $H^a(j_x^! A^\bullet) \cong H^{a-d}(j_S^! A^\bullet)_x$ the second condition may be expressed as:

$$H^i(j_S^! A^\bullet) = 0 \text{ for all } i < (n-d)/2$$

Equivalently, $A^\bullet$ is middle perverse if and only if for all $r \geq 0$,

(P1) $\dim\{x \in W |\ H^r(i_x^* A^\bullet) \neq 0\} \leq n - 2r$ and

(P2) $\dim\{x \in W |\ H^r(i_x^! A^\bullet) \neq 0\} \leq 2r - n$.

In these figures, "x" denotes regions of possibly nontrivial stalk cohomology and "c" denotes regions of possibly nontrivial stalk cohomology with compact support, $H^r(j_x^! A^\bullet)$.

19.3. Definition. The *category of perverse sheaves* (with middle perversity) is the full subcategory of $D_c^b(W)$ whose objects are (middle) perverse sheaves.

19.4. Theorem. [12] *The category of (middle) perverse sheaves forms an abelian subcategory of the derived category $D_c^b(W)$ that is preserved by Verdier duality.*

i	cod0	cod2	cod4	cod6	cod8
8	c	c	c	c	c
7			c	c	c
6				c	c
5					c
4					0
3					x
2				x	x
1			x	x	x
0	x	x	x	x	x
	$\underline{\underline{IC}}^\bullet$ support				

i	cod0	cod2	cod4	cod6	cod8
8	c	c	c	c	c
7		c	c	c	c
6			c	c	c
5				c	c
4					cx
3				x	x
2			x	x	x
1		x	x	x	x
0	x	x	x	x	x
	Perverse sheaf support				

FIGURE 15. Stalk and co-stalk cohomology of $\underline{\underline{IC}}^\bullet$ and perverse sheaves

Kernels and cokernels in this category can be described, see §21.8. If W is an algebraic variety, the simple objects are the shifted IC sheaves with irreducible local coefficients of irreducible subvarieties.

19.5. Other perversities. There is an abelian category of perverse sheaves for any perversity and even for the generalized perversities considered in [12], that is, for an arbitrary function $\bar{p}$ from $\{\text{strata}\} \to \mathbb{Z}$. For the case of perversity zero and top see §20.8.

19.6. Historical comment. The discovery of intersection homology was announced [46] in 1977, although the authors gave many lectures about intersection homology, starting in 1974-75. Deligne proposed his sheaf-theoretic construction of intersection homology in 1977 (eventually described in [50]). But it was not until 1979 that its connections with algebra and representation theory were discovered, which quickly led to a series of important developments.

In 1979 David Kazhdan and George Lusztig [75] posed two conjectures involving representations of Hecke algebras, Verma modules, and a collection of polynomials which have come to be known as Kazhdan Lusztig polynomials. They realized that these questions were related to the failure of Poincaré duality for Schubert varieties, which they had analyzed. Lusztig had heard MacPherson lecture about intersection homology and he suspected a relation with the KL polynomials. Separately, Raoul Bott suggested to Kazhdan that intersection homology might be relevant and that he and Lusztig should speak with MacPherson, at the same time telling MacPherson that he should speak with Kazhdan and Lusztig.

In the ensuing conversation MacPherson directed them to contact Deligne who had found a new sheaf-theoretic way to define intersection homology, which worked in the étale setting. Using Deligne's construction, Kazhdan and Lusztig were then able to prove [76] as they suspected, that the Kazhdan Lusztig polynomials coincide with the intersection cohomology local Poincaré polynomial of one Schubert variety at a point in another Schubert cell, thereby also proving that

the coefficients were nonnegative. This paper may be viewed as a step in the proof the K.L. conjectures. (See also Lusztig's more extensive comments in [86, 87].)

The remaining steps in the proof of the Kazhdan Lusztig conjectures were eventually proven by A. Beilinson and J. Bernstein [13] and independently by J. L. Brylinski and M. Kashiwara [23], by making the connection between intersection homology and the vast existing theory of $\mathcal{D}$-modules. On an algebraic manifold (in this case, the flag manifold) there is a ring $\mathcal{D}$ (or rather, a sheaf of rings) of differential operators (and an extensive literature, see [20, 93, 37, 72, 70, 71, 74, 73, 97]). To each $\mathcal{D}$-module there corresponds a sheaf of solutions, which is a constructible sheaf. Beilinson, Bernstein, Brylinski and Kashiwara showed that each Verma module can be associated to a certain holonomic $\mathcal{D}$-module with regular singularities, whose sheaf of solutions turns out to be the IC sheaf. This provided the link between the Kazhdan-Lusztig polynomials and Verma modules.

However, the category of $\mathcal{D}$-modules is an abelian category, whereas the (derived) category of constructible sheaves is not abelian, so it was conjectured that there must correspond an abelian subcategory of the derived category that "receives" the solution sheaves of individual $\mathcal{D}$-modules. Thus was born the category of perverse sheaves, with middle perversity. On the other hand, intersection homology is a topological invariant, so then the question arose as to whether this category of perverse sheaves could be constructed purely topologically, and for other perversities as well. The book [12] completely answers this question, giving a very general setting in which the category of perverse sheaves, an abelian subategory of the derived category, could be constructed.

20. Examples of perverse sheaves

20.1. IC of subvarieties. As above we consider the middle perversity $\bar{m}$ and a Whitney stratified space of dimension n with even codimension ($= k$ in Figure 16) strata. Let Y denote the closure of a single stratum, Y^o. Let $\mathcal{E}_Y$ be a local system on the stratum Y^o. Then the intersection complex $\underline{\underline{IC}}_Y^{\bar{m}}(\mathcal{E}_Y)[-\mathrm{cod(Y)}/2]$ is $\bar{m}$-perverse. Here are the support diagrams for an 8 dimensional stratified space with strata of codimension 0, 2, 4, 6, 8 where, as above, "x" denotes possibly nonzero stalk cohomology and "c" denotes possibly nonzero stalk cohomology with compact support. Adding these up gives the support diagram (Figure 16) for a perverse sheaf.

20.2. Let Y^o be a stratum of W (which is stratified by even dimensional strata). Let Y be its closure with inclusion $j_Y : Y \to W$. It is stratified by even dimensional strata. Let $A^\bullet$ be a perverse sheaf on Y. Then $Rj_*(A^\bullet)[-\mathrm{cod(Y)}/2]$ is a perverse sheaf on W.

20.3. Logarithmic perversity. Because the support conditions for (middle) perverse sheaves are relaxed slightly from those for $\underline{\underline{IC}}^\bullet$, there are several other perversities for which intersection cohomology forms a (middle) perverse sheaf. These include the *logarithmic* perversity $\bar{\ell}$, given by $\bar{\ell}(k) = k/2 = \bar{m}(k) + 1$ and its Verdier dual, the *sublogarithmic* perversity, $\bar{s}$ given by $\bar{s}(k) = \bar{m}(k) - 1$. See Figure 17.

i\k	0	2	4	6	8
8	c	c	c	c	c
7			c	c	c
6				c	c
5					c
4					0
3					x
2				x	x
1			x	x	x
0	x	x	x	x	x
$\dim(Y) = 8$					

i\k	0	2	4	6
6	c	c	c	c
5			c	c
4				c
3				0
2				x
1			x	x
0	x	x	x	x
$\dim(Y) = 6$				

i\k	0	2	4
4	c	c	c
3			c
2			0
1			x
0	x	x	x
$\dim(Y) = 4$			

i\k	0	2
2	c	c
1		0
0	x	x
dim(Y)=2		

i\k	0
0	cx
d(Y)=0	

FIGURE 16. Shifted IC of subvarieties

i	cod0	cod2	cod4	cod6	cod8
8	c	c	c	c	c
7				c	c
6					c
5					0
4					x
3				x	x
2			x	x	x
1			x	x	x
0	x	x	x	x	x
	$\underline{\underline{IC}}^{\bullet}_{\bar{\ell}}$ support				

i	cod0	cod2	cod4	cod6	cod8
8	c	c	c	c	c
7		c	c	c	c
6			c	c	c
5				c	c
4					c
3					0
2					x
1				x	x
0	x	x	x	x	x
	$\underline{\underline{IC}}^{\bullet}_{\bar{s}}$ support				

FIGURE 17. Support/cosupport of logarithmic and sublogarithmic IC sheaf

20.4. Hyperplane complements. (see also [68]) Let $\{H_1, H_2, \cdots, H_r\}$ be a collection of complex affine hyperplanes in $W = \mathbb{C}^n$. Stratify W according to the multi-intersections of the hyperplanes. The largest stratum is

$$W^o = W - \bigcup_{j=1}^{r} H_j$$

and it may have a highly nontrivial fundamental group. let $\mathcal{E}$ be a local coefficient system on this hyperplane complement. Then $\underline{\underline{IC}}^{\bullet}_{\bar{s}}(\mathcal{E})$, $\underline{\underline{IC}}^{\bullet}_{\bar{m}}(\mathcal{E})$ and $\underline{\underline{IC}}^{\bullet}_{\bar{\ell}}(\mathcal{E})$ are perverse sheaves on $W = \mathbb{C}^n$. These are surprisingly complicated objects, and even the case of middle perversity, when the hyperplanes are the coordinate hyperplanes, has been extensively studied. Notice, in this case,

that the space $W = \mathbb{C}^n$ is nonsingular, the hyperplane complement W^o is nonsingular, and the sheaf $\underline{\underline{IC}}(\mathcal{E})$ is constructible (with respect to this chosen stratification) but to analyze this sheaf we are forced to consider the singularities of the multi-intersections of the hyperplanes.

In the simplest case, $(\mathbb{C}, \{0\})$ the category of perverse sheaves is equivalent to the category of representations of the following quiver

$$\bullet \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \bullet$$

where $I - \alpha\beta$ and $I - \beta\alpha$ are invertible.

For $\mathbb{C}^2$, $xy = 0$ (the coordinate axes) the perverse category is equivalent to the category of representations of the quiver

$$\begin{tikzcd} \bullet \arrow[r, shift left] \arrow[d, shift left] & \bullet \arrow[l, shift left] \arrow[d, shift left] \\ \bullet \arrow[r, shift left] \arrow[u, shift left] & \bullet \arrow[l, shift left] \arrow[u, shift left] \end{tikzcd}$$

with similar conditions ($I - \alpha\beta$ and $I - \beta\alpha$ invertible) on each of the horizontal and vertical pairs, such that all possible ways around the outside of the square commute.

20.5. Small and semismall maps. Let M be a compact complex algebraic manifold and let $\pi : M \to W$ be an algebraic mapping. Then π is said to be *semismall* [21] if

$$\mathrm{cod_W}(\{\mathrm{x} \in \mathrm{W} |\ \dim \pi^{-1}(\mathrm{x}) \geq \mathrm{k}\}) \geq 2\mathrm{k}.$$

In other words, if the map has been stratified then for each stratum $S \subset W$ the dimension of the fiber over S is $\leq \frac{1}{2}$ the codimension of S. The map is *small* [50] if, for each singular stratum S, $\dim \pi^{-1}(x) < \frac{1}{2}\mathrm{cod(S)}$ (for all $x \in S$).

If π is small then $R\pi_*(\underline{\underline{\mathbb{Q}}})$ is a self dual sheaf on W whose support satisfies the support conditions of (middle) intersection cohomology. It follows from the axiomatic characterization that there is a canonical isomorphism (in $D^b_c(W)$), $R\pi_*(\underline{\underline{\mathbb{Q}}}) \cong \underline{\underline{IC}}_{\bar{m}}(W)$. In other words, the intersection cohomology of W is canonically isomorphic to the ordinary cohomology of M.

If π is semi-small then $R\pi_*(\underline{\underline{\mathbb{Q}}})$ is (middle) perverse.

Let $W = \{P \subset \mathbb{C}^4 |\ \dim(P) = 2, \dim(P \cap \mathbb{C}^2) \geq 1\}$ be the singular Schubert variety in the Grassmannian of 2-planes in 4-space. It has a singularity when $P = \mathbb{C}^2$. A resolution of singularities is $\widetilde{W} = \{(P, L) |\ P \in W, \text{ and } L \subset P \cap \mathbb{C}^2 \subset \mathbb{C}^4\}$. Then $\pi : \widetilde{W} \to W$ is a small map so $R\pi_*(\mathbb{Q}) \cong \underline{\underline{IC}}^\bullet_W$ hence $IH^*(W) \cong H^*(\widetilde{W})$.

20.6. Sheaves on $\mathbb{P}^1$. (See [92].) Let us stratify $\mathbb{P}^1$ with a single zero dimensional stratum, N (the north pole, say). The support diagram for middle perversity sheaves is the following:

$i\backslash$cod	0	2
2	c	c
1		cx
0	x	x

The skyscraper sheaf supported at the point, $\underline{\underline{\mathbb{Q}}}_N[-1]$ is perverse. We also have the following:

$\underline{\underline{\mathbb{Q}}}_N[-1]$:

2		
1		cx
0		

$\underline{\underline{\mathbb{Q}}}_{\mathbb{P}^1}$:

2	c	c
1		
0	x	x

$j_!(\underline{\underline{\mathbb{Q}}}_U)$:

2	c	c
1		c
0	x	

$j_*(\underline{\underline{\mathbb{Q}}}_U)$:

2	c	
1		x
0	x	x

The first sheaf is self dual. The second sheaf is self dual. The third and fourth sheaves are dual to each other. It turns out that there is one more indecomposable perverse sheaf on this space, which is not an IC sheaf, and its support diagram is the full diagram. It is self dual. Here is how to construct it. Take a closed disk and put the constant sheaf on the interior, zero on the boundary, except for one point (or even one segment). Then map this disk to the 2-sphere, collapsing the boundary to the N pole, and push this sheaf forward.

If we started with zero on the boundary and pushed forward we would get the sheaf $Rj_!(\mathbb{Q}_U)$. If we started with the full constant sheaf on the disk and pushed forward we would get the sheaf $Rj_*(\mathbb{Q})$. This new sheaf has both stalk cohomology and compact support stalk cohomology in degree 1, at the singular point. Verdier duality switches these two types of boundary conditions, so when we have a mixed boundary condition as in this case, we obtain a self dual sheaf.

In this case the category of perverse sheaves is equivalent to the category of representations of the quiver

$$\bullet \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \bullet$$

where $\alpha\beta = \beta\alpha = I$. There are five indecomposable objects, one of which is has $\mathbb{Q} \oplus \mathbb{Q}$ on one of the vertices of the graph.

20.7. Homological stratifications. Although we have assumed the space W is Whitney stratified, all the preceding arguments remain valid for spaces with a homological stratification. Homological stratifications were used to prove ([50]) the topological invariance of intersection homology and are also required in the étale setting.

Suppose W is a locally compact Hausdorff space with a locally finite decomposition into locally closed subsets called "strata" which satisfy the condition of the frontier: the closure of each stratum is a union of strata. If $T \subset \overline{S}$ write $T < S$.

Such a decomposition is a homological stratification of W provided

- the strata are topological manifolds
- There is an open dense stratum $U \subset W$ so that $W - U$ has codimension ≥ 2
- For each stratum $j : S \to W$ and for every (finite dimensional) local coefficient system $\mathcal{E}$ on S and for every i, the sheaf $R^i j_*(\mathcal{E}) = \underline{\underline{H}}^i(Rj_*(\mathcal{E}))$ is constructible, that is, locally constant and finite dimensional on every stratum $T < S$.

If we restrict the coefficients to lie in a field k (so that the local coefficient systems are systems of k-vector spaces) then condition (1) may be replaced by the weaker assumption that each stratum S is a k-homology manifold[10], as required for arguments involving Poincaré-Verdier duality.

20.8. Perversity zero. (See also §21.2.) Let W be a stratified pseudomanifold of dimension n (with a fixed stratification). The category of Perverse sheaves on W with perversity zero, constructible with respect to this stratification, is equivalent to the category of sheaves on W that are constructible with respect to this stratification, that is, sheaves whose restriction to each stratum is locally trivial. In this case, the "abelian subcategory" defined by the perversity condition simply coincides with the abelian category structure of the category of (constructible) sheaves. Here is the support diagram for $IC^{\bar{0}}$ of a six dimensional pseudomanifold, where "x" denotes possibly nonzero cohomology and "c" denotes possibly nonzero compact support cohomology:

i	cod0	cod1	cod2	cod3	cod4	cod5	cod6
6	c	c	c	c	c	c	c
5				c	c	c	c
4					c	c	c
3						c	c
2							c
1							0
0	x	x	x	x	x	x	x
	$\underline{\underline{IC}}^{\bullet}_{\bar{0}}$ support						

The support diagram (Figure 18) for the constant sheaf is the same if the link of every stratum is connected. In general the "c" in the bottom of each column in the following support diagram is given by the reduced cohomology $h^0(L)$ of the link of the stratum.

If $j : X \to W$ is the inclusion of a stratum and E is a local system on X then $j_!E$ is a perversity zero sheaf on W. The second table in Figure 18 may be obtained by "summing" support diagrams for sheaves of the form $j_!(E)$ of strata of dimension 0, 1, 2, 3, 4, 5 and 6 (with no shift). A "pseudomanifold" has no strata of codiimension one but in these diagrams we have included the

[10]meaning that $H_r(S, S - x; k) = 0$ for $r \neq \dim(S)$ and $H_{\dim(S)}(S, S - x; k) \cong k$ for all $x \in S$.

i	cod0	cod1	cod2	cod3	cod4	cod5	cod6
6	c	c	c	c	c	c	c
5			c	c	c	c	c
4				c	c	c	c
3					c	c	c
2						c	c
1							c
0	x	x	x	x	x	x	x
	constant sheaf support						

i	cod0	cod1	cod2	cod3	cod4	cod5	cod6
6	c	c	c	c	c	c	c
5		c	c	c	c	c	c
4			c	c	c	c	c
3				c	c	c	c
2					c	c	c
1						c	c
0	x	x	x	x	x	x	xc
	perverse sheaf support						

FIGURE 18. Support/cosupport for constant sheaf and perversity $\bar{0}$ sheaf

possibility of codimension one strata to allow for pseudomanifolds with boundary, and to allow for stratification by the simplices of a triangulation.

20.9. Top perversity. For $\bar{p}(S) = \bar{t}(S) = \mathrm{cod(S)} - 2$ the support and cosupport diagram for a 6 dimensional stratified space are shown in Figure 19 and 20. We have shifted the degrees by 6 so that the dualizing sheaf (the sheaf of Borel Moore chains) is $\bar{t}$ perverse.

i	cod0	cod1	cod2	cod3	cod4	cod5	cod6
0	c	c	c	c	c	c	c
-1							0
-2							x
-3						x	x
-4					x	x	x
-5				x	x	x	x
-6	x	x	x	x	x	x	x
	$\underline{\underline{IC}}^\bullet_{\bar{t}}$ support						

FIGURE 19. Support/cosupport for $\bar{p} = \bar{t}$ sheaf

For the sheaf of chains, the uppermost "x" in each column Figure 20 represents the reduced homology $h_0(L)$ of the link of the stratum. If $j : X \to W$ is a stratum and E is a local system on X then $Rj_*(E)[\dim(X)]$ is a sheaf on W with top perversity.

Suppose $A^\bullet$ is a top-perverse sheaf with on an n-dimensional pseudomanifold W. Then $A^\bullet$ may be interpreted as a *cosheaf*, namely, the covariant functor on the category of open sets,

$$U \mapsto H^0_c(U; A^\bullet) \tag{20.9.1}$$

which assigns the group homomorphism $H^0_c(U; A^\bullet) \to H^0_c(V; A^\bullet)$ to each inclusion $U \subset V$ in a functorial way. (Recall that we have introduced a shift by n, the dimension of the space. With no

i	cod0	cod1	cod2	cod3	cod4	cod5	cod6
0	c	c	c	c	c	c	c
-1							x
-2						x	x
-3					x	x	x
-4				x	x	x	x
-5			x	x	x	x	x
-6	x	x	x	x	x	x	x
	$\underline{\underline{\mathbb{D}}}^\bullet$ support						

i	cod0	cod1	cod2	cod3	cod4	cod5	cod6
0	c	c	c	c	c	c	cx
-1						x	x
-2					x	x	x
-3				x	x	x	x
-4			x	x	x	x	x
-5		x	x	x	x	x	x
-6	x	x	x	x	x	x	x
	perverse sheaf support						

FIGURE 20. Support/cosupport for the Borel-Moore sheaf and perversity $\bar{t}$ sheaf

shift, this cosheaf would be $H_c^n(U; A^\bullet)$.) The "co-stalk" at a point $j_x : \{x\} \in W$ is the limit over neighborhoods U_x of x:

$$H^0(j_x^! A^\bullet) = \varprojlim H_c^0(U_x; A^\bullet).$$

The support conditions (see Figure 20) imply that the co-stalk cohomology vanishes except in this single degree and the cosheaf (20.9.1) determines $A^\bullet$ up to quasi-isomorphism.

20.10. BBDG numbering system. In their book [12] the authors modified the indexing system for cohomology in a way that vastly reduces the amount of notation and arithmetic involving indices. Although the new system is extremely simple, it is deceptively so, because it takes us one step further away from any intuition concerning perverse sheaves. The new system works best in the case of a complex algebraic (or analytic) variety W, stratified with complex algebraic (or analytic) strata, and counted according to their complex dimensions. The idea is to shift all degrees by $\dim_{\mathbb{C}}(W) = \dim(W)/2$ because cohomology is symmetric about this point. So the support conditions look like this:

new	old	cod0	cod2	cod4	cod6	cod8
4	8	c	c	c	c	c
3	7		c	c	c	c
2	6			c	c	c
1	5				c	c
0	4					cx
-1	3				x	x
-2	2			x	x	x
-3	1		x	x	x	x
-4	0	x	x	x	x	x
		Perverse sheaf support				

FIGURE 21. Deligne's numbering for middle perversity

If S is a stratum of dimension d and codimension c ($d + c = n = \dim(W)$) let $j_S : S \to W$ and $j_x : \{x\} \to W$. Then subtracing $n/2$ from the indices in §19.2 gives support conditions for a middle perverse sheaf $A^\bullet$ in the new indexing scheme:

$$\begin{aligned} H^i_{new}(j_S^* A^\bullet) &= 0 \text{ for all } i > -d/2 \\ H^i_{new}(j_S^! A^\bullet) &= 0 \text{ for all } i < -d/2 \end{aligned} \tag{20.10.1}$$

Changing the degree of cohomology implies a corresponding change in the perversity function. The authors of [12] chose to further simplify the numerology by removing the "-2" that occurs in IC calculations. They define the middle perversity to take values $p(S) = -\dim(S)/2$ for every even dimensional stratum S so that equations (20.10.1) become

$$\begin{aligned} H^i_{new}(j_S^* A^\bullet) &= 0 \text{ for all } i > p(S) \\ H^i_{new}(j_S^! A^\bullet) &= 0 \text{ for all } i < p(S). \end{aligned}$$

The IC sheaf is defined by replacing these strict inequalities by $\leq$ and $\geq$.

20.11. This change of numbering has important consequences. Suppose $j : V \to W$ is the inclusion of a closed algebraic subvariety V in an algebraic variety W and suppose $A^\bullet$ is a perverse sheaf on V. If V, W are of pure dimension v, w respectively then $Rj_*(A^\bullet)[v - w]$ is perverse on W, using the original numbering scheme and the degree shift is essential. On the other hand, in the new numbering scheme, $Rj_*(A^\bullet)$ is perverse on W without any degree shift.

If W is an *ind*-variety, say $W_0 \subset W_1 \subset W_2 \subset \cdots$ (inclusions of closed subvarieties or subschemes) then a perverse sheaf on some W_j becomes a perverse sheaf on W, in the original numbering scheme, but only by thinking of it as a collection of ever-more shifted sheaves on each W_k ($k \geq j$). In the new numbering scheme, perverse sheaves on each W_j are automatically perverse on W.

Important examples of perverse sheaves on *ind*-varieties include the affine Grassmannian and affine flag manifold, cf. §28.

20.12. More generally the authors of [12] consider a perversity to be a function of *dimension*:

A perversity is an integer valued function $p : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\leq 0}$ with $p(0) = 0$ and

$$p(d) \geq p(d+1) \geq p(d) - 1. \tag{20.12.1}$$

(See diagram in §29.6.) The category of perverse sheaves is defined to be (abelian) full subcategory of $D^b_c(W)$ consisting of complexes $A^\bullet$ so that for any stratum S (except the largest stratum),

- $H^n(i_x^*(A^\bullet)) = 0$ for all $n > p(S)$
- $H^n(i_x^!(A^\bullet)) = 0$ for all $n < p(S) + \dim(S)$

for some (and hence any) $x \in S$ while the IC sheaf satisfies

- $H^n(i_x^*(\underline{\underline{IC}}^\bullet)) = 0$ for all $n \geq p(S)$
- $H^n(i_x^!(\underline{\underline{IC}}^\bullet)) = 0$ for all $n \leq p(S) + \dim(S)$.

Each perversity involves its own "shift": for a space W of dimension n the stalk cohomology of the IC sheaf at points in the top stratum is nonzero in degree $p(n)$. This is consistent with Deligne's numbering system in which the middle perversity is $p(S) = -\dim(S)/2$.

21. t-structures and Perverse cohomology

21.1. Definition. An *indecomposable* object A in an abelian category is one that cannot be expressed nontrivially as a direct sum $A = B \oplus C$. A *simple* object A is one that has no nontrivial subobjects $B \to A$ (where the morphism is a monomorphism). An object is *semisimple* if it is a direct sum of simple objects. An object is *Artinian* if descending chains stabilize and is *Noetherian* if ascending chains stabilize. A category is Artinian (resp. Noetherian) if every object is. Each object in an Artinian Noetherian category can be expressed as a finite iterated sequence of extensions of simple objects.

21.2. The perversity zero t structure: ordinary sheaves. Let W be a stratified space. The category $Sh_c(W)$ of (ordinary) sheaves on W that are constructible with respect to this stratification is Artinian and Noetherian: If S is a constructible sheaf on W there is a largest stratum $j : X \to W$ so that the local system $j^*(S)$ is nonzero. So there is an exact sequence

$$0 \to j_! j^* S \to S \to \underline{\underline{\text{coker}}} \to 0$$

and the cokernel is supported on smaller strata. Continuing by induction we conclude that S is an iterated extension of sheaves of the type $j_!(A)$ where A is a local system on a single stratum, which is itself an iterated extension of simple local systems.

Now let $A^\bullet$ be a complex of sheaves. We have truncation functors

$$\begin{aligned} A^\bullet &= (\cdots \xrightarrow{d^{r-2}} A^{r-1} \xrightarrow{d^{r-1}} A^r \xrightarrow{d^r} A^{r+1} \xrightarrow{d^{r+1}} \cdots) \\ \tau_{\leq r}A^\bullet &= (\cdots \longrightarrow A^{r-1} \longrightarrow \underline{\underline{\ker}}(d^r) \longrightarrow 0 \longrightarrow \cdots) \\ \tau^{\geq r}A^\bullet &= (\cdots \longrightarrow 0 \longrightarrow \underline{\underline{\operatorname{coker}}}(d^{r-1}) \longrightarrow A^{r+1} \longrightarrow \cdots) \end{aligned} \tag{21.2.1}$$

Then there is a short exact sequence $0 \to \tau_{\leq 0}A^\bullet \to A^\bullet \to \tau^{\geq 1}A^\bullet \to 0$, and the cohomology sheaf of $A^\bullet$ is given by

$$\underline{\underline{H}}^r(A^\bullet) = \tau_{\leq r}\left(\tau^{\geq r}A^\bullet\right) = \tau^{\geq r}\left(\tau_{\leq r}A^\bullet\right)$$

To summarize, let $Sh_c(W)$ be the category of (ordinary) sheaves on W that are constructible with respect to this stratification. Then the the following holds:

21.3. Theorem. *The cohomology functor $\underline{\underline{H}}^r : D^b_c(W) \to Sh_c(W)$ is given by $\tau_{\leq r} \circ \tau^{\geq r}$ and also by $\tau^{\geq r} \circ \tau_{\leq r}$. The functor $\underline{\underline{H}}^0$ restricts to an equivalence of categories between $Sh_c(W)$ and the full subcategory of $D^b_c(W)$ whose objects are complexs $A^\bullet$ such that $\underline{\underline{H}}^r(A^\bullet) = 0$ for $r \neq 0$. This category is Artinian and Noetherian and its simple objects are the sheaves $j_!(\mathcal{E})$ where $\mathcal{E}$ is a simple local system on a single stratum $j : X \to W$.*

21.4. The simple observations in the preceding paragraph reflect a general principal. Fix a perversity $\bar{p}$. Let $\mathcal{P}(W)$ denote the category of $\bar{p}$-perverse sheaves on W that are constructible with respect to a given stratification. There are truncation functors, cf. §21.7,

$${}^{\bar{p}}\tau_{\leq r} \quad \text{and} \quad {}^{\bar{p}}\tau^{\geq r} : D^b_c(W) \to D^b_c(W)$$

which are cohomological, that is, they take distinguished triangles to exact sequences, and satisfy

(T1) ${}^{\bar{p}}\tau_{\leq r}(A^\bullet) = ({}^{\bar{p}}\tau_{\leq 0}(A^\bullet[r]))[-r]$.

From this, define the *perverse cohomology*

$${}^{\bar{p}}H^r(A^\bullet) = {}^{\bar{p}}\tau_{\leq r}\left({}^{\bar{p}}\tau^{\geq r}A^\bullet\right).$$

Then ${}^{\bar{p}}H^r : D^b_c(W) \to \mathcal{P}(W)$ and $A^\bullet \in \mathcal{P}(W)$ if and only if ${}^{\bar{p}}H^r(A^\bullet) = 0$ for all $r \neq 0$. In this case ${}^{\bar{p}}H^0(A^\bullet) = A^\bullet$. The same method of proof (cf. [12] 4.3.1) gives:

21.5. Theorem. *The cohomology functor ${}^{\bar p}H^r : D^b_c(W) \to \mathcal{P}(W)$ is given by ${}^{\bar p}\tau_{\le r} \circ {}^{\bar p}\tau^{\ge r}$ and also by ${}^{\bar p}\tau^{\ge r} \circ {}^{\bar p}\tau_{\le r}$. The functor ${}^{\bar p}H^0$ restricts to an equivalence of categories between $\mathcal{P}(W)$ and the subcategory of $D^b_c(W)$ whose objects are complex $A^\bullet$ such that ${}^{\bar p}H^r(A^\bullet) = 0$ for $r \neq 0$. This category is Artinian and Noetherian and its simple objects are the sheaves $Rj_*(\underline{\underline{IC}}^\bullet_{\bar p}(\mathcal{E}))$ where $\mathcal{E}$ is a simple local system on a single stratum X and where $j : \bar X \to W$ is the inclusion.*

In particular, a *semisimple* perverse sheaf is one which is a direct sum of (appropriately shifted) intersection cohomology sheaves of (closures of) strata.

21.6. t structures. The truncation functors are determined by the support and cosupport conditions. At this point it becames essential to shift to Deligne's numbering scheme (see §20.12), and for simplicity we consider only the case of middle perversity $p(S) = -\dim(S)/2$.

Define the full subcategory $D^b_c(W)_{\le 0} \subset D^b_c(W)$ to consist of constructible complexes $A^\bullet$ that satisfy the support condition, that is,

$$\dim_{\mathbb{C}}\{x \in W |\ H^i(j_x^* A^\bullet) \neq 0\} \le -i$$

and define $D^b_c(W)^{\ge 0}$ to consist of complexes $A^\bullet$ that satisfy the cosupport condition,

$$\dim_{\mathbb{C}}\{x \in W |\ H^i(j_x^! A^\bullet) \neq 0\} \le i.$$

Then $\mathcal{P}(W) = D^b_c(W)_{\le 0} \cap D^b_c(W)^{\ge 0}$. Define $D^b_c(W)_{\le m} = D^b_c(W)_{\le 0}[-m]$ and $D^b_c(W)^{\ge m} = D^b_c(W)^{\ge 0}[-m]$. In §21.7 we show:

(T2) $D^b_c(W)_{\le 0} \subset D^b_c(W)_{\le 1}$ and $D^b_c(W)^{\ge 0} \supset D^b_c(W)^{\ge 1}$.
(T3) $\mathrm{Hom}_{D^b_c(W)}(A^\bullet, B^\bullet) = 0$ for all $A^\bullet \in D^b_c(W)_{\le 0}$ and $B \in D^b_c(W)^{\ge 1}$
(T4) For any complex $X^\bullet$ there is a distinguished triangle

$$\begin{tikzcd} A^\bullet \arrow[rr] & & X^\bullet \arrow[dl] \\ & B^\bullet \arrow[ul, "{[1]}"] & \end{tikzcd}$$

where $A^\bullet \in D^b_c(W)^{\le 0}$ and where $B^\bullet \in D^b_c(W)^{\ge 1}$.

This determines $A^\bullet$ and $B^\bullet$ up to unique isomorphism in $D^b_c(W)$ and it also determines the $\bar p$-*perverse truncation functors*

$${}^{\bar p}\tau_{\le 0} X^\bullet := A^\bullet \qquad \text{and} \qquad {}^{\bar p}\tau^{\ge 1} X^\bullet := B^\bullet.$$

21.7. Proof of (T2), (T3), (T4). Statement (T2) is clear and (T3) comes from Lemma C.1. Statement (T4) is the essential technical step in the theory: for any $X^\bullet$ in $D^b_c(W)$ we must construct its "truncations" $A^\bullet$ in $D^b_c(W)^{\leq 0}$ and $B^\bullet$ in $D^b_c(W)^{\geq 1}$. If W is a manifold and if $X^\bullet$ is cohomologically locally constant on W then take $A^\bullet = \tau_{\leq 0}X^\bullet$ and $B^\bullet = \tau^{\geq 1}X^\bullet$ as in (21.2.1), that is, ${}^{\bar{p}}\tau = \tau$ is the usual truncation. Otherwise, assume by induction that we have defined these truncation functors on an open union U of strata, and consider the addition of a single stratum S. We may assume that $W = U \cup S$.

$$U \xrightarrow[j]{} W \xleftarrow[i]{} S$$

Given a complex $X^\bullet$ on W let $Y^\bullet[-1]$ be the mapping cone of $X^\bullet \to Rj_*{}^{\bar{p}}\tau^{\geq 1}j^*X^\bullet$ and let $A^\bullet[-1]$ be the mapping cone of $Y^\bullet \to Ri_*{}^{\bar{p}}\tau^{\geq 1}i^*Y^\bullet$.

$$\begin{tikzcd}
 & X^\bullet \arrow[rr] & & Rj_*{}^{\bar{p}}\tau^{\geq 1}j^*X^\bullet \arrow[dl, "{[1]}"] \\
A^\bullet \arrow[ur, dotted] \arrow[rr] & & Y^\bullet \arrow[ul] \arrow[dl] & \\
 & Ri_*{}^{\bar{p}}\tau^{\geq 1}i^*Y^\bullet \arrow[ul, "{[1]}"] & &
\end{tikzcd}$$

This gives the desired morphism $A^\bullet \to X^\bullet$ as the composition $A^\bullet \to Y^\bullet \to X^\bullet$. Then $B^\bullet$ is the third term in the triangle defined by $A^\bullet \to X^\bullet$. There is a lot to check that $A^\bullet, B^\bullet$ have the desired properties. This argument is in [12] p. 48.

For example, suppose we are in the situation of middle perversity when there are two strata, the smaller stratum of even codimension. Let $X^\bullet = Rj_*(\mathbb{Q})$. Then ${}^{\bar{p}}\tau^{\geq 1}j^*X^\bullet = 0$ so $Y^\bullet = X^\bullet$. Now the stalk cohomology of $i^*(Y^\bullet)$ equals the cohomology of the link (with no shift). So ${}^{p}\tau^{\geq 1}i^*(Y^\bullet)$ is the cohomology of the link in degrees (strictly) above the middle. Therefore the mapping cone, which is $A^\bullet$, keeps the cohomology of the link in degrees $\leq$ the middle: it is the $\underline{\underline{IC}}^\bullet_{\bar{\ell}}$ (logarithmic perversity, in the "classical" numbering scheme) sheaf, which is perverse (see §20.3). We end up with $0 \to A^\bullet \to X^\bullet \to B^\bullet \to 0$ where $A^\bullet$ is perverse and where $B^\bullet$ is in $D^b_c(W)^{\geq 1}$.

If instead, we start with $X^\bullet = Rj_!(\mathbb{Q})$ (which is already in $D^b_c(W)^{\leq 0}$) then $X^\bullet = Y^\bullet$ as before but $i^*Y^\bullet = 0$ so $A^\bullet = Y^\bullet = X^\bullet$.

21.8. Perverse cohomology. If $\phi : A^\bullet \to B^\bullet$ is a morphism (in $D^b_c(W)$) between two perverse sheaves, we may consider it to be a morphism of complexes $A^\bullet \to B^\bullet$. As such, it has a kernel and a cokernel. These are unlikely to be perverse, and moreover, they may change if we choose different (but quasi-isomorphic) representative complexes for $A^\bullet, B^\bullet$: as a morphism in the derived category, ϕ does not have a kernel or cokernel. It only has a mapping cone $M^\bullet$.

However, the kernel and cokernel of ϕ *in the category of perverse sheaves* are again perverse sheaves, and are well defined as elements of the derived category. Moreover, various constructions from the theory of abelian cartegories can be implemented. For example, suppose $A^\bullet_0 \xrightarrow{d} A^\bullet_1 \xrightarrow{d}$

$A_2^\bullet \xrightarrow{d} \cdots$ is a complex of perverse sheaves, that is, a complex such that $d \circ d = 0$ in the derived category. Then $\ker(d)/\operatorname{Im}(d)$ makes sense as a perverse sheaf, so we obtain the *perverse cohomology* ${}^{\bar p}H^r(A_\bullet^\bullet)$ of such a complex.

There is a beautiful way to see how the mapping cone $M^\bullet$ of a morphism $\phi : A^\bullet \to B^\bullet$ of perverse sheaves is broken into a (perverse) kernel and cokernel. The long exact sequence on perverse cohomology for the triangle $A^\bullet \to B^\bullet \to M^\bullet$ reads as follows:

$$\begin{array}{ccccccccccccc}
\cdots \longrightarrow & {}^{\bar p}H^{-1}(B^\bullet) & \longrightarrow & {}^{\bar p}H^{-1}(M^\bullet) & \longrightarrow & {}^{\bar p}H^{0}(A^\bullet) & \longrightarrow & {}^{\bar p}H^{0}(B^\bullet) & \longrightarrow & {}^{\bar p}H^{0}(M^\bullet) & \longrightarrow & {}^{\bar p}H^{1}(A^\bullet) & \longrightarrow \cdots \\
 & \| & & \| & & \| & & \| & & \| & & \| & \\
 & 0 & \longrightarrow & {}^{p}\ker(\Phi) & \longrightarrow & A^\bullet & \xrightarrow{\Phi} & B^\bullet & \longrightarrow & {}^{p}\operatorname{coker}(\Phi) & \longrightarrow & 0 &
\end{array}$$

Thus, the perverse kernel and cokernel are precisely the perverse cohomology sheaves of $M^\bullet$. (The same statement holds in the category of sheaves: the kernel and cokernel of a sheaf morphism $A \to B$ are isomorphic to the cohomology sheaves of the mapping cone.)

Perverse versions of other functors can be defined by using ${}^{\bar p}\underline{\underline{H}}^0$ to "project" the result into the category $\mathcal{P}(W)$. For example, if $j : U \to X$ and if $A^\bullet$ is perverse on U then ${}^{\bar p}j_*A^\bullet = {}^{\bar p}H^0(Rj_*A^\bullet)$ and ${}^{\bar p}j_!A^\bullet = {}^{\bar p}H^0(Rj_!A^\bullet)$. If we start with a local system $\mathcal{E}$ on the nonsingular part $U = U_2$ of a stratified space W with even codimension strata then with middle perversity,

$$ {}^{\bar p}j_!(\mathcal{E}) = \underline{\underline{IC}}^\bullet_{\bar s}(\mathcal{E}) \quad \text{and} \quad {}^{\bar p}j_*(\mathcal{E}) = \underline{\underline{IC}}^\bullet_{\bar \ell}(\mathcal{E}) $$

are the logarithmic and sub-logarithmetic intersection complexes. The (perverse) image of the first in the second is the middle IC sheaf, and it is sometimes referred to as

$$ {}^{\bar p}j_{!*}(\mathcal{E}) = \underline{\underline{IC}}^\bullet_{\bar m}(\mathcal{E}). $$

21.9. More generally, a t-structure on a triangulated category $\mathcal{D}$ is defined to be a pair of strictly full subcategories $\mathcal{D}_{\le 0}$ and $\mathcal{D}^{\ge 0}$ satisfying (T1), (T2), (T3), (T4) above, where $\mathcal{D}^{\ge m} = \mathcal{D}^{\ge 0}[-m]$ and $\mathcal{D}_{\le m} = \mathcal{D}_{\le 0}[-m]$. It is proven in [BBD] that under these hypotheses the *heart* $P = \mathcal{D}_{\le 0} \cap \mathcal{D}^{\ge 0}$ is an abelian full subcategory.

22. Algebraic varieties and the decomposition theorem

22.1. Lefschetz theorems. Suppose $W \subset \mathbb{CP}^N$ is a complex projective algebraic variety of complex dimension n. Let $L^j \subset \mathbb{CP}^n$ be a codimension j linear subspace. Let $Y^j = L^j \cap W$. If L^j is transverse to each stratum of a Whitney stratification of W then there are natural morphisms $\alpha : IH^r(W;\mathbb{Q}) \to IH^r(Y^j;\mathbb{Q})$ and $\beta : IH^s(Y^j;\mathbb{Q}) \to IH^{s+2j}(W;\mathbb{Q})$.

22.2. Theorem. *If L^1 is transverse to W then the restriction mapping $IH^r(W;\mathbb{Z}) \to IH^r(Y^1;\mathbb{Z})$ is an isomorphism for $r \le n-2$ and is an injection for $r = n-1$. If $j \ge 1$ and L^j is transverse to W then the composition $L^j : IH^{n-j}(W;\mathbb{Q}) \to IH^{n-j}(Y^j;\mathbb{Q}) \to IH^{n+j}(W;\mathbb{Q})$ is an isomorphism.*

These maps are illustrated in the following diagram.

$$
\begin{array}{ccccccccccc}
IH^0 & IH^1 & IH^2 & IH^3 & IH^4 & IH^5 & IH^6 & IH^7 & IH^8 & IH^9 & IH^{10} \\
IP^0 & \longrightarrow & L(IP^0) & \longrightarrow & L^2(IP^0) & \longrightarrow & L^3(IP^0) & \longrightarrow & L^4(IP^0) & \longrightarrow & L^5(IP^0) \\
 & & IP^1 & \longrightarrow & L(IP^1) & \longrightarrow & L^2(IP^1) & \longrightarrow & L^3(IP^1) & \longrightarrow & L^4(IP^1) \\
 & & & & IP^2 & \longrightarrow & L(IP^2) & \longrightarrow & L^2(IP^2) & \longrightarrow & L^3(IP^2) \\
 & & & & & & IP^3 & \longrightarrow & L(IP^3) & \longrightarrow & L^2(IP^3) \\
 & & & & & & & & IP^4 & \longrightarrow & L(IP^4) \\
 & & & & & & & & & & IP^5
\end{array}
$$

where (for $j \leq n$) the *primitive intersection cohomology* $IP^j \subset IH^j$ is the kernel of $\cdot L^{n-j+1}$. (It may also be identified with the cokernel of $\cdot L : IH^{j-2} \to IH^j$.) The resulting decomposition is called the *Lefschetz decomposition* for $r \leq n$, $IH^r \cong \oplus_{j=0}^{\lfloor r/2 \rfloor} L^j \cdot IP^{r-2j}$. The combination of Poincaré duality

$$IH^{n+r}(W;\mathbb{Q}) \cong \mathrm{Hom}(IH^{n-r}(W;\mathbb{Q}),\mathbb{Q})$$

and the Lefschetz isomorphism $L^r : IH^{n-r}(W;\mathbb{Q}) \cong IH^{n+r}(W;\mathbb{Q})$ induces a nondegenerate bilinear pairing on $IH^{n-r}(W;\mathbb{Q})$. With respect to this pairing the Lefschetz decomposition is orthogonal and its signature is described by the Hodge Riemann bilinear relations.

22.3. Hodge theory and purity. Let W be a complex projective algebraic variety. Then there is a natural decomposition $IH^r(W;\mathbb{C}) \cong \oplus_{a+b=r} IH^{a,b}(W)$ with complex conjugate $\overline{IH^{a,b}} \cong H^{b,a}$. The Lefschetz operator $\cap H^1$ induces $IH^{a,b} \to IH^{a+1,b+1}$.

Let X be a projective algebraic variety defined over a finite field k with $q = p^t$ elements. Then the étale intersection cohomology $IH^s_{\acute{e}t}(X(\bar{k});\mathbb{Q}_\ell)$ carries an action of $\mathrm{Gal}(\bar{k}/k)$ which is topologically generated by the Frobenius Fr. The eigenvalues of Fr on $IH^s_{\acute{e}t}(X)$ have absolute value equal to $\sqrt{q}^s$.

22.4. Modular varieties. Let G be a semisimple algebraic group defined over $\mathbb{Q}$ of Hermitian type, with associated symmetric space $D = G(\mathbb{R})/K$ where $K \subset G(\mathbb{R})$ is a maximal compact subgroup. Let $\Gamma \subset G(\mathbb{Q})$ be a torsion free arithmetic group. The space $X = \Gamma\backslash D$ is a Hermitian locally symmetric space and it admits a natural "invariant" Riemannian metric. Let $\mathcal{E}$ be a finite dimensional metrized local system on X. The L^2 cohomology $H^r_{(2)}(X;\mathcal{E})$ is defined to be the cohomology of the complex of differential forms

$$\Omega^j_{(2)}(X) = \left\{ \omega \in \Omega^j(X;\mathbb{C}) | \int_X \omega \wedge *\omega < \infty, \int_X (d\omega) \wedge *(d\omega) < \infty \right\}.$$

W. Baily and A. Borel [4] showed how to compactify the space X so as to obtain a complex algebraic variety $\overline{X}$. Their construction was modified by G. Shimura [112] and P. Deligne [30] to obtain a variety that is defined over the rational numbers or over a number field. The complex $\Omega^\bullet_{(2)}(X)$ may be interpreted as the global sections of a complex of sheaves $\underline{\underline{\Omega}}^r$ on the compactification $\overline{X}$, whose sections over an open set $U \subset \bar{X}$ consist of differential forms $\omega \in \Omega^r(U \cap X)$ such that $\omega, d\omega$ are square integrable. It is true but not obviouis that this is a fine sheaf. (A cutoff function may introduce large derivatives, see [131].) Since this sheaf is fine its cohomology equals the cohomology of its global sections, which are the L^2 differential forms on X. The following conjecture of S. Zucker ([131]) was proven by E. Looijenga [85] and independently by L. Saper and M. Stern [104].

22.5. Theorem. *There is a natural quasi-isomorphism of sheaves $\underline{\underline{\Omega}}^\bullet_{(2)}(X) \cong \underline{\underline{IC}}^\bullet_{\overline{X}}$ which induces an isomorphism $H^r_{(2)}(X) \cong IH^r(\overline{X}; \mathbb{C})$.*

Since the complex of sheaves $\underline{\underline{\Omega}}^\bullet_{(2)}$ is self dual, the proof consists of showing that it satisfies the support conditions of intersection cohomology. In [85] this is accomplished by applying the decomposition theorem (below) to a resolution of singularities.

Very roughly speaking, this result provides a link beween the (infinite dimensional) representation theory of $G(\mathbb{R})$ (and modular forms) and the étale intersection cohomology of $\overline{X}$, which admits an action of a certain Galois group.

22.6. Morse theory again. Suppose $W \subset M$ is a Whitney stratified complex algebraic or complex analytic subvariety of a complex manifold and that the strata closures are also complex analytic. The proof of the following theorem is in Appendix §16.

22.7. Theorem. *Let $A^\bullet$ be a constructible sheaf on $W \subset M$. Then $A^\bullet$ is (middle) perverse if and only if for each stratum X, each point $x \in X$ and each nondegenerate covector $\xi \in T^*_{X,x}M$ the Morse group $M^t(W, A^\bullet, \xi)$ vanishes except possibly in the single degree $t = \mathrm{cod_W(X)}$.*

Consequently if $f : M \to \mathbb{R}$ is a a Morse function in the sense of §15.2 and $x \in X \subset W$ is a nondegenerate critical point of f then the Morse group

$$H^i(W_{\leq v+\epsilon}, W_{\leq v-\epsilon}; A^\bullet)$$

is nonzero in at most a single degree, $i = \mathrm{cod_{\mathbb{C}}(X)} + \lambda$ where λ is the Morse index of $f|X$. The proof of this and other Morse-theoretic properties of perverse sheaves is discussed in Appendix 16 following [51].

22.8. Decomposition theorem. This incredibly useful theorem was first formulated by S. Gelfand and R. MacPherson [36] in 1980, and proven shortly thereafter by Beilinson, Bernstein, Deligne and Gabber; discussed in Springer's article [114] and published in [12]. An accessible proof, valid also in the complex analytic context may be found in [28], summarized in [130]. Let

$f : X \to Y$ be a proper complex projective algebraic map with X nonsingular. The decomposition theorem says that $Rf_*(\underline{\underline{IC}}^\bullet_X)$ breaks into a direct sum of intersection complexes of subvarieties of Y, with coefficients in various local systems, and with shifts. In many cases this statement is already enough to determine the constituent sum. More precisely,

(1) $Rf_*(\underline{\underline{IC}}^\bullet_X) \cong \bigoplus_i {}^{\bar{p}}\underline{\underline{H}}^i(Rf_*(\underline{\underline{IC}}^\bullet_X))[-i]$ (This says that the push forward sheaf is a direct sum of perverse sheaves; these are its own perverse cohomology sheaves, shifted.)
(2) Each ${}^{\bar{p}}\underline{\underline{H}}^i(Rf_*(\underline{\underline{IC}}^\bullet_X))$ is a semisimple perverse sheaf. (This says that it is a direct sum of $\underline{\underline{IC}}^\bullet$ sheaves of strata. In particular, each summand enjoys all the remarkable properties of intersection cohomology that were described in the previous section.)
(3) The big summands come in pairs, ${}^{\bar{p}}\underline{\underline{H}}^i(Rf_*(\underline{\underline{IC}}^\bullet_X)) \cong {}^{\bar{p}}\underline{\underline{H}}^{-i}(\mathbf{D}(Rf_*(\underline{\underline{IC}}^\bullet_X)))$ (This is because $\underline{\underline{IC}}^\bullet_X$ is self-dual, hence so is its pushforward.)
(4) If η is the class of a hyperplane on X then for all r,
$$\cdot\eta^r : {}^{\bar{p}}\underline{\underline{H}}^{-r}(Rf_*(\underline{\underline{IC}}^\bullet_X)) \to {}^{\bar{p}}\underline{\underline{H}}^r(Rf_*(\underline{\underline{IC}}^\bullet_X))$$
is an isomorphism. (This is the relative hard Lefschetz theorem.)
(5) If L is the class of a hyperplane on Y then for all s and all r,
$$\cdot L^s : H^{-s}(Y, {}^{\bar{p}}\underline{\underline{H}}^r(Rf_*(\underline{\underline{IC}}^\bullet_X))) \to H^s(Y, {}^{\bar{p}}\underline{\underline{H}}^r(Rf_*(\underline{\underline{IC}}^\bullet_X)))$$
is an isomorphism. (This is just the statement that each summand satisfies hard Lefschetz.)

Moreover, if the mapping f can be stratified then the resulting perverse sheaves are constructible with respect to the resulting stratification of Y. An example is given in §24.9.

22.9. The hard Lefschetz isomorphisms give rise to the Lefschetz decomposition into primitive pieces as above. The combination of the Lefschetz isomorphism and the Poincaré duality isomorphism gives a nondegenerate bilinear form on each $H^r(Rf_*(\underline{\underline{IC}}^\bullet)$. The Lefschetz decomposition is orthogonal with respect to this pairing, and the signature of the components is given by the Hodge Rieman bilinear relations.

22.10. If $f : X \to Y$ is a proper projective algebraic map, recall that the i-th cohomology sheaf of $Rf_*(\underline{\underline{IC}}^\bullet_X)$ is the constructible sheaf

$$R^i f_*(\underline{\underline{IC}}^\bullet_X) = \underline{\underline{H}}^i(Rf_*(\underline{\underline{IC}}^\bullet_X)$$

whose stalk at a point $y \in Y$ is the cohomology $H^i(f^{-1}(y); \underline{\underline{IC}}^\bullet_X | f^{-1}(y))$. Let $U \subset Y$ be the nonsingular part. Then the *invariant cycle theorem* says that

(6) The restriction map
$$IH^i(X) \to H^0(U, R^i f_*(\underline{\underline{IC}}^\bullet_X)) = \Gamma(U, R^i f_*(\underline{\underline{IC}}^\bullet_X))$$
is surjective.

Let $y \in Y$ and let B_y be the intersection of Y with a small ball around y. Let $y_0 \in B_y \cap U$. For simplicity assume X is nonsingular so that $\underline{\underline{IC}}^\bullet_X = \mathbb{Q}_X[n]$. Then the *local invariant cycle theorem* says

(7) the restriction mapping to the monodromy invariants
$$H^i(f^{-1}(B_y, \mathbb{Q}) = H^i(f^{-1}(y), \mathbb{Q}) \to H^0(B_y, R^i f_*(\mathbb{Q}_X[n])) \cong H^i(f^{-1}(y_0))^{\pi_1(U \cap B_y)})$$
is surjective.

22.11. Let us suppose that X is nonsingular with complex dimension n, so that (in Deligne's numbering system) $\underline{\underline{IC}}^\bullet_X \cong \mathbb{Q}_X[n]$. The decomposition theorem contains two hard Lefschetz theorems and they work against each other to limit the types of terms that can appear in this decomposition. Let $\eta \in H^2(X)$ denote a hyperplane class and let $L \in H^2(Y)$ denote a hyperplane class. Statement (4) says that, for each $j \geq 0$ the cup product with η^j induces an isomorphism
$$H^r(Y; {}^{\bar p}H^{-j}(Rf_*(\mathbb{Q}[n]))) \cong H^{r+2j}(Y; {}^{\bar p}H^j(Rf_*(\mathbb{Q}[n]))) \text{ for all } r.$$
Statement (2) says that ${}^{\bar p}H^j(Rf_*(\mathbb{Q}[n]))$ is a direct sum of intersection cohomology sheaves, each of which satisfies hard Lefschetz (with respect to L) so that, for any $t \geq 0$ and for all j, the cup product with L^t induces an isomorphism
$$H^{r-t}(Y; {}^{\bar p}H^j(Rf_*(\mathbb{Q}[n]))) \cong H^{r+t}(Y; {}^{\bar p}H^j(Rf_*(\mathbb{Q}[n]))).$$

22.12. Suppose $\pi : X \to Y$ is a resolution of singularities. The decomposition theorem says that $R\pi_*(\mathbb{Q}[n])$ is a direct sum of intersection cohomology sheaves of subvarieties. The stalk cohomology of this sheaf, at any nonsingular point $y \in Y$ is $H^*(\pi^{-1}(y); \mathbb{Q}[n])$ which is $\mathbb{Q}$ in degree $-n$. So the sheaf $\underline{\underline{IC}}^\bullet_Y$ is one of the summands, that is: the intersection cohomology of Y appears as a summand in the cohomology of any resolution.

22.13. Suppose X, Y are nonsingular and $f : X \to Y$ is an algebraic fiber bundle. Then $Rf_*(\mathbb{Q}_X[n])$ decomposes into a direct sum of perverse sheaves on Y, each of which is therefore a local system on Y, that is,
$$H^r(X; \mathbb{Q}) \cong \oplus_{i+j=r} H^i(Y; H^j(F))$$
where $H^j(F)$ denotes the cohomology of the fiber, thought of as a local system on Y. In other words, the Leray spectral sequence for this map degenerates (an old theorem of Deligne) and hard Lefschetz applies both to $H^j(F)$ and to $H^i(Y)$.

22.14. Three proofs. The first and original proof is in [BBD] and uses reduction to varieties in characteristic $p > 0$, purity of Frobenius, and Deligne's proof of the Weil conjectures. The second proof is due to Morihiko Saito, who developed a theory of mixed Hodge modules in order to extend the proof to certain analytic settings. The third proof is due to deCataldo and Migliorini, who used classical Hodge theory. Their proof works in the complex analytic setting and some people feel it is the most accessible of the three.

23. Cohomology of toric varieties

23.1. In 1915 Emmy Noether proved that if a Hamiltonian system is preserved by an 1-parameter infinitesimal symmetry (that is to say, by the action of a Lie group) then a certain corresponding "conjugate" function, or "first integral" is preserved under the time evolution of the system. Time invariance gives rise to conservation of energy. Translation invariance gives rise to conservation of momentum. Rotation invariance gives rise to conservation of angular momentum.

Figure 22. Emmy Noether

Today, this is known as the *moment map*: Suppose (M, ω) is a symplectic manifold, and suppose a compact lie group G acts on M and preserves the symplectic form. The infinitesimal action of

G in the direction of V defines a vector field X on M. Contract this with the symplectic form to obtain a 1-form $\theta = \iota_X(\omega)$. It follows that $d\theta = 0$. If the action of G is *Hamiltonian* then in fact, $\theta = df$ for some smooth function $f : M \to \mathbb{R}$ (defined up to a constant). This is the conserved quantity.

23.2. Moment map. In summary, if the action of a compact Lie group G on a smooth manifold M is Hamiltonian then there exists a *moment map*, that is, a smooth mapping $\mu : M \to \mathfrak{g}^*$ so that for each $X \in \mathfrak{g}$ the differential of the function $p \mapsto \langle \mu(p), X\rangle$ equals $\iota_X(\omega)$.

For example consider the Fubini Study metric $h(z, w) = \sum dz_i \wedge d\bar{z}_i$ on projective space. The real and imaginary part, $h = R + i\omega$ are respectively, positive definite and sympletic. Fix $a_0, a_1, \cdots, a_n \in \mathbb{Z}$. If $\lambda \in \mathbb{C}^\times$ acts on $\mathbb{C}^{n+1}$ by

$$\lambda \cdot (z_0, z_1, \cdots, z_n) = (\lambda^{a_0} z_0, \lambda^{a_1} z_1, \cdots, \lambda^{a_n} z_n)$$

then, restricting the action to (S^1), the resulting moment map $\mu : \mathbb{CP}^n \to \mathbb{R}$ is

$$\mu([z_0 : z_1 : \cdots : z_n]) = \frac{a_0|z_0|^2 + a_1|z_1|^2 + \cdots + a_n|z_n|^2}{(|z_0|^2 + \cdots + |z_n|^2)}.$$

If $(\lambda_0, \lambda_1, \cdots, \lambda_n) \in (\mathbb{C}^\times)^n$ acts on $\mathbb{C}^n$ by

$$(\lambda_0, \cdots, \lambda_n) \cdot (z_0, \cdots, z_n) = (\lambda_0 z_0, \cdots, \lambda_n z_n)$$

then, restricting the action to $(S^1)^n$ the restulting moment map $\mu : \mathbb{CP}^n \to \mathbb{R}^n$

$$\mu([z_0 : \cdots : z_n]) = \frac{(|z_0|^2, |z_1|^2, \cdots, |z_n|^2)}{(|z_0|^2 + \cdots + |z_n|^2)}$$

and it is the standard simplex contained in the hyperplane $x_0 + \cdots + x_n = 1$. These actions are Hamiltonian and the moment map collapses orbits of $(S^1)^n$.

23.3. Now let $X \subset \mathbb{CP}^N$ be an n-dimensional subvariety on which a torus $T = (\mathbb{C}^\times)^n$ acts with finitely many orbits. In this case the action extends to a linear action on projective space of the sort described above and the moment map image (for the action of $(S^1)^n$), $\mu(X) \subset \mu(\mathbb{CP}^n)$ is convex. In fact, the *convexity theorem* of Atiyah, Kostant, Guillemin, Sternberg ([2, 58])states that

23.4. Theorem. *The moment map image $\mu(X)$ is the convex hull of the images $\mu(x_i)$ of the T-fixed points in X. The image of each k-dimensional T-orbit is a single k-dimensional face of this polyhedron.*

It turns out, moreover, that the toric variety is a rational homology manifold if and only $\mu(X)$ is a simple polytope, meaning that each vertex is adjacent to exactly n edges.

Algebraic geometers prefer a presentation of a toric variety from a *fan*, a collection of homogeneous cones in Euclidean space. From a fan one constructs a convex polynedron by intersecting the fan with a ball centered at the origin, and then flattening the faces. The resulting convex

polyhedron is the dual of the moment map polyhedron. If the moment map polyhedron is *simple* then the fan-polyhedron is *simplicial*, meaning that the faces are simplices.

23.5. Definition. If Y is a complex algebraic variety define the intersection cohomology Poincaré polynomial

$$h(Y,t) = h_0 + h_1 t + h_2 t^2 + \cdots + h_n t^n$$

where $h_r = \text{rank } IH^r(Y;\mathbb{Q})$. If $y \in Y$ define the local Poincaré polynomial $h_y(Y,t) = \sum_{r\geq 0} \text{rank}\left(\underline{\underline{H}}^r(\underline{\underline{IC}}^\bullet)_y\right) t^r$.

If Y is defined over $\mathbb{F}_q$ we use the same notation for the Poincaré polynomial of the étale intersection cohomology.

23.6. Counting points. There is a very general approach to understanding the cohomology and intersection cohomology of an n-dimensional algebraic variety defined over a finite field $\mathbb{F}_q$, provided its odd degree cohomology groups vanish. The variety $Y/\mathbb{F}_q$ is said to be *pure* if the eigenvalues of Frobenius on $H^r(Y;\mathbb{Q}_\ell)$ have absolute value $\sqrt{q}^r$ with respect to any embedding into the complex numbers. The Weil conjectures (proven by Grothendieck [**?**] and Deligne [**?**]) say that

$$\sum_{r=0}^{2n}(-1)^r \text{Tr}(Fr_q : H^r(Y) \to H^r(Y)) = |Y(\mathbb{F}_q)|$$

the right hand side being the number of points that are fixed by the Frobenius morphism. The intersection cohomology of any projective algebraic variety is pure. If the variety Y is also nonsingular (so that $IH^*(Y) = Y^*(Y)$ and *Tate* (which means that the eigenvalues on H^r are in fact equal to $(\sqrt{q})^r$) then this gives

$$h(Y,\sqrt{q}) = \sum_{r=0}^{n} \text{rank} H^{2r}(Y) q^r = |Y(\mathbb{F}_q)|.$$

For example, if such a variety Y is defined over the integers, is nonsingular and $Y(\mathbb{C})$ has an algebraic cell decomposition with m_r cells of (complex) dimension r then $h_{2r+1} = 0$ and $h_{2r} = m_r$ accounts for $m_r q^r$ points over $\mathbb{F}_q$. In the case of a nonsingular toric variety whose moment map image is a convex polyhedron with f_r faces of dimension r this gives

$$h(Y,\sqrt{q}) = \sum_{s=0}^{n} \text{rank} H^{2s}(Y;\mathbb{C}) q^s = \sum_{r=0}^{n} f_r (q-1)^r$$

since each r-dimensional orbit is itself (isomorphic to) a torus of dimension r. The hard Lefschetz theorem says $h_{2s-2} \leq h_{2s}$ for $2s \leq n$ which in turn gives inequalities between the face numbers, as observed by Stanley in 1980.

23.7. If we wish to use intersection cohomology rather than ordinary cohomology in the Weil conjectures then the formula becomes

$$\sum_{r=0}^{2n}(-1)^r \mathrm{Tr}(Fr_q : IH^r(Y) \to IH^r(Y)) = |Y(\mathbb{F}_q)|_{\mathrm{mult}}$$

where each point $y \in Y(\mathbb{F}_q)$ is counted with a multiplicity equal to the (alternating sum of) trace of Frobenius acting on the stalk of the intersection cohomology at $y \in Y(\mathbb{F}_q)$. If this is pure and if the stalk cohomology vanishes in odd degrees, then this multiplicity equals the Poincaré polynomial $h_y(Y, \sqrt{q})$ of the stalk of the intersection cohomology. In conclusion, if the intersection cohomology of Y is Tate and vanishes in odd degrees then

$$\sum_{s=0}^{n} \mathrm{Tr}(Fr_q|(IH^{2s}(Y))) = \sum_{s=0}^{n} \mathrm{rank}\ IH^{2s}(Y)q^s = h(Y, \sqrt{q}) = \sum_{y \in Y(\mathbb{F}_q)} h_y(Y, \sqrt{q}). \tag{23.7.1}$$

Let us now try to determine these multiplicities h_y. If $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ define the truncation $\tau_{\leq r} f$ to be the polynomial $a_0 + \cdots + a_r x^r$ consisting of those terms of degree $\leq r$.

23.8. Lemma. *Let $Z \subset \mathbb{C}P^{N-1}$ be a projective algebraic variety of dimension d, with intersection cohomology Poincaré polynomial*

$$g(t) = g_0 + g_1 t + \cdots + g_{2d} t^{2d} = \sum_{r=0}^{2d} \dim(IH^r(Z))t^r.$$

Then the stalk of the intersection cohomology of the complex cone $Y = \mathrm{cone}_{\mathbb{C}}(Z) \subset \mathbb{C}P^N$ at the cone point $y \in Y$ has Poincaré polynomial

$$h_y(Y, t) = \tau_{\leq d}\left(g(t)(1 - t^2)\right) \tag{23.8.1}$$

Proof. The complex projective space $\mathbb{CP}^N$ is the complex cone over $\mathbb{CP}^{N-1}$. In fact, if we remove the cone point then what remains is a line bundle $E \to \mathbb{CP}^{N-1}$ whose first Chern class $c^1(E) \in H^2(\mathbb{CP}^{N-1})$ is the class of a hyperplane section. This is to say that there exists a section of this bundle that vanishes precisely on a hyperplane; it may be taken to be

$$s([z_0 : ... z_{N-1}]) = [z_0 : ... : z_{N-1}, \Sigma_j a_j z_j] \in \mathbb{CP}^N$$

for any fixed choice (not all zero) of $a_0, a_1, \cdots, a_{N-1} \in \mathbb{C}$. The vanishing of the last coordinate is a hyperplane in $\mathbb{CP}^{N-1}$. So this class may be used as a hard Lefschetz class.

If $Z \subset \mathbb{CP}^{N-1}$ is a projective algebraic variety then $\mathrm{cone}_{\mathbb{C}}(Z) \subset \mathbb{CP}^N$ is a singular variety and the link L of the cone point can be identified with the sphere bundle of this line bundle $E_Z \to Z$.

The Gysin sequence becomes

$$IH^{i-2}(Z) \xrightarrow{\cup c^1} IH^i(Z) \longrightarrow IH^i(L) \longrightarrow IH^{i-1}(Z) \xrightarrow{\cup c^1} IH^{i+1}(Z) \longrightarrow$$

It follows from the hard Lefschetz theorem for Z that $IH^i(L)$ is the primitive part of the intersection cohomology of Z for $i \leq d$, that is,

$$IH^i(L) \cong IP^i(Z) = \operatorname{coker}(\cdot c^1(E) : IH^{i-2}(Z) \to IH^i(Z))$$

for $i \leq \dim(Z)$, and hence its Poincaré polynomial is given by

$$g_0 + g_1 t + (g_2 - g_0)t^2 + (g_3 - g_1)t^3 + (g_4 - g_2)t^4 + \cdots + (g_d - g_{d-2})t^d = \tau_{\leq d} g(t)(1 - t^2). \qquad \square$$

23.9. Some geometry. Let $\mu : Y \to P \subset \mathbb{R}^m$ be the moment map corresponding to the action of a torus $T \cong (\mathbb{C}^\times)^m$ on a toric variety Y. If F is a face of P, the link of F can be realized as another conex polyhedron, $L_F = P \cap V$ where $V \subset \mathbb{R}^N$ is a linear subspace such that $\dim(V) + \dim(F) = N - 1$, which passes near F and through P. (For example, V may be taken to lie completely in the plane $F^\perp$.) In fact, L_F is the moment map image of a sub-toric variety Y_F on which a sub-torus T_F acts.

23.10. In the case of a toric variety Y, a given face F corresponds to a stratum S_F of the toric variety. The link of this stratum is therefore isomorphic to a circle bundle over a toric variety whose moment map image is the link L_F of the face F. Let $h(Y_F, t)$ be the intersection cohomology Poincaré polynomial of this "link" toric variety. Then equations (23.8.1) and (23.7.1) give:

23.11. Theorem. *The IH Poincaré polynomial of Y is*

$$h(Y, t) = \sum_F (t^2 - 1)^{\dim(F)} . \tau_{\leq n - \dim(F)} \left((1 - t^2) h(Y_F, t) \right)$$

23.12. In particular, the intersection cohomology only depends on the combinatorics of the moment map image $P = \mu(Y)$, and moreoer, the functions $h(Y_F, t)$ may be determined (inductively) from the moment map images $L_F = \mu_F(Y_F)$. The hard Lefschetz theorem (which says that $h_{2r} \geq h_{2r-2}$ for all $r \leq \dim(Y)$) then implies a collection of inequalities among the numbers of chains of faces.

23.13. Remarks. This formula simplifies if P is a simple polyhedron, to:

$$h(Y, t) = \sum_F (t^2 - 1)^{\dim(F)} = f(t^2 - 1)$$

where $f(s) = f_0 + f_1 s + \cdots + f_d s^d$ and f_j is the number of faces of dimension j. The polytopes considered here are always rational, meaning that the vertices are rational points in $\mathbb{R}^d$, or equivalently, the faces are the kernels of linear maps $\mathbb{R}^d \to \mathbb{R}$ with rational coefficients. Any simple (or simplicial) polytope can be perturbed by moving the faces (resp. the vertices) so as to make them rational. Therefore the inequalities arising from hard Lefschetz apply to all simple

polytopes. However a general polytope cannot necessarily be perturbed into a rational polytope with the same face combinatorics. The Egyptian pyramid, for example, has a square face. Lifting one of the vertices on this face, an arbitrarily small amount, will force the face to "break". In order to prove that the inequalities arising from hard Lefschetz for intersection cohomology can be applied to any polytope it was necessary to construct something like intersection cohomology in the non-rational case. This was constructed in [9] and the Hard Lefschetz property was provven in [69].

23.14. There is another way to prove this result using the decomposition theorem (which does not involve passing to varieties over a finite field). The singularities of the toric variety Y can be resolved by a sequence of steps, each of which is toric with moment maps that correspond to 'cutting off the faces" that are singular. For example, the Egyptian pyramid has a single singular point. The singularity is resolved by a mapping $\pi : \tilde{Y} \to Y$ as illustrated in this diagram:

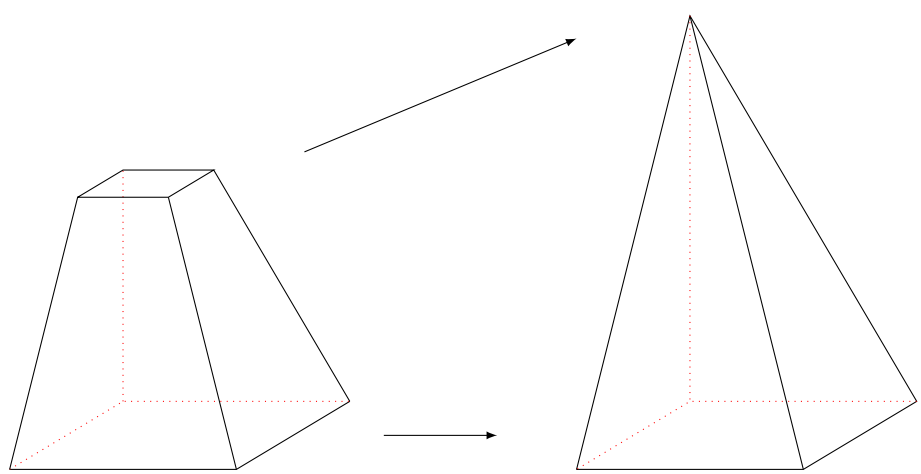


FIGURE 23. Moment map of a resolution

Let us examine the decomposition theorem for this mapping. The mapping is an isomorphism everywhere except over the singular point $y \in Y$ and $\pi^{-1}(y) \cong \mathbb{P}^1 \times \mathbb{P}^1$. The stalk cohomology of the pushforward $R\pi_*(\underline{\underline{\mathbb{Q}}}_{\tilde{Y}})$ is $(\mathbb{Q}, 0, \mathbb{Q} \oplus \mathbb{Q}, 0, \mathbb{Q})$. Put this into the support diagram for a 3 dimensional variety, see Figure 24. (In this figure, the degree i is the "usual" cohomology degree notation and the degree j is the "perverse degree" notation.)

From the decomposition theorem we know that one term will be $\underline{\underline{IC}}^\bullet_Y$ and that there are additional terms supported at the singular point y. From the support condition it is clear that $\mathbb{Q}[3]$ (on the bottom row) is part of the IC sheaf. It is not so clear how much of the $(\mathbb{Q} \oplus \mathbb{Q})[1]$ belongs to $\underline{\underline{IC}}^\bullet_Y$ and how much belongs to the other terms. However the $\mathbb{Q}[-1]$ (at the top of the column; in degree $j = 1$) is definitely not part of IC. By Poincaré and especially by Hard Lefschetz, it must be paired with one copy of $\mathbb{Q}$ in degree -1. So this leaves $\mathbb{Q}[3] \oplus \mathbb{Q}[1]$ (in degrees -3 and

i	j	cod0	cod2	cod4	cod6	$H^*(\pi^{-1}(y))$
6	3	c	c	c	c	
5	2			c	c	
4	1				c	$\mathbb{Q}$
3	0				0	0
2	-1				x	$\mathbb{Q} \oplus \mathbb{Q}$
1	-2			x	x	0
0	-3	x	x	x	x	$\mathbb{Q}$
		$\underline{\underline{IC}}^\bullet$ support				

FIGURE 24. Support of $R\pi_*(\underline{\underline{\mathbb{Q}}})$

1 respectively) for the IC sheaf. A closer inspection of this argument shows that these two terms constitute the primitive cohomology of the fiber, as we saw earlier.

Thus, the decomposition theorem singles out the primitive cohomology of the fiber as belonging to the IC sheaf. Now, assuming by induction that the formula holds for $IH^*(\tilde{Y}) = H^*(\tilde{Y})$ (which is less singular that Y) and knowing how these terms decompose, it is easy to conclude that the formula must also hold for $IH^*(Y)$.

24. Springer Representations

24.1. The flag manifold. Let $G = SL_n(\mathbb{C})$. It acts transitively on the set $\mathcal{F}$ of complete flags $0 \subset F^1 \subset \cdots \subset F^{n-1} \subset \mathbb{C}^n$ and the stabilizer of the standard flag is the "standared" Borel subgroup B of (determinant $= 1$) upper triangular matrices, giving an isomorphism $\mathcal{F} \cong G/B$. The Lie algebras are $\mathfrak{g}$ (matrices with trace $= 0$) and $\mathfrak{b} =$ upper triangular matrices with trace $= 0$. If $x \in G$ and $xBx^{-1} = B$ then $x \in B$. So we may identify $\mathcal{F}$ with the set $\mathcal{B}$ of all subgroups of G that are conjugate to B or equivalently to the set of all subalgebras of $\mathfrak{g}$ that are conjugate to $\mathfrak{b}$, that is, the *variety of Borel subalgebras* of $\mathfrak{g}$.

24.2. Definition. Let $\mathcal{N}$ be the set of all nilpotent elements in $\mathfrak{g}$. Define

$$\begin{array}{ccc} \widetilde{\mathcal{N}} = \{(x \in \mathcal{N}, A \in \mathcal{B}) |\ x \in \mathrm{Lie}(A)\} & \xrightarrow{\phi} & \mathcal{B} \\ \pi \big\downarrow & & \\ \mathcal{N} & & \end{array}$$

The mapping π is proper and its fibers $\mathcal{B}_x = \pi^{-1}(x)$ are called *Springer fibers*. In a remarkable series of papers [Springer 1976, 1978], T. A. Springer constructed an action of the symmetric group W on the cohomology of each Springer fiber $\mathcal{B}_x$, even though W does not actually act on $\mathcal{B}_x$.

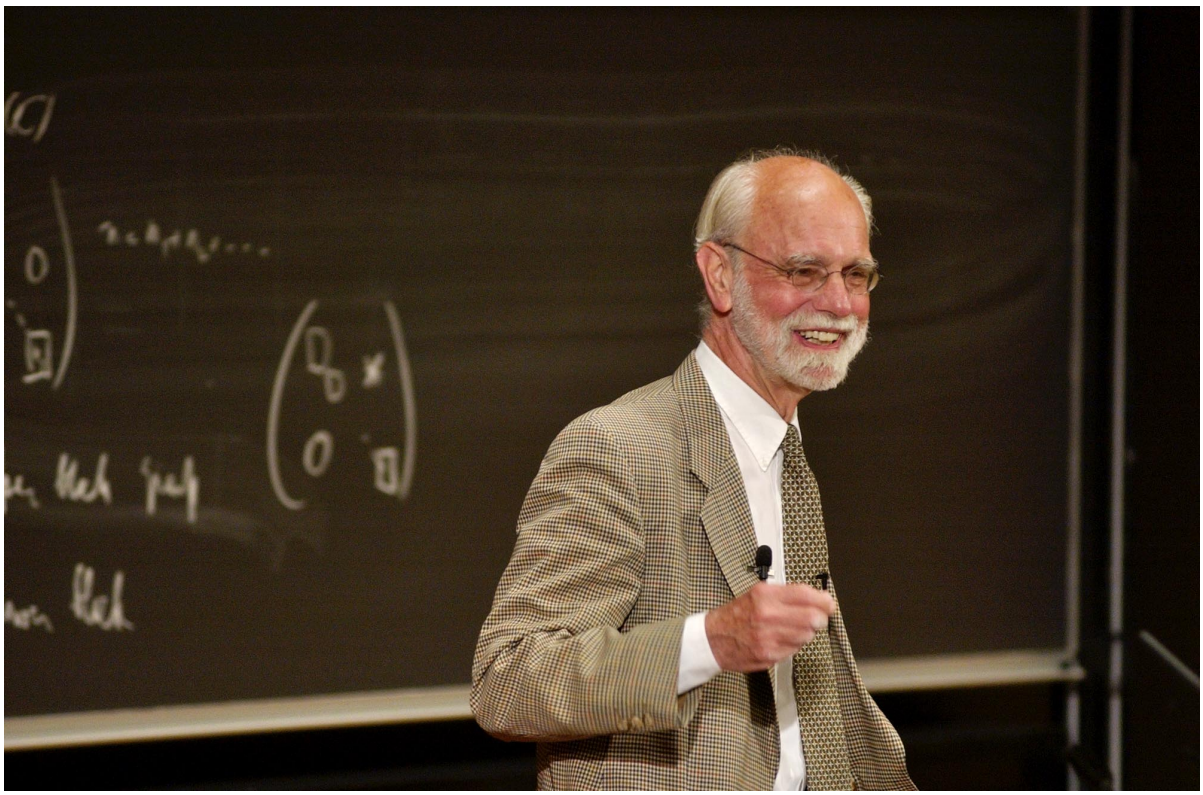


FIGURE 25. Tonny Springer

Let A be the subgroup that preserves a flag $F_A = (0 = A^0 \subset A^1 \subset \cdots \subset A^n = \mathbb{C}^n)$ then the following are equivalent:

(1) $(x, A) \in \widetilde{\mathcal{N}}$
(2) $x \in \mathrm{Lie}(A)$
(3) $\exp(x)$ preserves the flag F_A
(4) the vectorfield V_x (defined by x) on the flag manifold $\mathcal{F}$ vanishes on F_A
(5) $xA^j \subset A^{j-1}$ for $1 \leq j \leq n$.

So the Springer fiber $\pi^{-1}(x)$ is the zero set of the vectorfield V_x; it is the set of all flags that are preserved by x and is often referred to as the *variety of fixed flags*. For the subregular nilpotent $x \in \mathfrak{g}$ the Springer fiber turns out to be a string of $n-1$ copies of $\mathbb{P}^1$, each joined to the next at a single point. [picture]

24.3. Lemma. *The mapping $\phi : \widetilde{\mathcal{N}} \to \mathcal{B}$ identifies $\widetilde{\mathcal{N}}$ with the cotangent bundle to the flag manifold.*

Proof. The tangent space at the identity to $\mathcal{F}$ is $T_I(G/B) = \mathfrak{g}/\mathfrak{b}$. So its dual space is

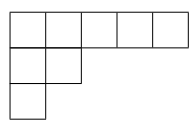
$$T_I^*(G/B) = \{\phi : \mathfrak{g} \to \mathbb{C} |\ \phi(\mathfrak{b}) = 0\}.$$

The canonical inner product $\langle , \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ given by $\langle x, y\rangle = \mathrm{Trace}(xy)$ is symmetric and nondegenerate. Using this to identify $\mathfrak{g}^*$ with $\mathfrak{g}$ gives

$$T_I^*(G/B) \cong \{x \in \mathfrak{g} |\ \langle x, \mathfrak{b}\rangle = 0\} = \mathfrak{n}$$

is the algebra of strictly upper triangular matrices, that is, the nilradical of $\mathfrak{b}$. So for each Borel subgroup $A \subset G$, the cotangent space $T_A^*(G/B) \cong \mathfrak{n}(A)$ is naturally isomorphic to the nilradical of $\mathrm{Lie}(A)$. But this is exactly the fiber, $\phi^{-1}(A)$. □

24.4. The group G acts on everything in the diagram (24.2). It acts transitively on $\mathcal{B}$ and it acts with finitely many orbits on $\mathcal{N}$, each of which is a *nilpotent conjugacy class*. These form a Whitney stratification of $\mathcal{N}$ by complex algebraic strata. It follows from Jordan normal form that each nilpotent conjugacy class corresponds to a partition $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ with $\sum \lambda_i = n$ or equivalently to a Young frame



Stratum closure relations correspond to refinement of partitions with the largest stratum corresponding to the case of a single Jordan block ($\lambda_1 = n$) and the smallest stratum corresponding to $0 \in \mathfrak{N}$, which is the partition $1 + 1 + 1 \cdots + 1 = n$.

24.5. The *Grothendieck simultaneous resolution* is the pair

$$\widetilde{\mathfrak{g}} = \{(x, A) \in \mathfrak{g} \times \mathcal{B} |\ x \in A\} \xrightarrow{\widehat{\pi}} \mathfrak{g}$$

Given $(x, A) \in \widetilde{\mathfrak{g}}$ choose $h \in G$ which conjugates A into the standard Borel subgroup B. It conjugates x into an element $x' \in \mathrm{Lie}(B)$ and the diagonal entries $\alpha = \alpha(x) \in \mathfrak{t}$ are well defined where $\mathfrak{t}$ is the set of diagonal matrices with trace $= 0$. On the other hand, the characteristic polynomial $ch(x)$ of x is determined by the diagonal matrix α but is independent of the order of the entries. The set of possible characteristic polynomials forms a vector space, with coordinates given by the coefficients of the characteristic polynomial, which provides an example of a remarkable theorem of Chevalley that the quotient $\mathfrak{t}/W$ is again an affine space. The nilpotent elements $\mathcal{N}$ in $\mathfrak{g}$ map to zero in $\mathfrak{t}/W$. In summary we have a diagram

$$\begin{tikzcd}
\widetilde{\mathcal{N}} \arrow[r, hook] \arrow[d, "\pi"'] & \widetilde{\mathfrak{g}} \arrow[r, "\alpha"] \arrow[d, "\widehat{\pi}"] & \mathfrak{t} \arrow[d, "p"] \\
\mathcal{N} \arrow[r, hook] & \mathfrak{g} \arrow[r, "ch"'] & \mathfrak{t}/W
\end{tikzcd}$$

24.6. Theorem (Grothendieck, Lusztig, Slowdowy)**.** *The map $\widehat{\pi} : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ is small. The map $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ is semi-small. For each $A \in \mathfrak{t}$ the map $\widehat{\pi} : \alpha^{-1}(A) \to ch^{-1}(p(A))$ is a resolution of singularities.*

24.7. Adjoint quotient. There is another way to view the map ch. Each $x \in \mathfrak{g}$ has a unique Jordan decomposition $x = x_s + x_n$ into commuting semisimple and nilpotent elements. Then x_s is conjugate to an element of $\mathfrak{t}$, and the resulting element is well defined up to the action of W. The quotient $\mathfrak{t}/W$ turns out to be isomorphic to the geometric invariant theory quotient $\mathfrak{g}//G$ and the map $ch : \mathfrak{g} \to \mathfrak{t}/W$ is called the adjoint quotient map.

The vector space $\mathfrak{t}$ consists of diagonal matrices $A = \mathrm{diag}(a_1, \cdots, a_n)$ with trace zero. The *reflecting hyperplanes* are the subspaces $H_{ij} = \{A | a_i = a_j\}$ where two entries coincide and they are permuted by the action of W. Their image in $\mathfrak{t}/W$ is the discriminant variety $Disc$ consisting of all (characteristic) polynomials with multiple roots. The complement of the set $\cup_{i \neq j} H_{ij}$ is sometimes called the *configuration space of n ordered points in $\mathbb{C}$*; its fundamental group is the *colored braid group*. The complement of the discriminant variety in $\mathfrak{t}/W$ is the configuration space of n unordered points, and its fundamental group is the braid group.

Suppose $x \in \mathfrak{g}^{rs} \subset \mathfrak{g}$ is regular and semisimple, meaning that its eigenspaces $E_1, E_2, \cdots, E_n$ are distinct and form a basis of $\mathbb{C}^n$. Then the flag $E_1 \subset E_1 \oplus E_2 \subset E_1 \oplus E_2 \oplus E_3 \cdots$ is fixed by x and every fixed flag has this form, for some ordering of the eigenspaces. Therefore there are $n!$ fixed flags and the symmetric group permutes them according to the regular representation.

24.8. Springer's representation. Since $\widehat{\pi}$ is a small map, we have a canonical isomorphism

$$R\widehat{\pi}_*(\underline{\underline{\mathbb{Q}}}_{\widetilde{\mathfrak{g}}}[n]) \cong \underline{\underline{IC}}^\bullet(\mathfrak{g}; \mathcal{E}),$$

the intersection cohomology sheaf on $\mathfrak{g}$, constructible with respect to a stratification of $\widehat{\pi}$, with coefficients in the local system $\mathcal{E}$ over the regular semisimple elements whose fiber at $x \in \mathfrak{g}^{rs}$ is the direct sum $\oplus_F \mathbb{Q}_F$ of copies of $\mathbb{Q}$, one for each fixed flag. The symmetric group W acts on $\mathcal{E}$ which induces an action on $\underline{\underline{IC}}^\bullet(\mathcal{E})$ and therefore also on the stalk cohomology at each point $y \in \mathfrak{g}$, that is, on

$$\underline{\underline{H}}^r(R\widehat{\pi}_*(\underline{\underline{\mathbb{Q}}}_{\widetilde{\mathfrak{g}}})_y = H^r(\widehat{\pi}^{-1}(y)) \cong \underline{\underline{IH}}^r(\mathfrak{g}; \mathcal{E})_y.$$

For $y \in \mathcal{N}$ this action of W on $H^*(\pi^{-1}(y))$ turns out to coincide with Springer's representation. The decomposition theorem for the semismall map π provides an enormous amount of information about these representations.

24.9. Decomposition theorem for semismall maps. Recall that a proper algebraic morphism $f : X \to Y$ is semismall if it can be stratified so that

$$2 \dim_{\mathbb{C}} f^{-1}(y) \leq \mathrm{cod(S)}$$

for each stratum $S \subset Y$, where $y \in S$. This implies that $d = \dim(X) = \dim(Y)$ and, if X is nonsingular, that $Rf_*(\mathbb{Q}_X)[d]$ is perverse on Y. A stratum $S \subset Y$ is said to be *relevant* if

$2d = \mathrm{cod(S)}$ where $d = \dim f^{-1}(y)$. In this case, the top degree cohomology $H^{2d}(f^{-1}(y)$ forms a local system L_S on the stratum S. The following result is due to W. Borho and R. MacPherson[21].

24.10. Proposition. *Suppose $f : X \to Y$ is semismall and X is nonsingular of complex dimension d. Then the decomposition theorem has the following special form:*

$$Rf_*(\mathbb{Q}_x)[d] \cong \bigoplus_S \underline{\underline{IC}}^\bullet_{\bar{S}}(L_S)$$

where the sum is over those strata S that are relevant (with no shifts, if we use Deligne's numbering). In particular, the endomorphism algebra of this sheaf $\mathrm{End}(Rf_*(\mathbb{Q}_X)[d]) \cong \bigoplus_S \mathrm{End}_S(L_S)$ is isomorphic to the direct sum of the endomorphism algebras of the individual local systems L_S.

Proof. The top stratum, Y^o is always relevant. If no other strata is relevant then the map is *small*. By §20.5 this implies $Rf_*(\mathbb{Q}_X[d]) \cong \underline{\underline{IC}}^\bullet_Y(L_{Y^o})$. So there is only one term in the decompsition theorem. Now suppose the next relevant stratum has codimension c so that the fiber over points in this stratum has (complex) dimension $d = c/2$ (and in particular, the complex codimension c is even). One term in the decomposition theorem is $\underline{\underline{IC}}^\bullet_Y(L_{Y^o})$. Consider the support diagram (e.g. if $c = 4$):

i	j	$\mathrm{cod}_{\mathbb{C}}0$	$\mathrm{cod}_{\mathbb{C}}1$	$\mathrm{cod}_{\mathbb{C}}2$	$\mathrm{cod}_{\mathbb{C}}3$	$\mathrm{cod}_{\mathbb{C}}4$	$H^*(f^{-1}(y))$
8	4	c	c	c	c	c	
7	3			c	c	c	
6	2				c	c	
5	1					c	
4	0					0	$H^4(f^{-1}(y))$
3	-1					x	$H^3(f^{-1}(y))$
2	-2				x	x	$H^2(f^{-1}(y))$
1	-3			x	x	x	$H^1(f^{-1}(y))$
0	-4	x	x	x	x	x	$H^0(f^{-1}(y))$
		$\underline{\underline{IC}}^\bullet$ support					

From this diagram we can see that a new summand must be added to the decomposition, and it is the local system $H^4(f^{-1}(y)) = H^{2d}(f^{-1}(y) = L_S$ arising from the top cohomology of the fiber, that is, from the irreducible components of the fiber. This local system on S gives rise to the summand $\underline{\underline{IC}}^\bullet_{\bar{S}}(L_S)$ over the whole of the closure $\bar{S} \subset Y$. Let $U_S = \cup_{T \geq S}$ be the open set consisting of S together with the strata larger than S. We have constructed an isomorphism of $Rf_*(\mathbb{Q}_X)$ with

$$Rf_*(\mathbb{Q}_X[d]) \cong \underline{\underline{IC}}^\bullet_Y(L_{Y^o}) \oplus \underline{\underline{IC}}^\bullet_{\bar{S}}(L_S)$$

over the open set U_S. The exact same method as in Proof No.1 shows that this isomorphism extends uniquely to an isomorphism over the larger open set that contains additional (smaller) strata until we come to the next relevant stratum. Continuing in this way by induction gives the desired decomposition.

Finally, if L_R, L_S are local systems on distinct strata R, S of Y then

$$\mathrm{Hom}_{D^b_c(Y)}(\underline{\underline{IC}}^\bullet_{\bar{R}}(L_R), \underline{\underline{IC}}^\bullet_{\bar{S}}(L_S)) = 0$$

which implies that the endomorphism algebra of this direct sum decomposes into a direct sum of endomorphism algebras. □

24.11. Some conclusions. Let $d = \dim(G/B) = n(n-1)/2$. For the SL_n adjoint quotient $R\pi_*(\mathbb{Q}_{\widetilde{N}}[d]) \cong \oplus_S \underline{\underline{IC}}^\bullet(\overline{S}; L_S)$

(1) Every stratum S is relevant: for $x \in S$, $2\dim(\mathcal{B}_x) = \mathrm{cod(S)}$.
(2) The odd cohomology of each Springer fiber $H^{2r+1}(\mathcal{B}_x) = 0$ vanishes.
(3) Let S be a stratum corresponding to some partition (or Young frame) $\lambda = \lambda(S)$. Then the Springer action on the top cohomology $H^{\mathrm{cod(S)}}(\mathcal{B}_x)$ is the irreducible representation ρ_λ corresponding to λ (modulo possible notational normalization involving transpose of the partition and tensoring with the sign representation). [Note: this representation is not (cannot) be realized via permutations of the components of $\mathcal{B}_x$, but the components give, nevertheless, a basis for the representation.]
(4) Every irreducible representation of W occurs in this decomposition, and it occurs with multiplicity one.
(5) The local systems occurring in the decomposition theorem (SL_n case only!) are all trivial.
(6) Putting these facts together, let S_λ denote the stratum in $\mathcal{N}$ corresponding to the partition λ. Then, using "classical" degree indices, the decomposition theorem in this case becomes:

$$R\pi_*(\mathbb{Q}_{\widetilde{N}}) \cong \oplus_\lambda \underline{\underline{IC}}^\bullet(\overline{S}_\lambda)[-2d_\lambda] \otimes V_\lambda$$

where V_λ is the (space of the) irreducible representation ρ_λ of W and d_λ is the complex dimension of the stratum S_λ.
(7) Applying stalk cohomology at a point $x \in S_\mu$ to this formula gives:

$$H^i(\mathcal{B}_x) \cong \oplus_{\lambda \geq \mu} \underline{\underline{IH}}^{i-2d_\lambda}_y(\overline{S}_\lambda) \otimes V_\lambda.$$

Consequently, if an irreducible representation ρ_λ of W occurs in $H^*(\mathcal{B}_x)$ then the stratum S_μ containing x is in the closure of the stratum S_λ corresponding to ρ.
(8) More generally, suppose $R < S$ are strata corresponding to partitions μ, λ respectively and let $x \in R$. Then $\dim \underline{\underline{IH}}^{i-2d_\lambda}(\overline{S})_x$ is the multiplicity of the representation ρ_λ in the cohomology $H^i(\mathcal{B}_x)$ of the Springer fiber $\mathcal{B}_x$. In fact the Poincaré polynomial of these multiplicities

$$P_{\lambda,\mu}(t) = \sum_i \mathrm{mult}(\rho_\lambda, H^{2i}(\mathcal{B}_x))t^i = \sum_i \mathrm{rank}(\underline{\underline{IH}}^{2i}(\overline{S})_x)t^i$$

turns out to be the Kostka-Foulkes polynomial.

(9) For $x = 0 \in \mathcal{N}$ the Springer fiber $\mathcal{B}_x = G/B$ is the full flag variety and the representation of W is the regular representation. Moreover, the full endomorphism algebra

$$\mathrm{End}(R\pi_*(\mathbb{Q}_{\widetilde{\mathcal{N}}}) \cong \mathbb{C}[W]$$

is isomorphic to the full group-algebra of the Weyl group, with its regular representation.

Even for the W action on the full flag manifold $\mathcal{B} \cong G/B$ these results are startling. In this case it had been shown by Borel and Leray that the action of W was the regular representation, but which irreducible factors appeared in which degrees of cohomology had appeared to be a total mystery. Many computations were done by hand and the result appeared to be random. The above conclusions explain that the multiplicity of each representation ρ_λ in $H^i(G/B)$ is given by the rank of the local intersection cohomology, in degree i, at the origin $o \in \mathcal{N}$ of the stratum S that corresponds to the partition λ.

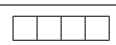
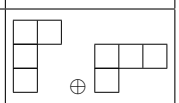
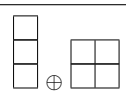
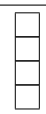
degee	rank	Young
$H^{12}(G/B)$	1	
$H^{10}(G/B)$	3	
$H^8(G/B)$	5	
$H^6(G/B)$	6	
$H^4(G/B)$	5	
$H^2(G/B)$	3	
$H^0(G/B)$	1	

FIGURE 26. Springer representations for SL_4. Each rep occurs as often as its dimension

25. Iwahori Hecke Algebra

25.1. The (Iwahori) Hecke algebra of the Weyl group of an algebraic group is usually defined using generators and relations (see Proposition 25.5) without motivation or explanation for the

mysterious formula $(T_s - q)(T_s + 1) = 0$. In this section we explain the geometric nature of this equation.

25.2. Let G be a finite group. The group algebra $\mathbb{C}[G]$ may be considered as the set of all functions $f : G \to \mathbb{C}$. The convolution of two functions $f, f' : G \to \mathbb{C}$ is the function

$$(f * f')(x) = \frac{1}{|G|} \sum_{h \in G} f(xh^{-1}) f'(h) = \frac{1}{|G|} \sum_{a \in G} f(a) f'(a^{-1}x).$$

this product is associative. If $H \subset G$ is a subgroup the Hecke algebra is

$$\mathcal{H}(G, H) = \{\phi : G \to \mathbb{C} |\ \phi(kgk') = \phi(g) \text{ for all } k, k' \in H\} \cong \mathbb{C}[H \backslash G / H]$$

with algebra structure given by convolution. If $\rho : H \to \mathrm{GL}(V)$ is a representation, the *induced representation* is

$$\mathrm{Ind}_H^G(\rho) = \{\phi : G \to V |\ \phi(hx) = \rho(h)\phi(x) \ \text{ for all } h \in H, x \in G\} \cong \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$$

with action $(g.\phi)(x) = \phi(xg^{-1})$. The Frobenius reciprocity theorem says:

$$\mathrm{Hom}_G(\mathrm{Ind}_H^G V, W) \cong \mathrm{Hom}_H(V, \mathrm{Res}_H^G(W)).$$

Consequently there is a natural identifiction

$$\mathcal{H}(G, H) \cong \mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathbb{1}), \mathrm{Ind}_H^G(\mathbb{1})).$$

Now let $G = \mathrm{SL}_n(\mathbb{F}_q)$ and $H = B$ the collection of upper triangular matrices (or determinant one). Let $W = S_n$ be the symmetric group which may be thought of as acting on the standard basis vectors $\{e_1, \cdots, e_n\}$. It is generated by the "simple reflections" $S = \{s_1, \cdots, s_{n-1}\}$ where s_i exchanges e_i and e_{i+1}. The *length* $\ell(w)$ of an element $w \in W$ is the minimum number of elements required to express w as a product of simple reflections, and it is well defined. The Bruhat decomposition says that $G = \coprod_{w \in W} BwB$. Each B orbit $BwB/B \subset G/B$ is isomorphic to an affine space of dimension $\ell(w)$.

25.3. Definition. The Hecke algebra $\mathcal{H} = \mathcal{H}(G, B)$ is the algebra of B-bi-invariant functions on G. It has a basis consisting of functions

$$\phi_w = \mathbb{1}_{BwB}.$$

The unit element in $\mathcal{H}$ is the function $\phi_1 = \mathbb{1}_B$. In this algebra we will use the following normalization for convolution of bi-invariant functions $f, f' : G \to \mathbb{C}$,

$$(f * f')(x) = \frac{1}{|B|} \sum_{h \in G} f(xh^{-1}) f'(h).$$

25.4. Lemma. *If $s \in S$ is a simple reflection and if $w \in W$ then the following holds:*

$$\begin{cases} \phi_w * \phi_{w'} = \phi_{ww'} & \text{if } \ell(w) + \ell(w') = \ell(ww') \\ \phi_s * \phi_s = (q-1)\phi_s + q\phi_1 & \\ \phi_s * \phi_w = (q-1)\phi_w + q\phi_{sw} & \text{if } \ell(sw) < \ell(w) \end{cases}$$

Proof. (The third equation follows from the second by induction.) The key nontrivial point (proven in [65]; see for example, [25]) is the following geometric property of these double cosets:

$$\ell(w) + \ell(w') = \ell(ww') \implies (BwB)(Bw'B) = Bww'B$$
$$\ell(ws) = \ell(w) - 1 \implies (BwB)(BsB) \subset (BwB) \cup (BwsB).$$

Following [25], for $f \in \mathcal{H}$ let $\epsilon(f) = \frac{1}{|B|}\sum_{g\in G} f(g)$ so that $\epsilon(f * f') = \epsilon(f)\epsilon(f')$ and $\epsilon(\phi_w) = q^{\ell(w)}$. If $\ell(w) + \ell(w') = \ell(ww')$ then $\phi_w * \phi_{w'}$ is supported on $Bww'B$ and is B bi-invariant. Apply epsilon to conclude that $\phi_w * \phi_{w'} = \phi_{ww'}$. Similarly, $\phi_s * \phi_s$ is supported on $(BsB) \cup B$ so it equals $\alpha\phi_s + \beta\phi_1$ for some $\alpha, \beta \in \mathbb{C}$. Apply ϵ to conclude that $q^2 = \alpha q + \beta$. Evaluate at $x = I \in G$ to get $\phi_s * \phi_s(I) = |BsB|/|B| = q = \alpha.0 + \beta.1$ So $\alpha = q - 1$. □

The same holds for any semisimple algebraic group G defined over $\mathbb{F}_q$, where B is a Borel subgroup and W is the Weyl group and S denotes the set of simple reflections,

25.5. Proposition. *The (Iwahori-) Hecke algebra $\mathcal{H}(W, S)$ is the free $\mathbb{Z}[q, q^{-1}]$ module with basis elements ϕ_w for $w \in W$ and relations*

- $\phi_s\phi_w = \phi_{sw}$ *if* $sw > w$
- $(\phi_s - q)(\phi_s + 1) = 0$.

If $q = 1$ this is the group algebra $\mathbb{Z}[W]$ which is to say that the Hecke algebra is a *deformation* of the group algebra. The relations in Proposition 25.5 are often presented as an unmotivated definition of the Hecke algebra. More generally, if (W, S) is a Coxeter group[11] with resulting order $>$ then the Hecke algebra of (W, S) is defined to be the $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ algebra generated by symbols T_w and satisfying the relations in the box. (The reason for introducing $\sqrt{q}$ in the coefficients will become clear in the next paragraph.)

25.6. Kazhdan-Lusztig polynomials. Kazhdan and Lusztig [75] discovered a mysterious new basis for the Hecke algebra that appeared to be closely related to infinite dimensional representations of the Lie algebra $\mathfrak{g}$ of SL_n. Each element $\phi_w \in \mathcal{H}$ is invertible and the algebra $\mathcal{H}$ admits an involution defined by

$$\iota(q^{1/2}) = q^{-1/2} \text{ and } \iota(\phi_w) = (\phi_{w^{-1}})^{-1}.$$

[11] that is, a group W with a finite set S of generators; with relations $(ab)^{m(a,b)} = 1\ \forall a, b \in S$, where $1 \le m(a, b) \le \infty$ and $m(a, a) = 2$

25.7. Theorem (Kazhdan, Lusztig)**.** *For each $w \in W$ there is a unique element $c_w \in \mathcal{H}$ and a uniquely determined polynomial P_{yw} for $y \leq w$ such that: $\iota(c_w) = c_w$; $P_{ww} = 1$; P_{yw} (for $y < w$) is a polynomial of degree $\leq \frac{1}{2}(\ell(w) - \ell(y) - 1)$; and*

$$c_w = q^{-\ell(w)/2} \sum_{y \leq w} P_{yw}(q)\phi_y$$

Existence and uniqueness of c_w and P_{yw} is easily proven by induction. Kazhdan and Lusztig conjectured that the coefficients of P_{yw} were nonnegative integers. They further conjectured that, in the Grothendieck group of Verma modules,

$$[L_w] = \sum_{y \leq w} (-1)^{\ell(w)-\ell(y)} P_{yw}(1)[M_y]$$

where M_w is the Verma module corresponding to highest weight $-\rho - w(\rho)$ and L_w is its unique irreducible quotient. This second conjecture became known as the *Kazhdan-Lusztig conjecture* and was eventually proven by J. L. Brylinski and M. Kashiwara [23] and independently by A. Beilinson and D. Bernstein [13]. This circle of ideas became extremely influential in representation theory. But what exactly is the meaning of c_w and P_{yw}? The answer (Theorem 26.3 below) eventually came from intersection cohomology.

26. Algebra of correspondences

26.1. Let us return to the complex picture with $G = \mathrm{SL}_n(\mathbb{C})$, B the Borel subgroup of upper triangular matrices and W the symmetric group. let $X = G/B$ be the flag manifold, or equivalently, the variety of Borel subgroups of G. The group G decomposes as a disjoint union $G = \coprod_{w]\ inW} BwB$. It follows that the group B acts on X with finitely many orbits, and these are indexed by the elements of W. For each $w \in W$ the *Schubert cell* or *Bruhat cell* $X_w = BwB/B \subset G/B$ indexed by w contains X_y in its closure iff $y < w$ in the Bruhat order. Similarly the group G acts on $X \times X$ with finitely many orbits, each of which contains a unique point (B, wB) (thinking of the standard Borel subgroup B as being the basepoint in the flag manifold) for some $w \in W$. It consists of pairs of flags $(F_1 \subset F_2 \subset \cdots F_n = \mathbb{C}^n)$ and $(F'_1 \subset F'_2 \subset \cdots \subset F'_n = \mathbb{C}^n)$ that are in relative position $w \in W$, meaning that there exists an ordered basis $(e_1, e_2, \cdots, e_n)$ of $\mathbb{C}^n$ so that, for each $1 \leq i \leq n$

$$\langle e_1, e_2, \cdots, e_i \rangle = F_i \quad \text{and} \quad \langle e_{w(1)}, e_{w(2)}, \cdots, e_{w(i)} \rangle = F'_i.$$

Equivalently, two flags F, F' are in relative position w if

$$\dim(F_i \cap F'_j) = |([1, i] \cap \sigma([1, j]))|.$$

If $q_2 : X \times X \to X$ denotes projection to the second factor then this orbit, let us denote it by $\mathcal{O}_w$ fibers over X with fiber equal to X_w; in particular it is simply connected. In other words, each G orbit on $X \times X$ intersects the fiber X in a single B orbit.

26.2. Algebra of correspondences. The following geometric construction of the Hecke algebra was discovered independently by R. MacPherson, G. Lusztig, Brylinski and Kashiwara, Beilinson and Bernsstein and is described in Springer's 1982 article [114]. It is convenient here to change notation in the Hecke algebra, setting

$$t = \sqrt{q}.$$

Consider the derived category $D^b_{c,even}(X \times X)$ of sheaves $A^\bullet$ on $X \times X$, cohomologically constructible with respect to this stratification such that $\underline{\underline{H}}^i(A^\bullet) = 0$ for i odd. For for such a sheaf $A^\bullet$ let $\underline{\underline{H}}^{2i}(A^\bullet)_w$ denote the stalk cohomology of $A^\bullet$ at a point in $\mathcal{O}_w$ and define

$$h : D^b_{c,even}(X \times X) \to \mathcal{H}$$
$$h(A) = \sum_{w \in W} \sum_{i \geq 0} \dim(\underline{\underline{H}}^i(A^\bullet)_w) t^i \phi_w$$

Note that all elements of $\mathcal{H}$ obtained in this way have nonnegative coefficients so the image of h ends up in a sort of "positive cone" in the Hecke algebra. Consider the following diagram of correspondences.

$$\begin{tikzcd} & X \times X \times X \arrow[dl, "q_{12}"'] \arrow[d, "q_{13}"] \arrow[dr, "q_{23}"] & \\ X \times X & X \times X & X \times X \end{tikzcd}$$

If $A^\bullet, B^\bullet \in D^b_c(X \times X)$ define their convolution product

$$A^\bullet \circ B^\bullet = Rq_{13!}\left(q_{12}^*(A^\bullet) \otimes q_{23}^*(B^\bullet)\right).$$

26.3. Theorem. *Let $j_w : \mathcal{O}_w \to X$ and $\bar{j}_w : \overline{\mathcal{O}_w} \to X$ denote the inclusions. Then*

$$h\left(j_{w!}(\underline{\underline{\mathbb{C}}}_{\mathcal{O}_w})[\ell(w)]\right) = \phi_w \quad \text{and } h(\underline{\underline{IC}}^\bullet_w) = c_w$$

where $\underline{\underline{IC}}^\bullet_w = R\bar{j}_{w*}\left(\underline{\underline{IC}}^\bullet_{\overline{\mathcal{O}}_w}\right)$. Moreover, $h(\underline{\underline{IC}}^\bullet_w \circ \underline{\underline{IC}}^\bullet_v) = h(\underline{\underline{IC}}^\bullet_w).h(\underline{\underline{IC}}^\bullet_v)$ for all $v, w \in W$.

In other words, each sheaf $A^\bullet \in D^b_{c,even}(X \times X)$ gives an element $h(A^\bullet)$ of the Hecke algebra. The constant sheaf on an orbit $\mathcal{O}_w$ gives the classical basis element ϕ_w, while the sheaf $\underline{\underline{IC}}^\bullet_w$ gives the Kazhdan Lusztig basis element c_w. Consequently the polynomial $P_{y,w}$ is the local intersection cohomology Poincaré polynomial of X_w at a point in X_y (originally proven in [76]). It vanishes in odd degrees and its coefficients are non-negative. Using "classical" indexing for sheaf cohomology

(and $q = t^2$),

$$P_{yw}(q) = \sum_{i \geq 0} \dim \underline{\underline{IH}}_y^{2i}(\overline{X}_w) q^i.$$

This stalk cohomology vanishes in odd degrees and the highest power of t that can occur here is $t^{\ell(w)-\ell(y)-1}$. Besides making the essential connection with geometry this result is a "categorification" of the Hecke algebra: it replaces numbers and coefficients with (cohomology) groups. It implies that the coefficients of P_{yw} are non-negative integers.

26.4. The mapping h in Theorem 26.3 is multiplicative on sums of $\underline{\underline{IC}}^\bullet$ sheaves, and the resulting elements c_w of the Hecke algebra form a basis of the Hecke algebra. However, the convolution operation on the sheaves $j_{w!}$ (corresponding to elements $\phi_w \in \mathcal{H}$) does not necessarily agree with Hecke multiplication, essentially because of the minus signs that arise in the product formula.

Let $s = (1, 2) \in S_3$ denote the simple reflection that exchanges 1 and 2. Then $c_s = \phi_1 + \phi_s$. Let us compute the convolution of this element with itself. Then BsB consists of pairs of flags (E, F) in $\mathbb{C}^3$ such that $F_1 \neq E_1$, $F_1 \subset E_2$, $F_2 = E_2$. So the union $(BsB) \cup (B1B)$ consists of pairs of flags (E, F) such that $F_1 \subset E_2$ is arbitrary. Let us denote this relationship by $E \xrightarrow{1+s} F$. We can assume that E is the standard flag, so that the flag F is completely determined by $F_1 \subset E_2$. This set is isomorphic to $\mathbb{P}^1$ hence $\underline{\underline{IC}}_s^\bullet = \underline{\underline{\mathbb{C}}}_{\overline{\mathcal{O}}_s}$.

Now let us compose this element $1 + s$ with itself. The correspondence consists of triples

$$E \xrightarrow{1+s} F \xrightarrow{1+s} F'$$

of flags and it is apparently a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. Now let us understand the mapping $\pi_{13} : X \times X \times X \to X \times X$, in other words, we project the tripe (E, F, F') of the correspondence to the pair (E, F'). Taking E to be the standard flag, the second flag F' is determined by $F_1' \subset E_2$ and apparently all possible subspaces occur. If $F_1' \subset E_2$ then the fiber consists of all 1-dimensional subspaces $F_1 \subset E_2$, which is a $\mathbb{P}^1$, and this is independent of which point F_1' was chosen. So the pushforward (which is also the pushforward with compact supports) of the constant sheaf gives $\underline{\underline{\mathbb{Q}}} \oplus \underline{\underline{\mathbb{Q}}}[2]$ on the closed set $(BsB) \cup (B1B)$. In other words, this calculation says that

$$(\phi_1 + \phi_s).(\phi_1 + \phi_s) = (q + 1)(\phi_1 + \phi_s)$$

from which it follows that $\phi_s.\phi_s = (q - 1)\phi_s + q\phi_1$, which is one of the defining formulas for $\mathcal{H}$.

Similarly, the product $(\phi_1 + \phi_s).\phi_s$ corresponds to a map of triples $E \xrightarrow{1+s} F \xrightarrow{s} F'$ for which $F_1, F_1' \subset E_2$ but $F_1 \neq F_1'$. When we project to the pair $E \xrightarrow{1+s} F'$ we find that the fiber over each point consists of $\mathbb{P}^1 - \{pt\}$ so the pushforward with compact supports yields $\underline{\underline{\mathbb{Q}}}[2]$ which gives the formula $(\phi_1 + \phi_s).\phi_s = q(1 + \phi_s)$.

These calculations are made using the "classical" indexing for intersection cohomology. If Deligne's indexing is used then the Hecke algebra should be redefined (as Lusztig often does) as follows: $(T_s - q^{\frac{1}{2}})(T_s + q^{-\frac{1}{2}}) = 0$.

26.5. The proof of Theorem 26.3 is tedious but does not require sophisticated methods; it is completely worked out in online notes [103] (following an outline of T. A. Springer [114], as communicated to him by R. MacPherson) and [61] . Here is the outline. It is obvious from the definitions that $h(Rj_{w!}(\mathcal{O}_w)) = \phi_w$. Now consider $h(\underline{\underline{IC}}_s^\bullet)$. The support condition for intersection cohomology implies that $h(\underline{\underline{IC}}_w)$ is a linear combination of $h(j_{w!}\underline{\underline{\mathbb{C}}}_w)$ with coefficients that are polynomials which satisfy the degree restriction.

The same argument as in the previous section §26.4 shows: if s is a simple reflection with corresponding orbit closure $\overline{\mathcal{O}}_s \subset X \times X$ then

$$h(\mathbb{C}_{\overline{\mathcal{O}}_s} \circ \underline{\underline{IC}}_w^\bullet) = (\phi_s + 1).h(\underline{\underline{IC}}_w^\bullet) \in \mathcal{H}$$

although the argument needs to be modified slightly when $sw < w$. Next, one verifies that $h(\underline{\underline{IC}}_w^\bullet)$ is preserved by ι. One way to prove this is to consider the Bott-Samelson resolution of the Schubert variety X_w. It is obtained as a sequence of blowups by simple reflections. One checks at each stage of the induction that the result is preserved by ι. This proves that $h(\underline{\underline{IC}}_w)$ is preserved by ι and satisfies the combinatorial conditions that uniquely identify it as $c_w \in \mathcal{H}$.

Finally, again using the Bott Samelsom resolution, the decomposition theorem and induction, one proves that $h(\underline{\underline{IC}}_w \circ \underline{\underline{IC}}_{w'}) = c_w c_{w'}$.

26.6. Digression: Hecke algebra and modular forms. There is a strong parallel between the theory of the (Iwahori-) Hecke algebra above, and the Hecke algebra in the theory of modular forms. Let $G = \mathrm{SL}_n(\mathbb{R})$, let $K = O(n)$ and let $\Gamma_0 = \mathrm{SL}_n(\mathbb{Z})$. Then $D = G/K$ be the (contractible) symmetric space of positive definite symmetric matrices of determinant one. Let $X = \Gamma_0 \backslash D$. This is the moduli space of Riemannian tori. (For each lattice $L \subset \mathbb{R}^n$ of determinant one we get a torus $\mathbb{R}^n/L$ and an invariant Riemannian metric on it.) Each $g \in G_{\mathbb{Q}} = \mathrm{SL}_n(\mathbb{Q})$ gives a correspondence on this space as follows. Let $\Gamma' = \Gamma_0 \cap (g^{-1}\Gamma_0 g)$ and let $X' = \Gamma' \backslash D$. Then the correspondence $X' \to X \times X$ is given by $\Gamma' x \mapsto (\Gamma_0 x, \Gamma_0 g x)$. It is well defined and each of the projections $X' \to X$ is a finite covering. Moreover, the isomorphism class of this correspondence depends only on the double coset $\Gamma_0 g \Gamma_0$. (Replacing g by γg where $\gamma \in \Gamma_0$ does not change the correspondence. Replacing g by $g\gamma$ changes the correspondence but it gives an isomorphic correspondence.) Therefore points in the double coset space

$$\Gamma_0 \backslash \mathrm{SL}_n(\mathbb{Q}) / \Gamma_0$$

may be interpetred as defining correspondences on X, which therefore acts on the homology, cohomology, functions etc. of X. So the same is true of linear combinations of double cosets. In summary, the Hecke algebra of compactly supported functions (meaning, functions with finite support) on $\Gamma_0 \backslash \mathrm{SL}_n(\mathbb{Q})/\Gamma_0$ acts on $H^*(X)$ by correspondences. The composition of correspondces turns out to coincide with convolution of compactly supported functions. Such functions are called Hecke operators.

This construction makes more sense in the adèlic setting where natural Haar measures can be used in order to define the algebra structure and the action without resorting to correspondences.

In this setting there is an equality

$$\mathrm{SL}_n(\mathbb{Z})\backslash \mathrm{SL}_n(\mathbb{R})/K \cong \mathrm{SL}_n(\mathbb{Q})\backslash \mathrm{SL}_m(\mathbb{A}_{\mathbb{Q}})/K.\,\mathrm{SL}_n(\widehat{\mathbb{Z}})$$

and the Hecke algebra is the convolution algebra of locally constant functions with compact support

$$f \in C_c^\infty(\mathrm{SL}_n(\mathbb{Q})\backslash \mathrm{SL}_n(\mathbb{A}_f)/\mathrm{SL}_n(\mathbb{Q}))$$

where $\mathbb{A}_{\mathbb{Q}}$ denotes the adèles of $\mathbb{Q}$ and $\mathbb{A}_f$ the finite adèles.

27. The affine theory

27.1. Affine Weyl group. The symmetric group S_n has Dynkin diagram : [diagram] It is generated by simple reflections $s_1, \cdots, s_{n-1}$ with the relations

(1) $s_i^2 = 1$
(2) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for $(1 \le i \le n-2)$
(3) $s_i s_j = s_j s_i$ if $|i-j| > 1$.

It can be interpreted as acting on $\mathbb{R}^{n-1} = \{\sum x_i = 0\} \subset \mathbb{R}^n$ with s_i acting as reflection across the hyperplane $x_i = x_{i+1}$. This decomposes $\mathbb{R}^{n-1}$ into Weyl chambers, one for each element of S_n.

The affine symmetric group $\widetilde{S}_n$ has Dynkin diagram: It is generated by simple reflections $s_0, s_1, \cdots, s_n$ with the same relations as S_n and the additional relations (corresponding to edges $s_0 s_1$ and $s_n s_0$). It can be interpreted as acting on $\mathbb{R}^{n-1}$ by adding a reflecting hyperplane to the previous picture. Then it acts simply transitively on the alcoves. If we take the fundamental alcove as the basepoint (identity), then every alcove becomes labeled by a unique element of $\widetilde{S}_n$.

27.2. The affine symmetric group can also be described as $S_n \ltimes A$ where A is the root lattice of translations,

$$\Big\{(a_1, \cdots, a_n) \in \mathbb{Z}^n |\ \sum a_i = 0\Big\}$$

on which S_n acts by permutations. The group A may also be interpretred as the cocharacter group of the maximal torus $\mathcal{T}$ consisting of diagonal matrices of determinant one. The affine Weyl group can be described as

$$\{\omega : \mathbb{Z} \to \mathbb{Z} |\ \omega(i+n) = \omega(i) + n\}.$$

In this realization each element is determined by its value on $\{1, \cdots, n\}$ and so it may be written as $[\omega(1), \cdots, \omega(n)]$. Then elements in the lattice of translations are the elements $[a_1, \cdots, a_n]$ with $\sum a_i = 0$ and they act by addition, that is, $\omega(i) = a_i + i$ for $1 \le i \le n$. Then every element $\omega \in \widetilde{S}_n$ can be expressed as a permutation followed by a translation.

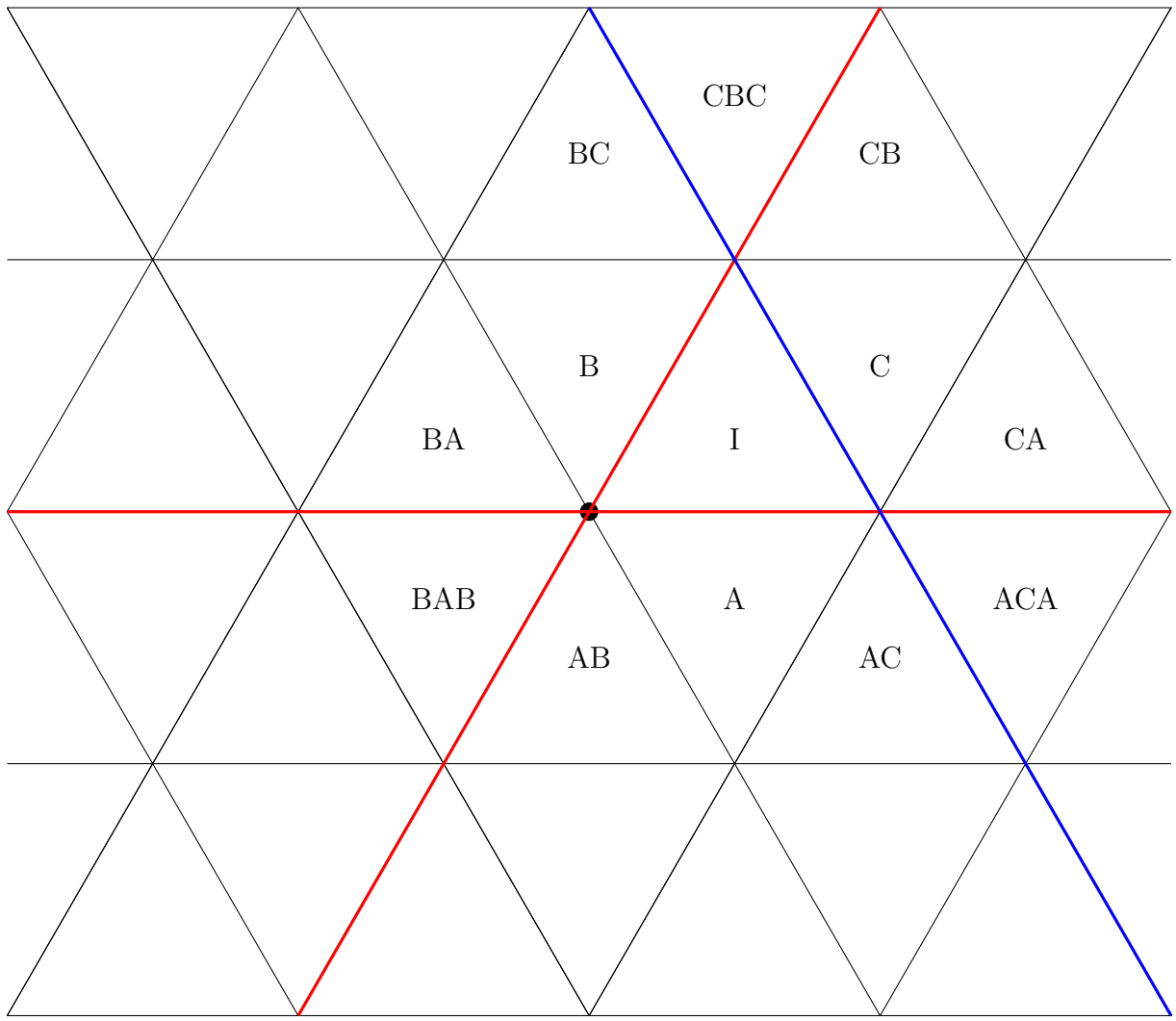


FIGURE 27. This diagram ilustrates the affine Weyl group for SL_3. The red lines are the reflecting hyperplanes of the finite Weyl group (generated by A and B). The blue line is the affine reflection, C.

27.3. Affine Hecke algebra. Let $G_{\mathbb{Q}_p} = \mathrm{SL}_n(\mathbb{Q}_p)$, as a locally compact topological group and set $K = G(\mathbb{Z}_p) = \mathrm{SL}_m(\mathbb{Z}_p)$. Let $B_{\mathbb{F}_p}$ be the Borel subgroup of $\mathrm{SL}_n(\mathbb{Z}/p\mathbb{Z})$. The Iwahori subgroup I_p is the preimage $\pi^{-1}(B_p)$ under the $\pmod p$ mapping $\phi : K \to SL_n(\mathbb{Z}/p\mathbb{Z})$, that is, it consists of $n \times n$ matrices with entries in $\mathbb{Z}_p$, whose diagonal entries are invertible in $\mathbb{Z}_p$, and whose lower diagonal entries are multiples of p. It is compact and open in $G_{\mathbb{Q}_p}$.

27.4. Definition. The Iwahori Hecke algebra is the convolution algebra of locally constant complex valued functions

$$f \in C_c^\infty(I_p \backslash G_{\mathbb{Q}_p}/I_p)$$

with compact support on $G_{\mathbb{Q}_p}$ that are bi-invariant under I_p.

Haar measure μ on $G_{\mathbb{Q}_p}$ is normalized so that $\mu(I_p) = 1$.

The Bruhat decomposition in this case reads $G_{\mathbb{Q}_p} = \coprod_{w \in W_a} I_p w I_p$ where W_a is the affine Weyl group. Then $\mathcal{H}_I$ is generated by characteristic functions ϕ_w for $w \in W_a$ and with the same relations as before: $\phi_s^2 = (q-1)\phi_s + q\phi_1$. The Kazhdan Lusztig canonical basis c_w is defined exactly as before. The Kazhdan Lusztig theorem works in this context as well and it gives a basis of $\mathcal{H}_I$ consisting of elements c_w for $w \in W_a$ the affine Weyl group.

The field $\mathbb{Q}_p$ is analogous to the field $\mathbb{F}_q((T))$ of formal Laurent series (meaning formal power series with finitely many negative powers of T and coefficients in $\mathbb{F}_q$). The ring $\mathbb{Z}_p$ corresponds to $\mathbb{F}_q[[T]]$ (the ring of formal power series). Reduction modulo T gives a homomorphism $\phi : \mathbb{F}_q[[T]] \to \mathbb{F}_q$ and the Iwahori subgroup $I_{p((T))} = \phi^{-1}(B_q)$ is defined similarly. Then an Iwahori Hecke algebra over $\mathbb{F}_q$ is defined to be the convolution algebra of locally constant complex valued functions

$$f \in C_c^\infty(I_{p[[T]]} \backslash G_{\mathbb{F}_q((T))}/I_{p[[T]]})$$

with compact support that are bi-invariant under the Iwahori subgroup.

All of this has a complex analog following the same procedure as in the finite case. Instead of the flag manifold over $\mathbb{C}$ one uses the "affine flag manifold" $\mathrm{SL}_n(\mathbb{C}((T)))/I$ where $\mathbb{C}((T))$ is the field of formal Laurent series (that is, power series with finitely many negative powers of T) and where $I = \phi^{-1}(B)$ is the Iwahori subgroup defined by $\pmod T$ reduction,

$$\phi : \mathrm{SL}_n(\mathbb{C}[[T]]) \to \mathrm{SL}_n(\mathbb{C}).$$

The quotient $\mathrm{SL}_n((T))/I$ is infinite dimensional but it is an increasing limit of finite dimensional complex algebraic varieties, and each I orbit of an element $w \in W_a$ is a (generalized) "Schubert cell" or Bruhat cell, of dimension $\ell(w)$. The Kazhdan Lusztig polynomials P_{yw} have non-negative coefficients and they may be interpreted as the local intersection cohomology Poincaré polynomials of one Schubert cell at a point in another Schubert cell.

However, the sheaf-convolution construction does not work in this setting because the orbits of I on $X \times X$ (where X denotes the affine flag manifold) have infinite dimension and infinite codimension. Instead, another approach is needed, which will be described later in the case of the affine Grassmannian.

27.5. Overview. There are strong analogies between these constructions over different fields. The following chart gives some idea of the parallels between the different cases.

Field	$\mathbb{C}$	$\mathbb{F}_q$	$\mathbb{Q}_p$	$\mathbb{C}((T))$	$\mathbb{F}_q((T)$
Group	$SL_n(\mathbb{C})$	$\mathrm{SL}_n(\mathbb{F}_q)$	$\mathrm{SL}_n(\mathbb{Q}_p)$	$\mathrm{SL}_n(\mathbb{C}((T)))$	$\mathrm{SL}_n(\mathbb{F}_q((T)))$
symbol	$G_{\mathbb{C}}$	G_q	$G_{\mathbb{Q}_p}$	$G_{((T))}$	$G_{q((T))}$
Borel/Iwahori	$\begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{pmatrix}$	$\begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{pmatrix}$	$\phi^{-1}(B)$	$\phi^{-1}(B)$	$\phi^{-1}(B_q)$
symbol	$B_{\mathbb{C}}$	$B_{\mathbb{F}_q}$	I_p	$I_{[[T]]}$	$I_{q[[T]]}$
Weyl group	S_n	S_n	W_a	W_a	W_a
Bruhat decomp	$\coprod_{w\in W} BwB$	$\coprod_{w\in W} BwB$	$\coprod_{w\in W_a} IwI$	$\coprod_{w\in W_a} IwI$	$\coprod_{w\in W_a} IwI$
Flag manifold	$G_{\mathbb{C}}/B_{\mathbb{C}}$	$G_{\mathbb{F}_q}/B_{\mathbb{F}_q}$	$G_{\mathbb{Q}_p}/I_p$	$G_{((T))}/I_{((T))}$	$G_{q((T))}/I_{q[[T]]}$
Parabolic/Parahoric	$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix}$	$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix}$	$\mathrm{SL}_n(\mathbb{Z}_p)$	$\mathrm{SL}_n(\mathbb{C}[[T]])$	$\mathrm{SL}_n(\mathbb{F}_q[[T]])$
symbol	$P_{\mathbb{C}}$	$P_{\mathbb{F}_q}$	K_p	$G_{[[T]]}$	$G_{q[[T]]}$
Bruhat decomp			$\coprod_{a\in A} K_p a K_p$	$\coprod_{a\in A} G_{[[T]]} a G_{[[T]]}$	$\coprod_{a\in A} G_{q[[T]]} a G_{q[[T]]}$
Grassmannian	$G_{\mathbb{C}}/P_{\mathbb{C}}$	$G_{\mathbb{F}_q}/P_{\mathbb{F}_q}$	$G_{\mathbb{Q}_p}/K_p$	$G_{((T))}/G_{[[T]]}$	$G_{q((T))}/G_{q[[T]]}$

27.6. Definition. The affine Grassmannian is the quotient

$$X = \mathrm{SL}_n(\mathbb{C}((T)))/\,\mathrm{SL}_n(\mathbb{C}[[T]]).$$

If we think of $\mathbb{C}((T))^n = \cup_{N=0}^{\infty} t^{-N}\mathbb{C}[[T]]$ then a *lattice* in $\mathbb{C}((T))^n$ is a $\mathbb{C}[[T]]$ submodule $M \subset \mathbb{C}((T))^n$ (meaning that it is preserved under multiplication by T) such that

$$T^{-N}\mathbb{C}[[T]]^n \supset M \supset T^N\mathbb{C}[[T]]^n$$

for sufficiently large N, and which satisfies the determinant one conditon, $\wedge^n M = \mathbb{C}[[T]]$. The affine Grassmannian is the set of all such lattices. In fact the group $\mathrm{SL}_n(\mathbb{C}((T)))$ acts transitively on the set of such lattices and the stabilizer of the standard lattice $\mathbb{C}[[T]]^n$ is the parahoric subgroup $\mathrm{SL}_n(\mathbb{C}[[T]])$.

$$\begin{pmatrix} T^{-1} & & & \\ 0 & T^2 & & \\ 0 & 0 & T^{-3} & \\ 0 & 0 & 0 & T^2 \end{pmatrix} K = \begin{array}{|c|c|c|c|c|} \hline T^{-4} & & & & \\ \hline T^{-3} & & & \bullet & \\ \hline T^{-2} & & & \bullet & \\ \hline T^{-1} & \bullet & & \bullet & \\ \hline T^{0} & \bullet & & \bullet & \\ \hline T^{1} & \bullet & & \bullet & \\ \hline T^{2} & \bullet & \bullet & \bullet & \bullet \\ \hline T^{3} & \bullet & \bullet & \bullet & \bullet \\ \hline \end{array}$$

27.7. Its stratification. The affine Grassmannian X is an infinite increasing union of projective varieties. It has two interesting stratifications. The first, is by orbits of the Iwahori subgroup $I_{[[T]]}$. These orbits are indexed by the group of translations A in affine Weyl group and the orbit $X_a = I_{[[T]]}aK$ is a Schubert cell: it is an affine space of dimension $\ell(a)$ with $X_y \subset \overline{X}_w$ iff $y < w$ in the Bruhat order on W_a. In other words (setting $I = I_{[[T]]}$ for brevity)

$$X = \coprod_{a\in A} IaK/K \text{ because } \mathrm{SL}_n(\mathbb{C}((T))) = IAK$$

which is the analog of the Iwasawa decomposition of G. The lattice of translations A may be identified with the group of all diagonal matrices $\mathrm{diag}(T^{a_1}, T^{a_2}, \cdots T^{a_n})$ such that $\sum_i a_i = 0$. Such an element may be interpreted as a cocharacter of the maximal torus $\mathcal{T}$ of diagonal matrices, that is, $A \cong \chi_*(\mathcal{T})$.

The second stratification is by orbits of the subgroup $K = \mathrm{SL}_n(\mathbb{C}[[T]])$. The (finite dimensional) Bruhat decomposition $\mathrm{SL}_n(\mathbb{C}) = \coprod_{w\in W} BwB$ implies that $K = \coprod_{w\in W} IwI$. So each K orbit is a union of $n!$ Schubert cells. Let $\mathcal{T}$ be the torus of diagonal matrices (of determinant one) in SL_n. Then W acts on $\mathcal{T}$ and on its group of cocharacters $A = \chi_*(\mathcal{T})$, which we have identified with the lattice of translations in the affine Weyl group W_a. A fundamental domain for this action is the *positive cone* $A_+ \subset \mathcal{T}_+$. So the Bruhat decomposition becomes, in this case:

$$\mathrm{SL}_n(\mathbb{C}((T))) = \textstyle\coprod_{a\in A_+} KaK \quad \text{and } X = \coprod_{a\in A_+} KaK/K.$$

The strata are no longer cells, but each stratum has the structure of a vector bundle over a nonsingular projective algebraic variety, so it is simply connected.

For example, in the affine Grassmannian for PGL_3 consider the K orbit of the lattice represented by

$$xK = \begin{pmatrix} T^{-2} & & \\ 0 & 1 & \\ 0 & 0 & 1 \end{pmatrix} K = \begin{array}{|c|c|c|c|}\hline T^{-3} & & & \\ \hline T^{-2} & \bullet & & \\ \hline T^{-1} & \bullet & & \\ \hline T^{0} & \bullet & \bullet & \bullet \\ \hline T^{1} & \bullet & \bullet & \bullet \\ \hline T^{2} & \bullet & \bullet & \bullet \\ \hline \end{array}$$

The leading term T^{-2} determines a line in $\mathbb{C}^3$. But, acting by elements of K we can also obtain lattices like this: $\begin{array}{|c|c|c|c|}\hline T^{-3} & & & \\ \hline T^{-2} & \bullet & & \\ \hline T^{-1} & \bullet & \star & \star \\ \hline T^{0} & \bullet & \bullet & \bullet \\ \hline T^{1} & \bullet & \bullet & \bullet \\ \hline T^{2} & \bullet & \bullet & \bullet \\ \hline \end{array}$ which means that the orbit KxK/K has the structure of a two dimensional vector bundle over $\mathbb{C}P^2$.

27.8. Two more views of the affine Grassmannian. Let $\mathbb{C}[T]$ be the ring of polynomials and let $\mathbb{C}(T)$ be the field of rational functions $p(T)/q(T)$. There is a natural map

$$G(\mathbb{C}(T))/G(\mathbb{C}[T]) \to G(\mathbb{C}((T)))/G(\mathbb{C}[[T]]).$$

It turns out to be an isomorphism. We can also consider $G(\mathbb{C}(T))$ to be the *loop group*

$$LG = \left\{ f : S^1 \to G | f \in \mathbb{C}(T) \right\}$$

consisting of all mappings which are rational functions. (Similarly one could consider analytic, smooth, or continuous functions; the results are homotopy equivalent). If LG^+ denotes mappings that can be extended (holomorphically, or as a polynomial) over the origin in $\mathbb{C}$ then the quotient

$$LG/LG^+ \cong G(\mathbb{C}(T))/G(\mathbb{C}[T])$$

is sometimes referred to (by physicists) as the *fundamental homogeneous space*.

28. Perverse sheaves on the affine Grassmannian

28.1. Spherical Hecke algebra. As in the previous section we take $G = \mathrm{SL}_n$, although the following is valid for any semisimple algebraic group defined over the appropriate ring. The Hecke algebra

$$\mathcal{H}(G(\mathbb{Q}_p)//G(\mathbb{Z}_p)) \text{ resp. } \mathcal{H}(G(\mathbb{F}_q((T))//G(\mathbb{F}_q[[T]])) \text{ etc.}$$

of locally constant compactly supported bi-invariant functions is called the *spherical Hecke algebra*, that is,

$$\boxed{\mathcal{H}(G(\mathbb{Q}_p)//G(\mathbb{Z}_p)) = C_c^\infty(\mathrm{SL}_n(\mathbb{Z}_p)\backslash \mathrm{SL}_n(\mathbb{Q}_p)/\mathrm{SL}_n(\mathbb{Z}_p)).}$$

Recall that the representation ring $R(G)$ of a (complex) reductive group G is isomorphic isomorphic to the Weyl invariants

$$R(G) \cong \mathbb{Z}[\chi^*(\mathcal{T})]^W$$

in the group of characters of a maximal torus $\mathcal{T}$. In fact, a fundamental domain for the action of W on $\chi^*(\mathcal{T})$ is given by the positive Weyl chamber, $\chi^*(\mathcal{T})_+$. To such a character $\lambda \in \chi^*(\mathcal{T})_+$ one associates the irreducible representation V_λ with highest weight λ. Its trace is a character of $\mathcal{T}$.

As a consequence, there are many equivalent ways to view this Hecke algebra.

(1) By theorems of Satake and MacDonald, there is a natural isomorphism

$$\mathcal{H}(G(\mathbb{Q}_p)//G(\mathbb{Z}_p)) \cong \mathbb{C}[X_*(\mathcal{T})]^W$$

of the Hecke algebra with the Weyl invariants in the group algebra of the cocharacter group of the maximal torus.

(2) This in turn may be identified with the Weyl invariants $\mathbb{C}[X^*(\widehat{\mathcal{T}})]^W$ in the characters of the dual torus.

(3) Which, by the adjoint quotient map, is isomorphic to the group of conjugation-invariant polynomial functions on PGL_n. (Recall that we previously identified this as a polynomial algebra, given by the coefficients of the characteristic polynomial.)
(4) This may be identified with $\mathbb{C} \otimes K(\mathrm{Rep}_{\mathrm{PGL_n}})$ (that is, the Grothendieck group of the category of finite dimensional (rational) representations of PGL_n) by associating, to any representation ρ its character (or trace), which is a Weyl invariant polynomial function.
(5) In fact, these identifications can be made over the integers. See [Gross] who considers the Hecke algebra $\mathcal{H}_G$ of $\mathbb{Z}$-valued functions on this double coset and describes isomorphisms

$$\mathcal{H}_G \longrightarrow \left(\mathcal{H}_{\mathcal{T}} \otimes \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]\right)^W \longleftarrow \mathrm{Rep}(\widehat{G}) \otimes \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}].$$

Here $\widehat{G}$ is the *Langlands dual group* of the group G. (The dual of SL_n is PGL_n.)

28.2. Digression: Langlands Dual group. (from Wikipedia, who took it from [114])

A *root datum* consists of a quadruple $(X^*, \Phi, X_*, \Phi^\vee)$ where X^*, X_* are free abelian groups of finite rank together with a perfect pairing $\langle , \rangle : X^* \times X_* \to \mathbb{Z}$, where $\Phi \subset X^*$ and $\Phi^\vee \subset X_*$ are finite subsets, and where there is a bijection $\Phi \to \Phi^\vee$, denoted $\alpha \mapsto \alpha^\vee$, and satisfying the following conditions:

(1) $\langle \alpha, \alpha^\vee \rangle = 2$ for all $\alpha \in \Phi$
(2) The map $x \mapsto x - \langle x, \alpha^\vee \rangle \alpha$ takes Φ to Φ and
(3) the induced action on X_* takes $\Phi^\vee$ to $\Phi^\vee$.

If G is a reductive algebraic group over an algebraically closed field then it defines a root datum where X^* is the lattice of characters of a (split) maximal torus T, where X_* is the lattice of cocharacters of T, where Φ is the set of roots and $\Phi^\vee$ is the set of coroots.

A connected reductive algebraic group over an algebraically closed field K is determined up to isomorphism (see [113]) by its root datum and every root datum corresponds to such a group.

Let G be a connected reductive algebraic group over an algegraically closed field, with root datum $(X^*, \Phi, X_*, \Phi^\vee)$. Then the connected reductive algebraic group with root datum $(X_*, \Phi^\vee, X^*, \Phi)$ is called the Langlands dual group and it is denoted $G^\vee$ or sometimes ${}^L G$.

Langlands duality switches adjoint groups with simply connected groups. It takes type A_n to A_n but it switches types $\mathrm{Sp}(2n)$ with $\mathrm{SO}(2n+1)$. It preserves the type $\mathrm{SO}(2n)$. A maximal torus in the dual group $G^\vee$ may be identified with the dual of a maximal torus in G.

28.3. Lusztig's character formula. As above let $G = \mathrm{SL}_n(\mathbb{C}((T)))$, let $K = \mathrm{SL}_n(\mathbb{C}[[T]])$, let $I \subset K$ be the Iwahori subgroup. The affine flag manifold $Y = G/I$ fibers over the affine Grassmannian $X = G/K$ with fiber isomorphic to $K/I \cong \mathrm{SL}_n(\mathbb{C})/B(\mathbb{C})$ the finite dimensional flag manifold, which is smooth. Consequently the singularities of I-orbit closures in X are the same as the singularities of I orbit closures in Y. The K orbits on X are indexed by cocharacters

in the positive cone. If $\lambda = \mathrm{diag}(a_1, a_2, \cdots, a_n) \in \mathbb{Z}^n$ is in the positive cone (and $\sum a_i = 0$) let $x_\lambda = \mathrm{diag}(T^{a_1}, T^{a_2}, \cdots, T^{a_n}) \in \mathrm{SL}_n(\mathbb{C}((T)))$. The K orbit X_λ corresponding to a is $X_\lambda = Kx_\lambda K/K \subset G/K$. If $\mu \leq \lambda$ then the point x_μ lies in the closure of the stratum X_λ and the local intersection cohomology Poincaré polynomial

$$\sum_{i \geq 0} \dim(\underline{\underline{IH}}^{2i}_{x_\mu}(\overline{X}_\lambda))t^i = P_{\mu,\lambda}(t)$$

is given by the Kazhdan Lusztig polynomial $P_{\mu,\lambda}$ for the affine Weyl group. Lusztig [Lu] proves

28.4. Theorem. *Let $\mu \leq \lambda \in \chi_*(\mathcal{T})_+ \cong \chi^*(\mathcal{T}^*)_+$. Let V_λ be the representation (of ${}^LG(\mathbb{C})$) of highest weight λ. It decomposes into weight spaces $(V_\lambda)(\mu)$ under the action of the maximal torus. Then*

$$\dim(V_\lambda(\mu)) = P_{\mu\lambda}(1).$$

That is, the local intersection cohomology Euler characteristic of the affine Schubert varieties (and of the affine K orbits) equals the weight multiplicity in the irreducible representation. (If you wish to add up these polynomials in order to get the intersection cohomology of the whole orbit closure, then you must do so with a shift $t^{\ell(\lambda)-\ell(\mu)}$ corresponding to the codimension in X_λ of the I-orbit that contains the point x_μ.) Consequently Lusztig considers the full Kazhdan Lusztig polynomial $P_{\mu\lambda}(q)$ be a *q-analog of the weight multiplicity.* The individual coefficients were eventually shown (by R. Brylinski, Lusztig, others) to equal the multiplicity of the weight μ in a certain layer $V_\lambda^r/V_\lambda^{r-1}$ of the filtration of V_λ that is induced by the principal nilpotent element.

28.5. Moment map. The complex torus $\{(a_1, \cdots, a_n) \in \mathbb{C}^n | \prod_i a_i = 1\} \cong (\mathbb{C}^\times)^{n-1}$ acts on X with a the moment map (for the action of $(S^1)^{n-1}$) $\mu : X \to \mathfrak{a}^*$. Each fixed point $x_\lambda = (T^{\lambda_1}, \cdots, T^{\lambda_n})$ corresponds to a cocharacter $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n$ (with $\sum_i \lambda_i = 0$). The torus action preserves both stratifications and the image of each stratum closure is a convex polyhedron. We can put all this information on the same diagram. Fix $\lambda \in \chi_*(\mathcal{T})_+$. Let V_λ be the irreducible representation of PGL_3 with highest weight λ. Let X_λ be the K-orbit of the point x_λ. Then $\mu(X_\lambda)$ is the convex polyhedron spanned by the W orbit of the point λ. The lattice points μ inside this polyhedron correspond to the weight spaces $V_\lambda(\mu)$. At each of these points the Kazhdan Lusztig polynomial $P_{\mu,\lambda}(t)$ gives the dimension of this weight space.

For SL_3 the moment map image gives the triangular lattice which can be interpreted as the weight lattice for PGL_3. The moment map image of the first stratum is a hexagon and in fact the first stratum has the structure of a vector bundle over the flag manifold $F_{1,2}(\mathbb{C}^3)$. The closure of this stratum consists of adding a single point, so it is the Thom space of this bundle. The local intersection cohomology at the vertex (which appears at the origin in the diagram) is the primitive cohomology of the flag manifold (that is, $1 + t$).

Figure 28 represents the moment map image of the affine Grassmannian for SL_3; it also equals the weight diagram for $\mathrm{PGL}_3(\mathbb{C})$. The dotted red lines are reflecting hyperplanes for the Weyl group. The red dot is a highest weight λ for PGL_e; the other dots are their Weyl images. The

blue hexagon is the outline of the moment map images; it is also the collection of weights in the irreducible representation of highest weight λ. The 1 and $1+t$ beneth the dots are the Kazhdan Lusztig polynomials.

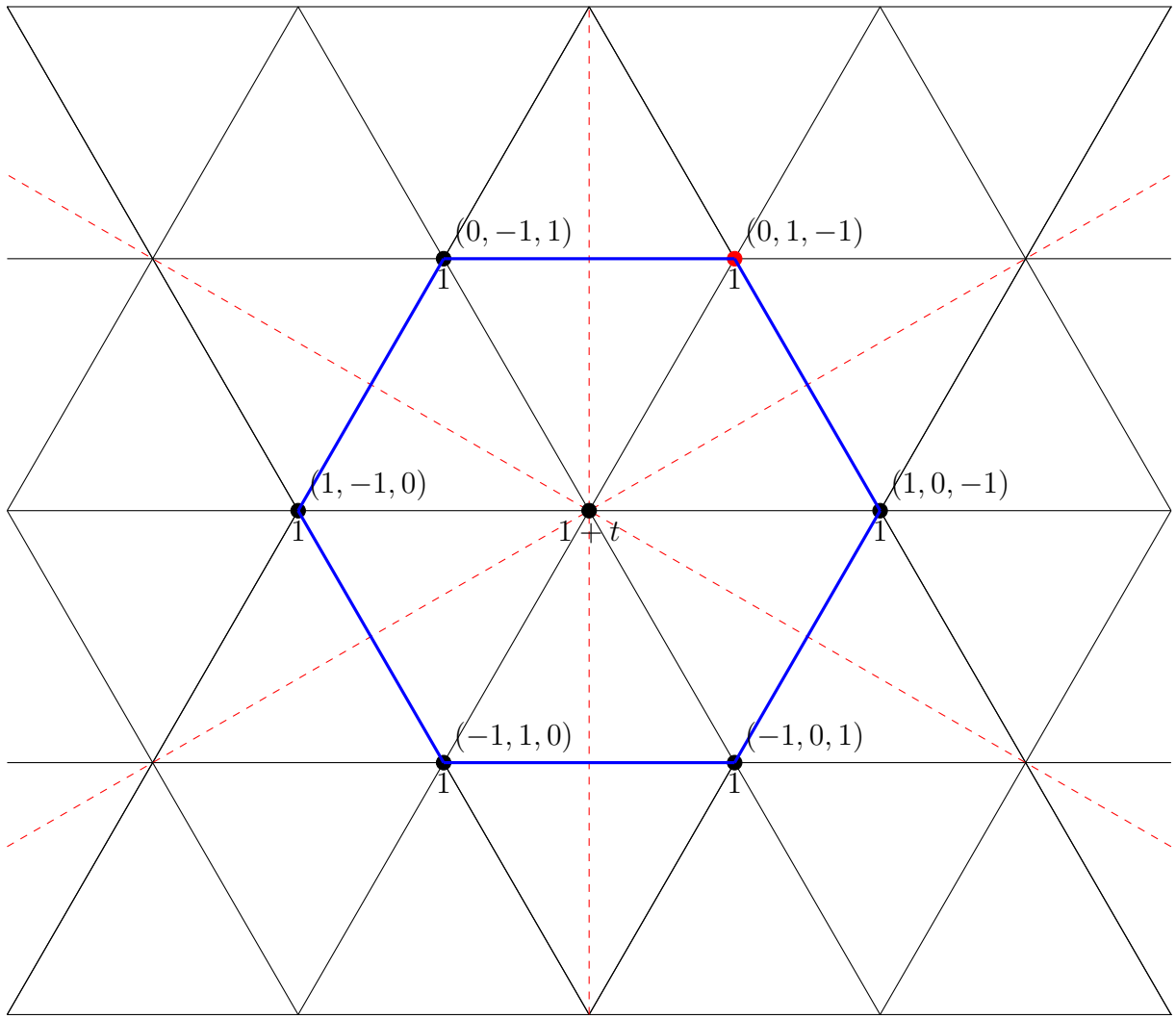


FIGURE 28. Affine moment map

28.6. Perverse sheaves on X. Throughout this section for simplicity let $G = \mathrm{SL}_n(\mathbb{C}((T)))$, $K = \mathrm{SL}_n(\mathbb{C}[[T]])$ and $X = G/K$. We would like to imitate the construction with the flag manifold, and create a convolution product for sheaves on $X \times X$ that are constructible with respect to the

stratification by G orbits, that is, if $p_{ij} : X \times X \times X \to X \times X$ as before, set

$$A^\bullet \circ B^\bullet = Rp_{13*}(p_{12}^*(A^\bullet) \otimes p_{23}^*(B^\bullet)).$$

Unfortunately the orbits of G on $X \times X$ have infinite dimension and infinite codimension so this simply does not make sense. V. Ginzburg and (later) K. Vilonen and I. Mirkovič found a way around this problem.

Let $\mathcal{P}(X)$ denote the category of perverse sheaves, constructible with respect to the above orbit stratification of X, cf. §20.11. (Ginzburg shows this is equivalent to the category of K-equivariant perverse sheaves on X.) Since each stratum is simply connected the local systems associated to these sheaves are trivial. It turns out that the intersection cohomology sheaves live only in even degrees and this implies that every perverse sheaf is isomorphic to a direct sum of $\underline{\underline{IC}}^\bullet$ sheaves of stratum closures.

Mirkovič and Vilonen define a tensor product structure on $\mathcal{P}(X)$ as follows. Consider the diagram

$$X \times X \xleftarrow{p} G \times X \xrightarrow{q} G \times_K X \xrightarrow{m} X$$

Here, $k.(g, x) = (gk, k^{-1}x)$ so that $G \times_K X$ is a bundle over X whose fibers are copies of X, and $m(g, x) = gx$. If $A^\bullet, B^\bullet \in P(X)$ it turns out that there exists $C^\bullet$ a perverse sheaf on $G \times_K X$, constructible with respect to the G orbits on this space, such that

$$q^*(C^\bullet) = p^*(\pi_1^*(A^\bullet) \otimes \pi_2^*(B^\bullet)$$

where $\pi_1, \pi_2 : X \times X \to X$ are the two projections. Then set $A^\bullet \circ B^\bullet = Rm_*(C^\bullet)$.

28.7. Theorem. *If $A^\bullet, B^\bullet \in \mathcal{P}(X)$ then so is $A^\bullet \circ B^\bullet$. The functor $h : A^\bullet \mapsto H^*(X; A^\bullet)$ is exact and it induces an equivalence of categories*

$$\mathcal{P}(X) \sim \mathrm{Rep}({}^L G)$$

which takes $A^\bullet \circ B^\bullet$ to the tensor product $h(A^\bullet) \otimes h(B^\bullet)$ of the associated representations. If $\lambda \in \chi_*(\mathcal{T})_+$ then $h(\underline{\underline{IC}}^\bullet(\overline{X}_\lambda)) = V_\lambda$ is the irreducible representation of highest weight λ.

28.8. Although it sounds intimidating, the convolution product of sheaves is exactly parallel to the previous case of the finite (dimensional) flag manifold. Consider the weight diagram for SL_3 and the moment map image of torus fixed points in the affine Grassmannian for PGL_3. The coordinate lattices are indicated on Figure 29, for example, the point $(1, 0, 0)$ corresponds to the lattice $sK =$

T^{-2}			
T^{-1}	•		
T^0	•	•	•
T^1	•	•	•
T^2	•	•	•

where s is a certain simple reflection in the affine Weyl group. So the

orbit of $G(\mathbb{C}[[T]])$ in $X \times X$ corresponding to this element consists of the set of pairs of lattices $L_0 \xrightarrow{s} L$ that are in relative position s, that is,

$$\left\{(L_0, L_1)|\ L_0 \subset L_1 \subset T^{-1}L_0 \text{ and } \ \dim(L_1/L_0) = 1\right\}.$$

If we fix L_0 to be the standard lattice, then we see that the orbit $\mathcal{O}_s \subset X$ is isomorphic to $\mathbb{P}^1$ so its $\underline{\underline{IC}}$ sheaf is the constant sheaf.

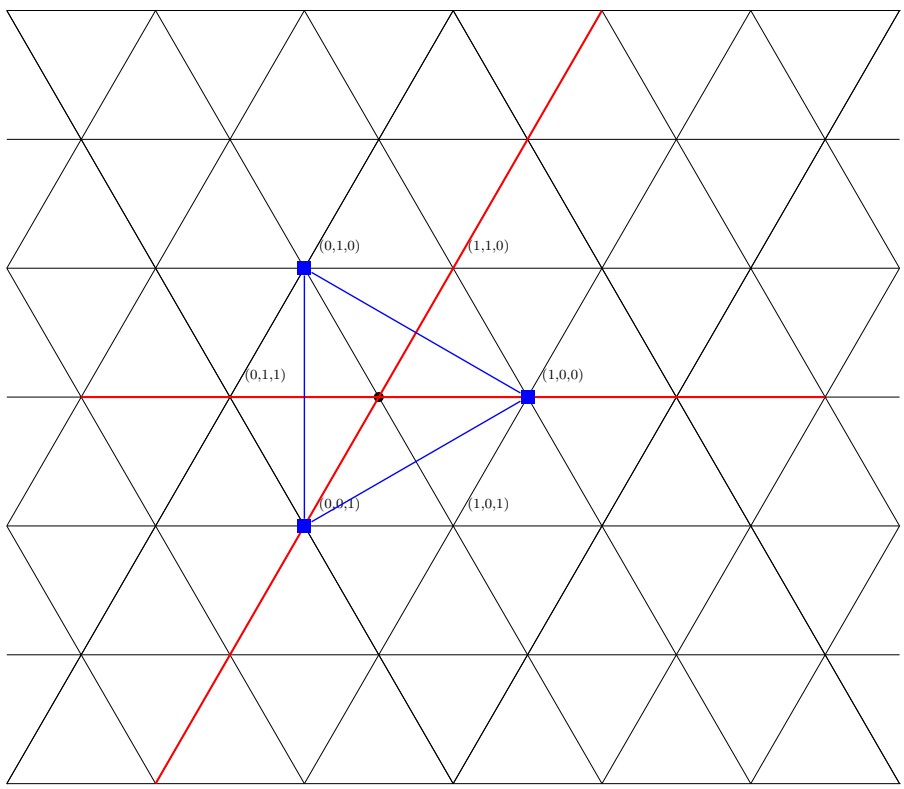


FIGURE 29. SL_3: Reflecting hyperplanes and moment map image of $\mathcal{O}_{(1,1,0)}$

Let us consider the convolution product of this sheaf with itself. Thus the total space of the correspondence consists of triples of lattices

$$C = \left\{(L_0, L_1, L_2)|\ L_0 \xrightarrow{s} L_1 \xrightarrow{s} L_2\right\}$$

in their appropriate relative positions. Apply π_{13} to obtain the correspondence

$$\pi_{23}(C) = \left\{(L_0, L_2)|\ L_0 \subset L_2 \subset T^{-2}L_0 \ \text{ and } \ \dim(L_2/L_0) = 2\right\}.$$

Again, taking L_0 to be the standard lattice, we need to understand the decomposition into orbits of $R\pi_{23*}(\underline{\underline{\mathbb{Q}}}_C)$. There are two types of such lattices: the first type considered in §27.7 consists

of those lattices in the orbit xK that was, that is, lattices like this:

T^{-3}			
T^{-2}	$\bullet$		
T^{-1}	$\bullet$	$\star$	$\star$
T^{0}	$\bullet$	$\bullet$	$\bullet$
T^{1}	$\bullet$	$\bullet$	$\bullet$
T^{2}	$\bullet$	$\bullet$	$\bullet$

which project under the moment map to an image that contains the W-translates of $(2,0,0)$. The second type is lattices like this:

T^{-3}			
T^{-2}			
T^{-1}	$\bullet$	$\bullet$	
T^{0}	$\bullet$	$\bullet$	$\bullet$
T^{1}	$\bullet$	$\bullet$	$\bullet$
T^{2}	$\bullet$	$\bullet$	$\bullet$

which project under the moment map to an image containing $(1,1,0)$. That is,

$$m : C \to \mathcal{O}_{(2,0,0)} \cup \mathcal{O}_{(1,1,0)}.$$

The map m is guaranteed to be semi-small and $Rm_*(\underline{\underline{\mathbb{Q}}}_C)$ breaks into a direct sum of copies of $\underline{\underline{IC}}$ sheaves of these two strata the multiplicities equal to the number of components of the fiber. One checks that the multiplicity equals one.

By taking the cohomology of these sheaes we obtain highest weight representations: $V_{(1,0,0)} = \mathrm{std}$, $V_{(1,1,0)} = \mathrm{std}^\vee$, and $V_{(2,0,0)} = \wedge^2(\mathrm{std})$ (where std is the standard representation) so that

$$\mathrm{std} \otimes \mathrm{std} \cong \mathrm{std}^\vee \oplus \wedge^2(\mathrm{std}).$$

28.9. Remarks. Theorem 28.7 should be regarded as a categorification of Satake's isomorphism. In fact, taking the K group of the Grothendieck group on both sides gives

$$K(\mathcal{P}(X)) \cong \mathcal{H}(G,K) \cong \chi_*(\mathcal{T})^W$$

which is the classical Satake isomorphism.

If we could duplicate the construction in the finite dimensional case we would consider G orbits on $X \times X$. Ginzburg, Mirkovič and Vilonen replace $X \times X$ with $G \times_K X$ which is a fiber bundle over X with fiber isomorphic to X. They replace the G orbits with the strata $S_{\lambda,\mu}$ which is a fiber bundle over X_μ with fiber isomorphic to X_λ.

It is totally nonobvious that the convolution of perverse sheaves is perverse. This depends on the fact that the mapping $m : G \times_K X \to X$ is semi-small in a very strong sense. For each $\lambda, \mu \in \chi_*(\mathcal{T})_+$ let $S_{\lambda,\mu} = p^{-1}(X_\lambda) \times_K X_\mu$. These form a stratification of $G \times_K X$. It turns out that the restriction $m : \overline{S}_{\lambda,\mu} \to X$ is semi-small (onto its image, which is a union of strata X_τ). This implies that $Rm_*(\mathcal{E}[d_{\lambda,\mu}])$ is perverse, for any locally constant sheaf $\mathcal{E}$ on $S_{\lambda,\mu}$ (where $d_{\lambda,\mu} = \dim(S_{\lambda,\mu})$.

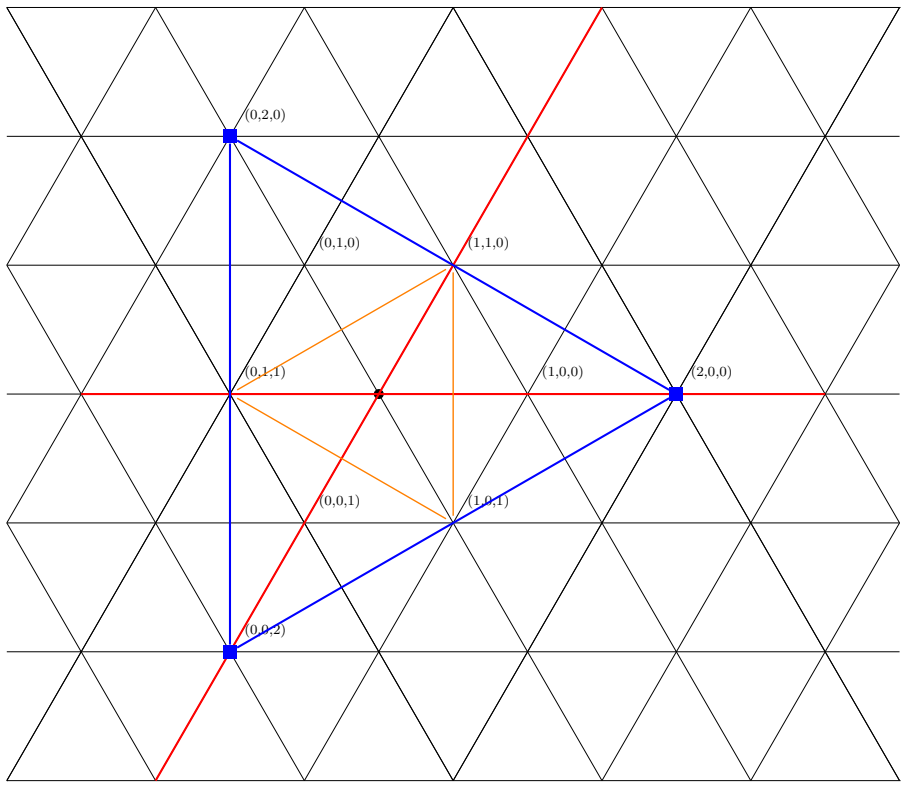


FIGURE 30. $\mathcal{O}_{(1,0,0)} \circ \mathcal{O}_{(1,0,0)} = \mathcal{O}_{(2,0,0)} + \mathcal{O}_{(1,1,0)}$

28.10. Tannakian category. (see [31]) There is a general theorem that a reductive algebraic group (say, over an algebraically closed field) can be recovered from its cartegory of representations. More generally if $\mathcal{C}$ is a tensor category, that is, an abelian category together with a "tensor" structure ($A, B \in \mathcal{C} \implies A \circ B \in \mathcal{C}$) that is commutative and associative in a functorial way (satisfies the *associativity constraint* and the *commutativity constraint*, and if $h : \mathcal{C} \to \{VS\}$ is a rigid *fiber functor* (that is, an exact functor to vector spaces such that $h(A \circ B) = h(A) \otimes h(B)$) then the group of automorphisms of h (that is, the group of natural transformations $h \to h$) is an algebraic group G whose category of representations is equivalent to the original category $\mathcal{C}$.

It turns out that $\mathcal{P}(X)$ is such a Tannakian category and that h is a rigid fiber functor. Therefore the group of automorphisms of h is isomorphic to the Langlands dual group ${}^L G$.

29. Cellular perverse sheaves

29.1. The theory of cellular perverse sheaves was developed by R. MacPherson and described in lectures [90, 91] and was explained in detail in the thesis [124, 123] of M. Vybornov. Cellular perverse sheaves interpolate between cellular sheaves and cellular cosheaves in a remarkable way. The theory was extended and simplified in [101, 122, 125].

Suppose K is a finite simplicial complex[12]. We will be sloppy and identify K with its geometric realization $|K|$. Each simplex σ is a closed subset of K and we denote its interior by σ^o which we refer to as a *cell*. If $\sigma \subseteq \tau$ we write $\sigma \leq \tau$ and similarly for proper inclusions. We also write $\sigma \leftrightarrow \tau$ if *either* $\sigma \leq \tau$ or $\tau \leq \sigma$. The barycenter of a simplex σ is denoted $\hat{\sigma}$. An r-simplex θ in the barycentric subdivision K' is the span $\theta = \langle \hat{\sigma}_0, \hat{\sigma}_1, \cdots, \hat{\sigma}_r \rangle$ where $\sigma_0 < \sigma_1 < \cdots < \sigma_r$. If S is a collection of vertices of K then its *span* consists of those simplices σ such that every vertex of σ is in S.

29.2. Simplicial derived category. (See also §2.1, §2.5.) We summarize some results in the thesis of Allen Shepard [110] that were developed in a seminar at Brown University in 1977-78. We consider sheaves of $\mathbb{Q}$-vector spaces but the coefficients may be taken in any field.

Let $D^b_c(K)$ denote the bounded derived category of sheaves on K that are cohomologically constructible with respect to the stratification by cells (i.e. interiors of simplices).

Recall §2.1 that a simplicial sheaf A assigns to each simplex σ a finite dimensional $\mathbb{Q}$-vector space A_σ and to each face $\sigma < \tau$ a *restriction* homomorphism $s_{\sigma\tau} : A_\sigma \to A_\tau$ in a way that is compatible with compositions. Morphisms of simplicial sheaves are required to be compatible with these restriction mappings. The category of simplicial sheaves is equivalent to the subcategory of $D^b_c(K)$ whose objects have stalk cohomology only in degree 0 (that is, the category of $\bar{0}$-perverse sheaves).

If A, B are simplicial sheaves, the sheaf $\underline{\underline{\mathrm{Hom}}}(A, B)$ is the simplicial sheaf

$$\underline{\underline{\mathrm{Hom}}}(A, B)(\sigma) = \mathrm{Hom}(A|St(\sigma), B|St(\sigma))$$

where $St(\sigma)$ denotes the open star of σ.

So the group of sections $\underline{\underline{\mathrm{Hom}}}(A, B)(\sigma)$ consists of homomorphisms $A(\sigma) \to B(\sigma)$ that are compatible with all corestriction maps emanating from σ.

Let C be a cell (or stratum) of K. Let $i : C \to K$ be the inclusion. The closure of its image is a simplex σ and $Ri_*(\underline{\underline{\mathbb{Q}}}_C) = i_*(\underline{\underline{\mathbb{Q}}}_C) = \underline{\underline{\mathbb{Q}}}_\sigma$ is the constant sheaf on the (closed) simplex σ. To streamline notation we denote this by $\underline{\underline{\mathbb{Q}}}_{\sigma *}$ and set $\underline{\underline{\mathbb{Q}}}_{\sigma !} = i_!(\underline{\underline{\mathbb{Q}}}_C)$, the extension by zero. A key exercise is to verify the following:

In the category Sh^{Δ}_K of simplicial sheaves each $\underline{\underline{\mathbb{Q}}}_{\sigma *}$ is injective and every injective sheaf is a direct sum of such *elementary* injectives. $\square$

[12]Statements in this section hold without modification for any finite strongly regular cell complex, meaning that the closure of each cell is homeomorphic to a closed ball, and the intersection of two (closed) cells is a cell, or else empty. Beware that the Schubert cell decomposition of the flag and Grassmann manifold is not regular.

Every simplicial sheaf of $\mathbb{Q}$ vector spaces has a canonical injective imbedding,

$$A \to I^0(A) = \bigoplus_\sigma (A_\sigma)_{\sigma *} = \bigoplus_\sigma A_\sigma \otimes \underline{\underline{\mathbb{Q}}}_{\sigma *}$$

which, by repeated application to the cokernel gives rise to a canonical injective resolution

$$A \to I^0(A) \to I^1(A) \cdots .$$

where $I^k(A)$ is the sum over flags with codimension one steps (cf. [110] §1.4),

$$I^k(A) = \bigoplus_{\tau_k < \tau_{k-1} < \cdots < \tau_0} A_{\tau_k} \otimes \underline{\underline{\mathbb{Q}}}_{\tau_k *}.$$

So its global sections $\Gamma(K, I^k(A))$ may be identified with the simplicial k-cochains $C^k(K'; A)$ of the first barycentric subdivision K' of K.

Consequently any (bounded below) complex $A^\bullet$ of simplicial sheaves has a canonical injective resolution $A^\bullet \xrightarrow{\sim} T^\bullet$ where $T^\bullet$ is the single complex associated to the double complex $I^p(A^q)$. This gives canonical models for derived functors such as the cohomology (derived functor of global sections),

$$H^i(K, A^\bullet) = R^i\Gamma(K, A^\bullet) = H^i(\Gamma(K, T^\bullet))$$

and

$$\mathrm{R}\underline{\underline{\mathrm{Hom}}}^\bullet(B^\bullet, A^\bullet) = \underline{\underline{\mathrm{Hom}}}^\bullet(B^\bullet, T^\bullet)$$

(Recall that $\underline{\underline{\mathrm{Hom}}}^\bullet(B^\bullet, T^\bullet)$ is defined as the single complex associated to the double complex $\underline{\underline{\mathrm{Hom}}}(B^i, T^j)$.) The main properties of these derived functors can then be proven directly using these canonical models. In [110], [74]§8.1.11 is proven:

> The natural functor (§2.1) $D^b(Sh_K^\Delta) \to D_c^b(K)$ from the bounded derived category of simplicial sheaves on K to the bounded derived category of sheaves that are (cohomologically) constructible with respect to the cell decomposition of K, is an equivalence of categories.

The dualizing sheaf $\mathbb{D}^\bullet$ has a canonical injective model in the category $D^b(Sh_K^\Delta)$. If $j : \sigma < \tau$ is a codimension one face, denote the composition $\underline{\underline{\mathbb{Q}}}_{\tau *} \to j_* j^* \underline{\underline{\mathbb{Q}}}_{\tau *} = \underline{\underline{\mathbb{Q}}}_{\sigma *}$ by $\partial_{\tau,\sigma}$. Choose an ordering of each simplex. Let $[\tau : \sigma] = \pm 1$ denote the incidence number which is $+1$ if the orientation of σ followed by an inward pointing vector agrees with the orientation of τ. Set $[\tau : \sigma] = 0$ if σ is not a codimension one face of τ. Then:

$$\mathbb{D}^{-j} = \bigoplus_{\dim(\sigma) = j} \underline{\underline{\mathbb{Q}}}_{\sigma *}$$

with $d : \mathbb{D}^{-j} \to \mathbb{D}^{-j+1}$ given by

$$\bigoplus_{\dim \tau = j} \bigoplus_{\dim \sigma = j-1} [\tau : \sigma] \partial_{\tau,\sigma}$$

It is independent, up to unique quasi-isomorphism, of the choice of orientation. ([110])

For any $A^\bullet$ in $D^b(Sh_K^\Delta)$ this gives a canonical construction of the dual $\mathrm{D}A^\bullet = \mathrm{Hom}^\bullet(A^\bullet, \mathbb{D}^\bullet)$, and the natural isomorphism $\mathrm{DD}A^\bullet \cong A^\bullet$ may be proven directly ([110]). Combinatorially, $\mathrm{DD}A^\bullet$ corresponds to the sheaf induced by $A^\bullet$ on the second barycentric subdivision of K.

If $f : A^\bullet \to B^\bullet$ is a morphism of injective simplicial sheaves then $\ker(f)$ is not necessarily injective, and so the truncation functor $\tau_{\leq a}$ does not preserve injectives.

29.3. Local calculations. Let $x \in \sigma$. It lies in the interior τ^o of some simplex $\tau \leq \sigma$. Let $j_x : \{x\} \to \sigma$. Then one checks that

- $H^0(j_x^*(\underline{\underline{\mathbb{Q}}}_{\sigma *})) = \mathbb{Q}$ and $H^n(j_x^*(\underline{\underline{\mathbb{Q}}}_{\sigma *})) = 0$ for $n \neq 0$
- $H^n(j_x^!(\underline{\underline{\mathbb{Q}}}_{\sigma *})) = 0$ for all n
- $H^n(j_x^*(\underline{\underline{\mathbb{Q}}}_{\sigma !})) = 0$ for all n
- $H^{\dim(\sigma)}(j_x^!(\underline{\underline{\mathbb{Q}}}_{\sigma !})) = \mathbb{Q}$ and $H^n(j_x^!(\underline{\underline{\mathbb{Q}}}_{\sigma !})) = 0$ for $n \neq \dim(\sigma)$

If $A^\bullet, B^\bullet \in D^b(Sh_K^\Delta)$ are (constructible) complexes of sheaves, and if $B^\bullet \to I^\bullet$ is an injective resolution of $B^\bullet$, recall that

$$\mathrm{Ext}^0(A^\bullet, B^\bullet) = \mathrm{Hom}_{D^b(Sh_K)}(A^\bullet, B^\bullet) = H^0(K; R\underline{\underline{\mathrm{Hom}}}^\bullet(A^\bullet, B^\bullet)) = H^0(K; \underline{\underline{\mathrm{Hom}}}^\bullet(A^\bullet, I^\bullet))$$

so $\mathrm{Ext}^i(A^\bullet, B^\bullet) = \mathrm{Ext}^0(A^\bullet, B^\bullet[i])$ is the i-th cohomology group of the single complex associated to the double complex $\mathrm{Hom}_{D^b(Sh_K)}(A^\bullet, I^\bullet)$. It is an exercise to calculate these Ext groups for simplices σ, τ, and they turn out to live in a single degree. Set $d(\sigma) = \dim(\sigma)$. Using the canonical injective resolution of $\underline{\underline{\mathbb{Q}}}_{\tau !}$ and noting that $\underline{\underline{\mathbb{Q}}}_{\tau *}$ is injective, we find:

$$\mathrm{Ext}^i(\underline{\underline{\mathbb{Q}}}_{\sigma *}, \underline{\underline{\mathbb{Q}}}_{\tau *}) = \begin{cases} \mathbb{Q} & \text{if } \ i = 0 \ \text{ and } \sigma \geq \tau \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{Ext}^i(\underline{\underline{\mathbb{Q}}}_{\sigma !}, \underline{\underline{\mathbb{Q}}}_{\tau !}) = \begin{cases} \mathbb{Q} & \text{if } \ i = d(\tau) - d(\sigma) \text{ and } \ \sigma \leq \tau \\ 0 & \text{otherwise} \end{cases}$$

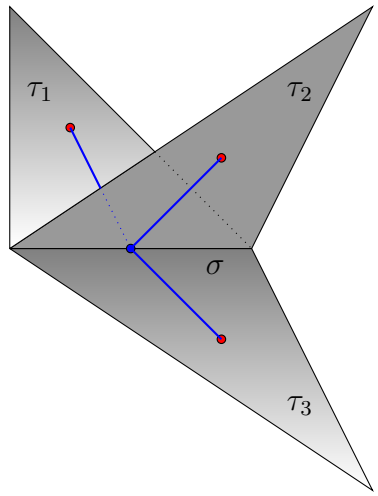
$$\mathrm{Ext}^i(\underline{\underline{\mathbb{Q}}}_{\sigma !}, \underline{\underline{\mathbb{Q}}}_{\tau *}) = \begin{cases} \mathbb{Q} & \text{if } \ i = 0 \text{ and } \ \sigma = \tau \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{Ext}^i(\underline{\underline{\mathbb{Q}}}_{\sigma *}, \underline{\underline{\mathbb{Q}}}_{\tau !}) = \begin{cases} \mathbb{Q} & \text{if } i = d(\tau) - d(\sigma \cap \tau) \text{ and } \sigma \cap \tau \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

29.4. Simplicial cosheaves. There are two ways to view a simplicial cosheaf R on a simplicial complex K. The first, as a pre-cosheaf that assigns to each simplex σ a vector space R_σ and to each incidence $\sigma < \tau$ a homomorphism $A_\tau \to A_\sigma$,in a way that is compatible with compositions. The second way involves the *dual cells* of K.

Let K' denote the first barycentric subdivision of K. For each simplex σ in K the dual cell $D(\sigma)$ is the union of those simplices x in K' spanned by $\hat{\sigma}$ and the barycenters $\hat{\tau}$ of simplices $\tau > \sigma$.

(Although it is contractible to $\hat{\sigma}$ the dual cell $D(\sigma)$ is not necessarily homeomorphic to a disk.) If $\sigma < \tau$ then $D(\tau) \subset D(\sigma)$ and we write $D(\tau) < D(\sigma)$.



Dual of σ in blue

A *sheaf* B on the dual cell decomposition assigns a vector space B_ξ to each dual cell $\xi = D(\sigma)$ and a homomorphism $B_\xi \to B_\eta$ whenever $\xi < \eta$ in a way that is compatible with compositions. So a sheaf on the dual cell decomposition is the same thing as a cosheaf on the cell decomposition.

29.5. Perversities. We follow the notation of [12] by modfying the definition of a perversity p to allow for codimension one strata and considering p to be a function of the *dimension* of the stratum. This results in a shift of cohomological degree that depends on p. We refer to the notation of §20.12.

Although it is possible to develop the theory of simplicial perverse sheaves for perversities that vary with the simplex, in these notes we assume for simplicity that the perversity $p(k) \leq 0$ is a function only of the dimension k of the simplex[13] with $p(0) = 0$ and $p(k-1) \geq p(k) \geq p(k-1)$ for all k.

Let C be a k-dimensional cell of K, with closure σ. Following [101] we say that C (or its closure σ or its dimension k) is of type ! if $p(k) = p(k-1)$ and is of type * if $p(k) = p(k-1) - 1$. The number $k = 0$ is considered to be both types. The simple objects in the category of perverse sheaves are the complexes $\underline{\underline{IC}}^\bullet_\sigma$. From §29.3, and observed in [101], for any simplex σ:

[13]Suppose K is a triangulation of a stratified space X and $A^\bullet \in D^b(X)$ is constructible with respect to the stratification $\mathcal{S}$. Then it is also constructible with respect to the triangulation K and (by [50] §4.1, or [12] §2.1.14) it is p-perverse with respect to the stratification S if and only if it is p-perverse with respect to the triangulation.

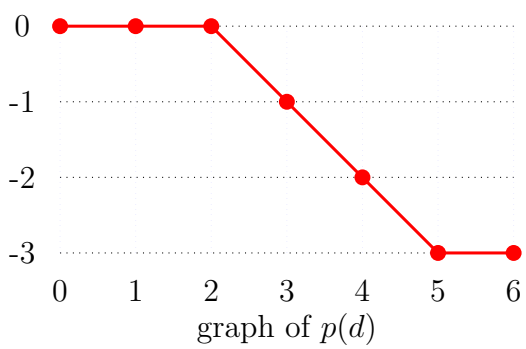
The intersection complex extending the constant sheaf $\underline{\underline{\mathbb{Q}}}_{\sigma^o}$ on the interior σ^o is:

$$\underline{\underline{IC}}_\sigma = \begin{cases} \underline{\underline{\mathbb{Q}}}_{\sigma *}[-p(\sigma)] & \text{if } \sigma \text{ is type } * \\ \underline{\underline{\mathbb{Q}}}_{\sigma !}[-p(\sigma)] & \text{if } \sigma \text{ is type } ! \end{cases} \tag{29.5.1}$$

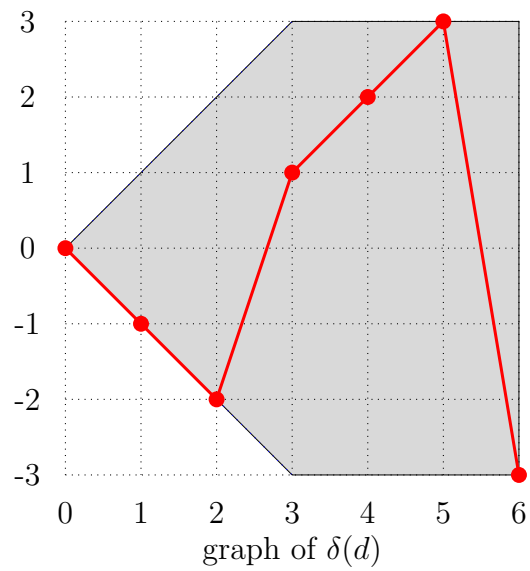
29.6. Perverse dimension. MacPherson's insight ([90, 91]) was to define the *perverse dimension* of a d-simplex σ (or of $C = \sigma^o$) to be

$$\delta(\sigma) = \begin{cases} -p(d) \geq 0 & \text{if } \sigma \text{ has type } * \\ -p(d) - d \leq 0 & \text{if } \sigma \text{ has type } ! \end{cases}$$

Therefore $|\delta(\sigma)|$ is the number of integers k with $1 \leq k \leq d$ of the same type as d. Then $\delta : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ is one to one, and for each d the image $\delta([0, d])$ is a subinterval of integers in $[-d, d]$ containing 0.



graph of $p(d)$

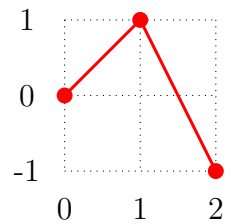


graph of $\delta(d)$

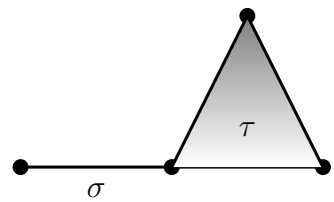
29.7. Partial order. There is a unique partial order on the set of simplices of K that is generated by the following *elementary relation*: $\sigma \succeq \tau$ if $\sigma \leftrightarrow \tau$ and $\delta(\sigma) = \delta(\tau) + 1$, in other words, two simplices σ, τ satisfy $\sigma \succeq \tau$ if there is a sequence of elementary relations $\sigma = \sigma_0 \succeq \sigma_1 \cdots \succeq \sigma_r = \tau$. It is easy to check the following.

Suppose $\sigma \succeq \tau$. If both σ, τ are type * then $\tau \subset \sigma$. If both are type ! then $\sigma \subset \tau$. If they are opposite types then σ is type * (so $\delta(\sigma) > 0$) and τ has type ! (so $\delta(\tau) < 0$) and $\sigma \cap \tau \neq \phi$.

Hence an arrow $\sigma \succeq \tau$ can extend no farther than adjacent simplices. This may happen if $\delta(\sigma)$ and $\delta(\tau)$ have opposite signs, for example, $\dim(\sigma) = 1$, $\dim(\tau) = 2$ and $\delta(0,1,2) = 0, 1, -1$ respectively.



graph of $\delta(d)$



$\sigma \succeq \tau$

Using §29.3 we find: if σ, τ are simplices then

$$\mathrm{Ext}^i(\underline{\underline{IC}}^\bullet_\sigma, \underline{\underline{IC}}^\bullet_\tau) = \begin{cases} \mathbb{Q} & \text{if } \sigma \succeq \tau \text{ and } i = \delta(\sigma) - \delta(\tau) \\ 0 & \text{otherwise.} \end{cases} \tag{29.7.1}$$

29.8. Perverse cells. For each simplex σ the corresponding *perverse cell* ${}^\delta\sigma$ is the following union of simplices in the barycentric subdivision K':

$${}^\delta\sigma = \bigcup \{\langle \hat\tau_0 \hat\tau_1 \cdots \hat\tau_r \rangle |\ \forall j,\ \delta(\tau_j) \le \delta(\sigma) \text{ and } \exists i \text{ with } \tau_i = \sigma\}$$

In this equation $\tau_0 < \tau_1 < \cdots < \tau_r$ form a partial flag of simplices hence $\delta(\tau_j)$ takes distinct values. The "boundary" $\partial({}^\delta\sigma)$ (also called the perverse link of σ) is the union of those simplices $\langle \hat\tau_0 \hat\tau_1 \cdots \hat\tau_r \rangle$ in ${}^\delta\sigma$ such that $\delta(\tau_j) < \delta(\sigma)$ for all j $(0 \le j \le r)$. (That is, remove the vertex $\hat\sigma$ from each simplex.) If σ is type ! then ${}^\delta\sigma \subset D(\sigma)$. If σ is type * then $\sigma \subset {}^\delta\sigma$ and $\partial\sigma \subset \partial^\delta\sigma$. The *interior* is ${}^\delta\sigma^o = {}^\delta\sigma - \partial^\delta\sigma$. If $C = \tau^o$ is the interior of τ we sometimes denote ${}^\delta\sigma^o$ by ${}^\delta C$. The key technical point in the theory is the following, which implies the functor $H^r_{{}^\delta C}$ from the category of perverse sheaves to vector spaces is exact.

29.9. Proposition. [90, 91][101] *Let A be a perverse sheaf on the simplicial complex K and let σ be a simplex of K with $C = {}^\delta\sigma^o$. The cohomology with support in ${}^\delta C$ vanishes:*

$$H^r_{{}^\delta C}(K, A) = 0 \quad \textit{unless} \quad r = -\delta(\sigma) \tag{29.9.1}$$

Proof. It suffices to prove equation (29.9.1) when $A = \underline{\underline{IC}}_\tau$ since these are the simple objects in the Noetherian category of perverse sheaves. If $\sigma \leftrightarrow \tau$ and $i : {}^\delta\sigma^o \cap \tau \to K$, we will show that

$$H^r_{\{{}^\delta\sigma^o\}}(\underline{\underline{IC}}_\tau) = H^r_{\{{}^\delta\sigma^o \cap \tau\}}(\underline{\underline{IC}}_\tau) = H^r(i^!\underline{\underline{IC}}_\tau) = 0 \quad \text{unless} \quad \sigma = \tau \text{ and } r = -\delta(\sigma) \tag{29.9.2}$$

in which case, $H^{-\delta(\sigma)}_{{}^\delta\sigma^o}(\underline{\underline{IC}}_\sigma) = \mathbb{Q}$.

If $\sigma = \tau$ is type * then ${}^\delta\sigma \cap \tau = \sigma$ and $\underline{\underline{IC}}_\sigma = \underline{\underline{\mathbb{Q}}}_\sigma[\delta(\sigma)]$ so $H^*_{\{\sigma\}}(\underline{\underline{IC}}_\sigma) = \mathbb{Q}[-p(\sigma)] = \mathbb{Q}[\delta(\sigma)]$.

If $\sigma = \tau$ is type ! then ${}^\delta\sigma \cap \tau = \hat{\sigma}$ and by §29.3,

$$H^r_{\{\hat{\sigma}\}}(\underline{\underline{IC}}_\sigma) = H^r_{\{\hat{\sigma}\}}(\mathbb{Q}_{\sigma!}[-p(\sigma)]) = \begin{cases} \mathbb{Q} & \text{if } r = \dim(\sigma) + p(\sigma) \\ 0 & \text{otherwise} \end{cases}$$

Next, suppose $\sigma \neq \tau$ and τ is type *. If $\sigma < \tau$ then $\delta(\sigma) < \delta(\tau)$ so ${}^\delta\sigma^o \subset {}^\delta\sigma \subset \partial\tau$ since $\hat{\tau}$. Let $j : \tau^o \to K$ denote the inlusion. Then $H^*(i^! Rj_* \underline{\underline{\mathbb{Q}}}_{\tau^o})$ is dual to $H^*(i^* Rj_! \underline{\underline{\mathbb{Q}}}_{\tau^o}) = 0$. If $\sigma > \tau$ then ${}^\delta\sigma^o \cap \tau = \phi$ so $H^*(i^! \underline{\underline{IC}}_\tau) = 0$.

Now suppose that τ is type !. Then $H^*(i^! \underline{\underline{IC}}_\tau) = H^*(i^! \underline{\underline{\mathbb{Q}}}_{\tau!})$ is dual to

$$H^*_c(i^* \underline{\underline{\mathbb{Q}}}_{\tau *}) = H^*_c({}^\delta\sigma^o \cap \tau) = H^*({}^\delta\sigma \cap \tau, \partial^\delta\sigma \cap \tau).$$

If σ is type *, or if σ is type ! and $\sigma < \tau$ then $\delta(\sigma) > \delta(\tau)$ so every maximal simplex in ${}^\delta\sigma \cap \tau$ and every maximal simplex in $\partial^\delta\sigma \cap \tau$ contains the vertex $\hat{\tau}$ hence both are contractible and the relative cohomology is trivial. In the remaining case, τ is type !, σ is type ! and $\sigma > \tau$ so ${}^\delta\sigma \cap \tau = \phi$. □

29.10. Perverse skeleton. MacPherson defines the r-th perverse skeleton:

$${}^\delta K_r = \bigcup \left\{ {}^\delta\sigma \mid \delta(\sigma) \leq r \right\}.$$

It is a closed subcomplex[14] of the barycentric subdivision of K and it is the span of the barycenters $\hat{\sigma}$ of simplices with $\delta(\sigma) \leq r$. Its interior is the disjoint union,

$${}^\delta K^o_r = {}^\delta K_r - {}^\delta K_{r-1} = \coprod_{\delta(\sigma)=r} {}^\delta\sigma^o.$$

Let $U_r = K - {}^\delta K_{r-1}$. This gives a filtration of K by open sets,

$$\cdots \subset U_{r+1} \subset U_r \subset U_{r-1} \subset \cdots \tag{29.10.1}$$

and ${}^\delta K^o_r = U_r - U_{r+1}$ is closed in U_r with inclusions

$${}^\delta K^o_r \xrightarrow[i_r]{} U_r \xleftarrow[j_r]{} U_{r+1}$$

The long exact sequence on cohomology (§7.4) for $A|U_r$ together with Proposition 29.9 says that the relative cohomology lives in a single degree:

For any perverse sheaf A we have:

$$H^t(U_r, U_{r+1}; A) = H^t(i_r^!(A|U_r)) = \begin{cases} \bigoplus_{\delta(\sigma)=r} H^r_{\delta_{\sigma^o}}(K; A) & \text{if } t = -r \\ 0 & \text{otherwise.} \end{cases} \tag{29.10.2}$$

[14]after appropriate shifts and translation of perversity conventions it coincides with the "basic set" $Q^{\bar{p}}_r$ of [49].

For each simplex σ let $A_\sigma = H^{-r}_{\delta_{\sigma^o}}(K;A)$ where $r=\delta(\sigma)$. From the exact sequences for the triple (U_{r-1},U_r,U_{r+1}) we obtain a chain complex (meaning $d\circ d=0$) whose cohomology is $H^*(K;A)$:

$$\begin{array}{ccccccc} \xrightarrow{d} & H^{-r-1}(U_{r+1},U_{r+2};A) & \xrightarrow{d} & H^{-r}(U_r,U_{r+1};A) & \xrightarrow{d} & H^{-r+1}(U_{r-1},U_r;A) & \xrightarrow{d} \\ & || & & || & & || & \\ & \bigoplus_{\delta(\sigma)=r+1} A_\sigma & \xrightarrow{d} & \bigoplus_{\delta(\sigma)=r} A_\sigma & \xrightarrow{d} & \bigoplus_{\delta(\sigma)=r-1} A_\sigma & \end{array} \tag{29.10.3}$$

Writing $d=\oplus_{\sigma,\tau}s_{\sigma\tau}$ leads to the following "combinatorial" definition.

29.11. Definition. A cellular perverse sheaf S on the simplicial complex K is a rule that assigns to each simplex σ a $\mathbb{Q}$-vector space S_σ and "attaching" homomorphism $s_{\sigma\tau}:S_\sigma\to S_\tau$ whenever $\sigma\leftrightarrow\tau$ and $\delta(\sigma)=\delta(\tau)+1$ such that $d\circ d=0$ in the resulting sequence

$$\xrightarrow{d} \bigoplus_{\delta(\sigma)=r+1} S_\sigma \xrightarrow{d} \bigoplus_{\delta(\sigma)=r} S_\sigma \xrightarrow{d} \bigoplus_{\delta(\sigma)=r-1} S_\sigma \xrightarrow{d} \tag{29.11.1}$$

The cohomology of the cellular perverse sheaf A is the cohomology of this sequence. The chain complex condition is equivalent to the statement that whenever $\delta(\sigma)=r+1$ and $\delta(\tau)=r-1$ then the sum over θ vanishes:

$$\sum_{\substack{\delta(\theta)=r\\ \sigma\leftrightarrow\theta\leftrightarrow\tau}} s_{\theta\tau}\circ s_{\sigma\theta}=0 \tag{29.11.2}$$

A morphism $A\to B$ of cellular perverse sheaves is a homomorphism $A_\sigma\to B_\sigma$ for every simplex σ, that commutes with the boundary maps $s_{\sigma\tau}$. The category of cellular perverse sheaves is denoted $\mathcal{P}^\Delta(K)$. By (29.9.2) the complex $\underline{\underline{IC}}_\sigma$ corresponds to the single non-zero assignment $S_\sigma=\mathbb{Q}$.

29.12. Proposition. [90, 101] *The cohomology functor $T=\bigoplus_\sigma H^{-\delta(\sigma)}_{\delta_{\sigma^o}}$ is exact and it defines an equivalence of categories*

$$\mathcal{P}(K)\xrightarrow{T}\mathcal{P}^\Delta(K)$$

between the category of perverse sheaves that are constructible with respect to the triangulation and the category of cellular perverse sheaves (with the same perversity).

Proof: The proof in [101] uses Koszul duality (see §30.6). A direct proof is tedious but the main point is that by (29.10.2) the spectral sequence for the filtration (29.10.1) collapses after the E_1 page with $E_1^{t,r}=H^t(U_r,U_{r+1};A)$ leaving only the diagonal terms, $E_2^{-r,r}$ on the E_2 page. These are precisely the cohomology groups of the sequence (29.10.3) (in degree $-r$). Therefore the inclusion

$U_r \to K$ induces an isomorphism

$$H^{-r}(\oplus_\sigma S^\bullet_\sigma) \cong H^{-r}(K; A)$$

of the cohomology in degree $-r$ of the above complex (29.11.1) with the cohomology of the perverse sheaf A. The same remark applies to subcomplexes of K. However a perverse sheaf on K is determined by the cohomology of every subcomplex of K. Similarly, a homomorphism between perverse sheaves $A \to B$ is a global section of the sheaf $\underline{\underline{\mathrm{RHom}}}^\bullet(A, B)$ and is determined by its restriction to every subcomplex. □

Remark. Rather than the "chain complex" relations (29.11.1) one might consider objects T which assign to each simplex σ a vector space T_σ and attaching homomorphisms $T_\sigma \to T_\tau$ whenever $\sigma \leftrightarrow \tau$ with $\delta(\sigma) = \delta(\tau) + 1$, that are compatible with compositions. The resulting category turns out to be the *Koszul dual* category, see §30.5.

29.13. Verdier duality. It is easy to check that Verdier duality in the category of sheaves on K constructible with respect to the triangulation exchanges the category of perverse sheaves with perversity p and associated function δ with the category of perverse sheaves with perversity $q = t-p$ and associated function $-\delta$ (where $t(d) = -d$ is the "top" perversity). It reverses the arrows of the quiver Q associated to δ (§30.2).

29.14. Remarks.
1. If $p = 0$ is the zero perversity then every cell is type ! and ${}^\delta\sigma = D(\sigma)$ is the dual of the simplex σ. The perverse link $\partial^\delta\sigma$ is the "usual" link of σ. A perverse sheaf A is a simplicial sheaf in the usual sense, except that signs must be added (which is always possible in this case) to the mappings $s_{\sigma\tau}$ so as to obtain $s_{\theta\tau}s_{\sigma\theta} = s_{\sigma\tau}$. The complex (29.11.1) computes the cellular cohomology of A. cf. §2.1.

If p is the top perversity then ${}^\delta\sigma = \sigma$ and a perverse sheaf A is a simplicial cosheaf (§29.4) in the usual sense, up to the same issue involving signs. The complex (29.11.1) computes its homology.

2. One might attempt to mimic Proposition 29.12 using the filtration by closed subsets,

$$\cdots \subset K_{r-1} \subset K_r \subset K_{r+1} \subset \cdots$$

in order to obtain a chain complex with terms

$$H^t(K_r, K_{r-1}; A) \cong \bigoplus_{\delta(\sigma)=r} H^t({}^\delta\sigma, \partial^\delta\sigma, A).$$

Unfortunately the relative cohomology $H^*({}^\delta\sigma, \partial^\delta\sigma; A)$ may live in several degrees as the perverse sheaf A varies. Let $K = \tau$ be the 3-simplex, let $p(3) = -1$ and $p(d) = 0$ otherwise, so that $\delta(3) = 1$ and $\delta(d) = -d$ for $d = 0, 1, 2$. Let σ be a vertex of τ. Then $H^0({}^\delta\sigma, \partial^\delta\sigma; \underline{\underline{IC}}_\sigma) = \mathbb{Q}$ but $H^1({}^\delta\sigma, \partial^\delta\sigma; \underline{\underline{IC}}_\tau) = \mathbb{Q}$.

3. In the example of §29.7 the sum (29.11.1) contains a single term, so that $s_{\sigma\tau} = 0$. In general, if σ, τ are simplices of K such that $\delta(\sigma)$ and $\delta(\tau)$ have opposite signs, and $\sigma \succeq \tau$ but $\sigma \not\leftrightarrow \tau$ then any composition

$$s(\sigma_r, \tau) \circ s(\sigma_{r-1}, \sigma_r) \circ \cdots \circ s(\sigma_1, \sigma_2) \circ s(\sigma, \sigma_1) ... = 0$$

of attaching homomorphisms from A_σ to A_τ will vanish because such a composition eventually passes through a triple $\theta_1 \succeq \theta_0 \succeq \theta_{-1}$ with $\delta = +1, 0, -1$ respectively and $\theta_1 \not\leftrightarrow \theta_{-1}$. Then θ_0 is a vertex and one of the other two is a 1-simplex which meets the remaining simplex in a single point, $\{\theta_0\}$. The chain complex requirement for this triple has only one term: $s(\theta_0, \theta_{-1})s(\theta_1, \theta_0) = 0$.

4. In the more general situation when K is a regular cell complex the preceding argument fails. In this case the definition of a cellular perverse sheaf requires the additional *locality axiom* ([90]): If $\sigma \succeq \tau$ and $\sigma \not\leftrightarrow \tau$ then any composition of morphisms from A_σ to A_τ vanishes.

29.15. Proposition. [101] *The natural functor $\mathcal{P}(K) \to D^b_c(K)$ induces an equivalence of derived categories $D^b(\mathcal{P}(K)) \xrightarrow{R} D^b_c(K)$.*

The proof in [101] uses Koszul duality (cf. §30.6, §30.8) together with the results of [12] (p. 84-85).

Every complex $A^\bullet$ in $D^b_c(K)$ is quasi-isomorphic to a complex of perverse sheaves. For, if σ is a simplex of K of type * then the elementary injective sheaf $\underline{\underline{\mathbb{Q}}}_{\sigma *}$ is isomorphic to the perverse sheaf $\underline{\underline{IC}}_\sigma$. If it has type ! then there is a triangle

$$\begin{tikzcd} \underline{\underline{IC}}_\sigma = \underline{\underline{\mathbb{Q}}}_{\sigma !} \arrow[rr] & & \underline{\underline{\mathbb{Q}}}_{\sigma *} \arrow[dl] \\ & B^\bullet \arrow[ul, "d"] & \end{tikzcd}$$

The complex $B^\bullet$ is supported on the boundary, $\partial\sigma$, so by induction it is quasi-isomorphic to a complex $\to \cdots \to C^{r-1} \to C^r$ of perverse sheaves. Then $\cdots C^{r-1} \to C^r \xrightarrow{d} \underline{\underline{\mathbb{Q}}}_{\sigma !}$ is a complex of perverse sheaves quasi-isomorphic to the elementary injective sheaf $\underline{\underline{\mathbb{Q}}}_{\sigma *}$ so the functor R above is essentially surjective. The content of [101] is showing that R induces an isomorphism on morphisms, that is, $\mathrm{Ext}^i_{\mathcal{P}(K)}(A, B) \cong \mathrm{Ext}^i(A, B)$. (The Yoneda Ext does not require enough injectives in the category $\mathcal{P}(K)$ for its definition while the $\mathrm{Ext}^i(A, B)$ is defined using injective resolutions in the full derived $D^b_c(K)$.)

30. The path algebra

30.1. Aside from standard definitions the material in this section is due to Vybornov [122, 124, 125, 126] and Polishchuk [101]. Throughout this section we retain the notation of §29: K is a finite simplicial complex, p is a perversity with associated function δ, For simplices σ, τ in K we write $\sigma \leftrightarrow \tau$ if either $\sigma < \tau$ or $\tau < \sigma$. We denote by $\mathcal{C}$ the category of cellular perverse sheave of $\mathbb{Q}$-vector spaces on K.

30.2. Associated to K, δ there is a quiver Q, or directed graph without oriented cycles, whose vertices are the simplices of K and whose arrows correspond to elementary relations $\sigma \succ \tau$ with $\delta(\sigma) = \delta(\tau) + 1$. There is a graded *path algebra* $\mathrm{F} = \bigoplus_{j \geq 0} \mathrm{F}_j$, which is generated by elements of degree ≤ 1. The vector space F_0 has a basis element $[\sigma]$ for each simplex σ in K (the "trivial paths'), and F_1 has a (canonical) basis consisting of the elementary relations, that is, $[s_{\sigma\tau}]$ where $\sigma \leftrightarrow \tau$ and $\delta(\sigma) = \delta(\tau) + 1$, subject to the following relations

(1) $[\sigma]^2 = [\sigma]$
(2) $[\sigma].[\tau] = 0$ unless $\tau = \sigma$.
(3) $[\tau].[s_{\sigma\tau}] = [s_{\sigma\tau}].[\sigma] = [s_{\sigma\tau}]$

These relations already imply that $[s_{\sigma\tau}].[s_{\alpha\beta}] = 0$ unless $\sigma = \beta$.

If $\sigma \succeq \tau$ and $\delta(\sigma) = \delta(\tau) + r$ (with $r \geq 1$) then there is a sequence of elementary relations $\sigma = \sigma_0 \succeq \sigma_1 \succeq \cdots \succeq \sigma_r = \tau$ connecting them. It follows that the vector space F_r has a (canonical) basis whose elements are paths of length r with decreasing δ. Products in F corresponds to concatenation of paths. Consequently a representation of the quiver Q (that is, a vector space A_σ for each simplex and a homomorphism $A_\sigma \to A_\tau$ for each elementary relation $\sigma \succ \tau$) is the same thing as a module over the path algebra F. Given a module M the associated vector spaces are given by $M_\sigma = [\sigma].M$.

Not every representation of Q corresponds to a perverse sheaf: the relations (29.11.2) are required to hold. From these relations we immediately conclude

30.3. Proposition. *The category $\mathcal{P}(K)$ of perverse sheaves, constructible with respect to the triangulation of K is naturally equivalent to the category $Mod - B$ of modules over the ring $B = \mathrm{F}/J$ where $J \subset \mathrm{F}$ is the two-sided homogeneous ideal generated by the vector subspace $E \subset \mathrm{F}_2$ spanned by the elements $u_{\sigma\tau}$ whenever $\delta(\sigma) = r + 1$ and $\delta(\tau) = r - 1$ defined by*

$$u_{\sigma\tau} = \sum_{\substack{\delta(\theta)=r \\ \sigma \leftrightarrow \theta \leftrightarrow \tau}} [s_{\theta\tau}].[s_{\sigma\theta}] \tag{30.3.1}$$

30.4. The Ext **algebra.** The ring B is a quadratic algebra. The Koszul dual of such an algebra is sometimes defined to be the Ext algebra of the simple B-modules. Following [101] we consider the graded algebra

$$A = \bigoplus_{\sigma,\tau} \mathrm{Ext}^*(\underline{\underline{IC}}_\sigma, \underline{\underline{IC}}_\tau).$$

(As in §29.3, Ext^i is computed in the full derived category $D^b_c(K)$.) By (29.7.1) we know that $\mathrm{Ext}^j(\underline{\underline{IC}}_\sigma, \underline{\underline{IC}}_\tau) = 0$ unless $\sigma \succeq \tau$ and $\delta(\sigma) = \delta(\tau) + j$, in which case this group is canonically isomorphic to $\mathbb{Q}$ and it has a canonical generator $t_{\sigma\tau} \in \mathrm{Ext}^{\delta(\sigma)-\delta(\tau)}(\underline{\underline{IC}}_\sigma, \underline{\underline{IC}}_\tau)$ that corresponds to the identity mapping $\mathbb{Q} \to \mathbb{Q}$ between the stalk cohomology of $\underline{\underline{IC}}_\sigma$ and $\underline{\underline{IC}}_\tau$ at points in the interior.

Similarly by considering the composition of two morphisms it is easy to see that the product $t_{\alpha\beta}t_{\sigma\tau}$ is zero unless $\alpha = \tau$ in which case it is $t_{\sigma\beta}$ ([101]). Equivalently, if $\delta(\sigma) = r+1$, $\delta(\tau) = r-1$ and if $\sigma \leftrightarrow \theta \leftrightarrow \tau$ and $\sigma \leftrightarrow \theta' \leftrightarrow \tau$ with $\delta(\theta) = \delta(\theta') = r$ then $t_{\theta\tau}.t_{\sigma\theta} = t_{\theta'\tau}.t_{\sigma\theta'}$. (Compare §29.14 (3).) We may conclude:

30.5. Proposition. *A module T over the Ext algebra A is a vector space T_σ attached to each simplex σ, and "attaching" morphisms $t_{\sigma\tau} : T_\sigma \to T_\tau$ whenever $\sigma \succeq \tau$ which are compatible with composition. Equivalently, the algebra A is isomorphic to the quotient F/I where I is the two-sided homogeneous ideal generated by the vector space $D \subset \mathrm{F}_2$ spanned by the following elements:*

$$[s_{\theta\tau}].[s_{\sigma\theta}] - [s_{\theta'\tau}].[s_{\sigma\theta'}] \tag{30.5.1}$$

for every σ, τ with $\delta(\sigma) = \delta(\tau)+2$ and every θ, θ' with $\delta(\theta) = \delta(\theta') = \delta(\tau)+1$ and $\sigma \leftrightarrow \theta \leftrightarrow \tau$ and $\sigma \leftrightarrow \theta' \leftrightarrow \tau$.

□

30.6. The canonical basis $\{[s_{\sigma\tau}]\}$ of F_1 determines an inner product $\langle [s_{\sigma\tau}], [s_{\alpha\beta}]\rangle = \delta_{\sigma\alpha}\delta_{\tau\beta}$ on F_1 and hence also on each F_r. Any vector subsapce $X \subset \mathrm{F}_2$ generates a homogeneous ideal $(X) \subset \mathrm{F}$ with quotient algebra $Y = \mathrm{F}/(X)$. The *quadratic dual* algebra is $Y^! = \mathrm{F}/V$ where $V = (X^\perp)$ is the homogeneous ideal generated by the complementary subspace $X^\perp \subset \mathrm{F}_2$.

30.7. Proposition. [123] *The ring B and the Ext algebra A are quadratic duals.*

Proof. In fact the subspaces $D, E \subset \mathrm{F}_2$ above are orthogonal complements. The vector space F_2 in the path algebra is the orthogonal direct sum of the subspaces $V_{\sigma\tau}$ spanned by paths of length two, that is,

$$V_{\sigma\tau} = \mathrm{span}\left\{[s_{x\tau}].[s_{\sigma x}] \middle| \ \sigma \succeq x \succeq \tau, \delta(\sigma) = \delta(x) + 1 = \delta(\tau) + 2\right\}. \tag{30.7.1}$$

We claim that within each $V_{\sigma\tau}$ the corresponding subspaces $D_{\sigma\tau}, E_{\sigma\tau} \subset \mathrm{F}_2$ are orthogonal complements. Fix σ, τ as in (30.7.1) and first suppose they are both of type *. This implies that $\sigma \leftrightarrow \tau$ and so, given σ, τ there are two or more choices for x. The resulting vectors (30.5.1) span the subspace $D_{\sigma\tau}$ that is orthogonal to the vector $u_{\sigma\tau}$ of (30.3.1) which spans $E_{\sigma\tau}$. The same holds if σ, τ are both of type !. If σ, τ are different types then $\delta(\sigma) = 1$, $\delta(\tau) = -1$, $\delta(x) = 0$ and either σ or τ is one-dimensional, cf. the example in §29.7. Then $\{x\} = \sigma \cap \tau$ is the unique point in the intersection so the sum (30.3.1) has a single term: it is the monomial $[s_{x\tau}].[s_{\sigma x}]$. Hence $V_{\sigma\tau} = E_{\sigma\tau}$ is one dimensional, while $D_{\sigma\tau} = 0$. □.

30.8. Injective objects in the category of modules over the Ext algebra A are sums of *elementary injectives* I_σ where

$$I_\sigma(\tau) = \begin{cases} \mathbb{Q} & \text{if } \tau \succeq \sigma \\ 0 & \text{otherwise} \end{cases}$$

with identity attaching morphisms. This category has enough injectives and it is not difficult to see that the derived category $D^b(Mod - A)$ is naturally equivalent to the full derived category $D^b_c(K)$. The general theory of Koszul duality [10] guarantees an equivalence of categories $D^b(Mod - A) \sim D^b(Mod - B)$ with the category of modules over the Koszul dual algebra. This is the main idea in the proof of Proposition 29.15, that is,

$$D^b_c(K) \simeq D^b(Mod - A) \simeq D^b(Mod - B) \simeq D^b(\mathcal{P}(K)) \simeq D^b(\mathcal{P}^\Delta(K)).$$

Part 3. Further Items

31. Equivariant sheaves

32. Vanishing cycles and all that

33. Constructible functions and index theorems

Part 4. Appendices

Appendix A. Essential[15] Category Theory

A.1. Definitions. A category $\mathcal{C}$ has a class of objects and, for any pair (X, Y) of objects, a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms, elements of which are denoted by arrows $X \to Y$. (It is common to abuse notation and write $X \in \mathcal{C}$ when X is an object of $\mathcal{C}$.) There is a unique identity morphism id_X in $\mathrm{Hom}_{\mathcal{C}}(X, X)$. Composition of morphisms is defined with

$$X \xrightarrow{f} Y \xrightarrow{g} Z \text{ denoted } g \circ f : X \to Z.$$

Composition is associative: $h(gf) = (hg)f$ and $\mathrm{id}_X \circ f = f = f \circ \mathrm{id}_Y$ whenever $f : X \to Y$.

A morphism $f : X \to Y$ is *monic* if: for any object Z and any two distinct $h_1 \neq h_2 \in \mathrm{Hom}(Z, X)$ the compositions $fh_x \neq fh_x$ are not equal. The morphism $f : X \to Y$ is *epi* if: for any object Z and any two distinct $h_1 \neq h_2 \in \mathrm{Hom}(Y, Z)$ the compositions $h_1 f \neq h_2 f$ are not equal.

If $\mathcal{C}, \mathcal{D}$ are categories, a (covariant) functor $F : \mathcal{C} \to \mathcal{D}$ is a rule that assigns to each object $X \in \mathcal{C}$ another object $F(X) \in \mathcal{D}$ and for any morphism $f : X \to Y$ in $\mathcal{C}$ another morphism $F(f) : F(X) \to F(Y)$ in $\mathcal{D}$, in a way that is compatible with composition and which takes identities to identites. The category $\mathcal{C}$ is a *full subcategory* if the functor F induces an isomorphism $\mathrm{Hom}_{\mathcal{C}}(x, y) \cong \mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$.

[15] Saunders MacLane recalled that the first paper [34] on category theory was initially rejected by the editor of a prestigious journal, who stated that he had never before seen a paper that was so entirely devoid of content.

A.2. A comment about categories. The Zermelo Fraenkel axioms avoid Russell's paradox by providing classes of objects that are "too big" to be sets. In a category $\mathcal{C}$ the collection of morphisms $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ between two objects X, Y is assumed to form a set, and so we may speak of two morphisms f, g being the same, and we write $f = g$. However the collection of objects may fail to form a set and so the statement that "$Y = X$" or "Y is the same object as X" does not make sense. Rather, "the morphism $f : X \to Y$ is an isomorphism" is the correct way to indicate an identification between two objects in a category. If the set $\mathrm{Hom}_{\mathcal{C}}(X, X)$ of self-isomorphisms of X is nontrivial, there will be many distinct isomorphisms between X and Y.

For example, all the objects in the category of 1-dimensional real vector spaces are isomorphic, however an isomorphism $h : \mathbb{R} \to V$ corresponds to a choice of a nonzero element $x \in V$ by setting $h(1) = x$.

A.3. Abelian categories. An Abelian category $\mathcal{C}$ is one in which every $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ has the structure of an Abelian group such that composition of morphisms is distributive over addition ($g(f + f')h = gfh + gf'h$) and furthermore

(1) There exists a zero object 0 so that $\mathrm{Hom}_{\mathcal{C}}(0, 0)$ is the Abelian group $\{0\}$
(2) Any two objects X, Y have a direct sum object $X \oplus Y$ with morphisms

$$X \underset{p_2}{\overset{p_1}{\leftrightarrows}} X \oplus Y$$

(3) Every $f : X \to Y$ has a kernel, an image and a cokernel in $\mathcal{C}$ with morphisms

$$\begin{tikzcd} \ker(f) \arrow[r] & X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[r] & \mathrm{coker}(f) \\ & & \mathrm{Im(f)} \arrow[ur] & & \end{tikzcd}$$

(4) If $f : X \to Y$ is a monomorphism then $\ker(\mathrm{coker}(f))$

A.4. Adjoint functors. If $\mathcal{C}, \mathcal{D}$ are categories, $f : \mathcal{C} \to \mathcal{D}$ and $g : \mathcal{D} \to \mathcal{C}$ are functors, then f and g are said to be adjoint if there are "natural" isomorphisms

$$\Phi_{AB} : \mathrm{Hom}_{\mathcal{C}}(A, g(B)) \to \mathrm{Hom}_{\mathcal{D}}(f(A), B)$$

for all A, B objects of $\mathcal{C}, \mathcal{D}$ respectively (where "natural" means that these isomorphism are compatible with all morphisms $A_1 \to A_2$ and $B_1 \to B_2$). We say that f is left adjoint to g and g is right adjoint to f.

In this case, f essentially determines g and vice versa, that is, given f and two adjoint functors g_1, g_2 then there exists a natural equivalence of functors between g_1, g_2. On a formal level this is a consequence of the Yoneda Lemma, but it can be seen directly. We have a natural isomorphism $\mathrm{Hom}_{\mathcal{C}}(A, g_1(B)) \cong \mathrm{Hom}_{\mathcal{C}}(A, g_2(B))$ for every $A \in \mathcal{C}, B \in \mathcal{D}$. Taking $A = g_1(B)$, the identity on the left side gives an element $\tau(B) \in \mathrm{Hom}_{\mathcal{C}}(g_1(B), g_2(B))$. It is an exercise to see that this defines a natural transformation $g_1 \implies g_2$, in other words, if $h : B \to B'$ is a morphism in $\mathcal{D}$ then

$g_2(h) \circ \tau(B) = \tau(B') \circ g_1(h)$. Since each Φ_{AB} is an isomorphism, it follows that τ is a natural equivalence.

Examples. If H is a subgroup of a finite group G then "restriction" is a more or less obvious functor from the category of representations of G to the category of representations of H; its adjoint turns out to be "induction", a much more subtle construction, and the adjointness property is classically known as the Frobenius reciprocity theorem,

$$\mathrm{Hom}_G(W, \mathrm{Ind}_H^G(V)) \cong \mathrm{Hom}_H(W|H, V).$$

Sheafification (§1.4) is adjoint to the inclusion of categories (sheaves $\to$ presheaves). If $f : X \to Y$ is a continuous mapping then Proposition 1.13 says that f^* is left adjoint to f_*. One might ask whether there is an adjoint to the functor $f_!$. This turns out to be a very subtle question, see §11.4.

A.5. Exact functors and injective objects. A functor $F : \mathcal{C} \to \mathcal{D}$ between Abelian categories $\mathcal{C}, \mathcal{D}$ is exact if it takes each short exact sequence $0 \to A \to B \to C \to 0$ to a short exact sequence $0 \to F(A) \to F(B) \to F(C) \to 0$. It is left exact $0 \to F(A) \to F(B) \to F(C)$ is exact. It is right exact if $F(A) \to F(B) \to F(C) \to 0$ is exact. For any object B in $\mathcal{C}$ the (contravariant) functor $A \mapsto \mathrm{Hom}_{\mathcal{C}}(A, B)$ is left exact.

An object I in $\mathcal{C}$ is *injective* if $A \mapsto \mathrm{Hom}_{\mathcal{C}}(A, I)$ is an exact functor. Equivalently, Suppose $f : A \to B$ is an injective morphism in $\mathcal{C}$. Then any morphism $h : A \to I$ extends to a morphism $\tilde{h} : B \to I$.

An object P is projective if $B \mapsto \mathrm{Hom}_{\mathcal{C}}(P, B)$ is exact. Equivalently, suppose $g : B \to A$ is a surjective morphism in $\mathcal{C}$. Then any morphism $k : P \to A$ factors through some $\tilde{k} : P \to B$.

$$\begin{tikzcd} B \arrow[dr, dotted, "\tilde{h}"] & \\ A \arrow[u, hook, "f"] \arrow[r, "h"'] & I \end{tikzcd} \qquad \begin{tikzcd} B \arrow[d, two heads, "g"'] & \\ A & P \arrow[l, "k"] \arrow[ul, dotted, "\tilde{k}"'] \end{tikzcd}$$

I injective $\qquad$ P projective

Any short exact sequence $0 \to A \to B \to C \to 0$ of injective (resp. projective) objects is split

(because the identity map $A \to A$ extends to a morphism $B \to A$ which is therefore a splitting.)

A.6. A *complex* $S^\bullet$ is a sequence $\cdots \to S^{r-1} \xrightarrow{d^{r-1}} S^r \xrightarrow{d^r} S^{r+1} \xrightarrow{d^{r+1}} \cdots$ where $d \circ d = 0$. Its *cohomology* is $H^r(S^\bullet) = \ker d^r / \operatorname{Im} d^{r-1}$. The complex $S^\bullet$ is *exact* iff $H^r(S^\bullet) = 0$ for all r. A morphism $S^\bullet \to T^\bullet$ of complexes is a collection of morphisms that commute with the differentials. A morphism $\phi : S^\bullet \to T^\bullet$ is a *quasi-isomorphism* if it induces isomorphisms on the cohomology objects $H^r(S^\bullet) \to H^r(T^\bullet)$ for all r.

We will assume throughout that all of our complexes are bounded from below, that is $S^r = 0$ if r is sufficiently small, usually if $r < 0$.

A.7. Magic Triangles. The *mapping cone* $C^\bullet = C(\phi)$ of a morphism $\phi : A^\bullet \to B^\bullet$ is the complex $C^r = A^{r+1} \oplus B^r$ with differential $d_C(a, b) = (d_A(a), (-1)^{\deg(a)}\phi(a) + d_B(b))$. It is the total complex of the double complex

$$\begin{tikzcd}
 & & \\
A^2 \arrow[r, "\phi"] \arrow[u, "d"] & B^2 \arrow[u, "d"'] \\
A^1 \arrow[r, "\phi"] \arrow[u, "d"] & B^1 \arrow[u, "d"'] \\
A^0 \arrow[r, "\phi"] \arrow[u, "d"] & B^0 \arrow[u, "d"']
\end{tikzcd}$$

from which we see that there are obvious morphisms $\beta : B^\bullet \to C^\bullet$ and $\gamma : C^\bullet \to A^\bullet[1]$. It is customary to denote this situation as a triangle of morphisms

$$\begin{tikzcd}
A^\bullet \arrow[rr, "\phi"] & & B^\bullet \arrow[dl, "\beta"] \\
 & C(\phi) \arrow[ul, "\gamma", "{[1]}"'] &
\end{tikzcd}$$

The following lemma says that the cone $C(\phi)$ serves as both kernel and cokernel of ϕ. Moreover, any side of a magic triangle determines the third vertex, up to quasi-isomorphiism:

A.8. Lemma. *(exercise) Let $\phi : A^\bullet \to B^\bullet$ be a morphism of complexes in an Abelian category. If ϕ is injective then there is a natural quasi-isomorphism* $\mathrm{coker}(\phi) \cong C(\phi)$. *If ϕ is surjective then there is a natural quasi-isomorphism $C(\phi) \cong \ker(\phi)[1]$. There are natural quasi-isomorphisms $A^\bullet[1] \cong C(\beta)$ and $B^\bullet[1] \cong C(\gamma)$. The morphisms β, γ induce a long exact sequence on cohomology*

$$\cdots \to \mathrm{H}^{r-1}(B^\bullet) \to \mathrm{H}^{r-1}(C(\phi)) \to \mathrm{H}^r(A^\bullet) \to \mathrm{H}^r(B^\bullet) \to \mathrm{H}^r(C(\phi)) \to \cdots .$$

The morphism ϕ is a quasi-isomorphism iff $C(\phi)$ is acyclic: $H^r(C(\phi)) = 0$ for all r.

A.9. Double complexes. Let $J^{\bullet\bullet}$ be a first quadrant double complex of objects in an Abelian category. This means $J^{pq} = 0$ for $p < 0$ or $q < 0$ and there are "horizontal" and "vertical" morphisms, $d_v : J^{pq} \to J^{p+1,q}$ and $d_h : J^{pq} \to J^{p,q+1}$ with $d_v d_h = d_h d_v$, $d_v d_v = 0$, and $d_h d_h = 0$. The associated single complex $S^\bullet$ is defined by $S^r = \oplus_{p+q=r} J^{pq}$ with total differential $d_S(c_{pq}) = (d_h + (-1)^q d_v)c_{pq} \in J^{pq}$.

A.10. Lemma. *Assume the rows of $J^{\bullet\bullet}$ are exact. Then the total complex $S^\bullet$ of $J^{\bullet\bullet}$ is acyclic: $H^r(S^\bullet) = 0$ for all r.*

Proof. A proper proof involves indices, signs, and an induction that is totally confusing and is best worked out in the privacy of your own home. To see how the argument goes, let us show that the total complex $S^0 \to S^1 \to S^2 \cdots$ is exact at S^2. Let $x = x_{02} + x_{11} + x_{20} \in S^2$ and suppose that $d_S x = 0$.

$$\begin{tikzcd}
 & \cdot & & \\
 & x_{20} \arrow[u] \arrow[r] & \cdot \arrow[u] & \\
 & y_{10} \arrow[u] \arrow[r] & x_{11} \arrow[u] \arrow[r] & \cdot \\
 & \cdot \arrow[u] \arrow[r] & y_{01} \arrow[u] \arrow[r] & x_{02} \arrow[u] \arrow[r] & \cdot
\end{tikzcd}$$

- Since $d_S x = 0$ we have:

$$d_v x_{20} = 0;\ d_v x_{11} + d_h x_{20} = 0;\ d_v x_{02} - d_h x_{11} = 0;\ d_h x_{02} = 0.$$

- Since the bottom row is exact there exists y_{10} so that $d_h y_{10} = x_{20}$.
- Now consider $x'_{11} = x_{11} - d_v y_{01}$. Check that $d_h(x'_{11}) = 0$.
- Since the first row is exact there exists y_{10}so that $d_h y_{10} = x'_{11}$.
- Now consider $x'_{20} = x_{20} + d_v y_{10}$. Check that $d_h x'_{20} = 0$.
- This last operation, d_h was an injective mapping so $x'_{20} = 0$.

- Check that $d_S y = x$ where $y = y_{01} + y_{10} \in S^1$.
- This argument shows $H^2(S^\bullet) = 0$. For degrees ≥ 3 it is necessary to use $d_v d_v = 0$. $\square$

A.11. Homotopy theory. Suppose $A^\bullet, B^\bullet$ are complexes in an Abelin category. Define the complex of abelian groups (or R modules)

$$\mathrm{Hom}^n(A^\bullet, B^\bullet) = \prod_s \mathrm{Hom}(A^s, B^{s+n})$$

with differential $d_{\mathrm{Hom}} f = d_B f + (-1)^{n+1} f d_A$ where $f : A^s \to B^{s+n}$.

Two morphisms $f, g : A^\bullet \to B^\bullet$ of complexes are said to be *homotopic* if there is a collection of mappings $h : A^r \to B^{r-1}$ so that $hd_A + d_B h = f - g$. This is an equivalence relation. Equivalence classes are referred to as homotopy classes of maps; the set of which is denoted $[A^\bullet, B^\bullet]$.

A cool fact is that homotopy classes of chain maps are exactly the cohomology of the $\mathrm{Hom}^\bullet$ complex:

A.12. Lemma. *(exercise) If $A^\bullet, B^\bullet$ are complexes in an Abelian category then (by changing the sign of half the differentials) a chain map $f : A^\bullet \to B^\bullet$ determines a cycle*

$$d_{Hom} f = 0 \in \mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$$

and consequently $H^n \mathrm{Hom}^\bullet(A^\bullet, B^\bullet) \cong [A^\bullet, B^\bullet[n]]$.

A.13. Lemma. *Let $C^\bullet$ be a (bounded below) complex and suppose that the cohomology $\mathbf{H}^r(C^\bullet) = 0$ for all r. Let $J^\bullet$ be a complex of injective objects. Then any morphism $f : C^\bullet \to J^\bullet$ is homotopic to zero, meaning that there exists $h : C^\bullet \to J^\bullet[-1]$ such that $d_J h + h d_C = f$.*

Proof. It helps to think about the diagram of complexes:

$$\begin{tikzcd}
0 \arrow[r] & C^0 \arrow[r, "d^0"] \arrow[d, "f"'] & C^1 \arrow[r, "d^1"] \arrow[d, "f"'] \arrow[dl, "h^1"'] & C^2 \arrow[r, "d^2"] \arrow[d, "f"'] \arrow[dl, "h^2"'] & C^3 \arrow[r] \arrow[d, "f"'] \arrow[dl, "h^3"'] & \cdots \\
0 \arrow[r] & J^0 \arrow[r, "d^0"'] & J^1 \arrow[r, "d^1"'] & J^2 \arrow[r, "d^2"'] & J^3 \arrow[r] & \cdots
\end{tikzcd}$$

The first step is easy, since $C^0 \to C^1$ is an injection and since J^0 is injective there exists $h^1 : C^1 \to J^0$ that makes the triangle commute, that is, $h^1 d^0 = f$. Now let us define $h^2 : C^2 \to J^1$. Consider the map $(f - d^0 h^1) : C^1 \to J^1$. It vanishes on $\mathsf{Im}(d^0) = \ker(d^1)$ because

$$(f - d^0 h^1) d^0 = f d^0 - d^0 h^1 d^0 = f d^0 - d^0 f = 0.$$

Therefore it passes to a vertical mapping in this diagram:

$$\begin{tikzcd} C^1 \arrow[r] \arrow[dr] & C^1/\ker(d^1) \arrow[r, hook] \arrow[d, "f - d^0h^1"] & C^2 \\ & J^1 & \end{tikzcd}$$

where the second horizontal mapping is an injection. Since J^1 is injective we obtain an extension $h^2 : C^2 \to J^1$ such that $h^2 \circ d^1 = f - d^0h^1$ so that $h^2d^1 + d^0h^1 = f$. Continuing in this way, the other h^r can be constructed inductively. □

A.14. Lemma. *Let $f : A^\bullet \to B^\bullet$ be a quasi-isomorphism of (bounded below) complexes. Then for any complex $J^\bullet$ of injectives, the induced map $[B^\bullet, J^\bullet] \to [A^\bullet, J^\bullet]$ on homotopy classes is an isomorphism. If $A^\bullet, B^\bullet$ are injective then f admits a homotopy inverse $g : B^\bullet \to A^\bullet$ (meaning that $gf \sim I$ and $fg \sim I$).*

Proof. Let $C(f)$ denote the cone of f. Its cohomology vanishes by Lemma A.8. Lemma A.13 gives that $\mathrm{Hom}^r(C(f), J^\bullet) = 0$ for all r, It follows from the long exact sequence in Lemma A.8 that $\mathrm{Hom}^r(A^\bullet, J^\bullet) \cong \mathrm{Hom}^r(B^\bullet, J^\bullet)$ for all r.

If $A^\bullet, B^\bullet$ are injective, to construct a homotopy inverse,the first statement gives an isomorphism $[A^\bullet, A^\bullet] \to [B^\bullet, A^\bullet]$. So the identity $A^\bullet \to A^\bullet$ corresponds to some $h : B^\bullet \to A^\bullet$ such that $h \circ f \sim Id$, implying that f has a left homotopy-inverse. We also need a right homotopy inverse. Consider mapping $B^\bullet$ into the triangle $A^\bullet \to B^\bullet \to C(f) \to \cdots$, giving an exact sequence $\cdots \to [B^\bullet, A^\bullet] \to [B^\bullet, B^\bullet] \to [B^\bullet, C(f)] \to \cdots$. Since $C(f)$ is injective and its cohomology vanishes, the identity is homotopic to zero. Hence $[B^\bullet, C(f)] = 0$ so that $[B^\bullet, A^\bullet] \cong [B^\bullet, B^\bullet]$ with the map given by $g \mapsto f \circ g$. Therefore there exists $g : B^\bullet \to A^\bullet$ so that $f \circ g \sim Id$ meaning that f has a right inverse in the homotopy category. If a mapping has both a left inverse and a right inverse then it has an inverse (in other words, h and g are homotopic, so either of them will behave as a homotopy inverse to f). □

A.15. Injective resolution. A category $\mathcal{C}$ has *enough injectives* if every object A embeds in some injective object I. In this case, there is an exact sequence, classically called an *injective resolution*,

$$0 \to A \to I^0 \to I^1 \to \cdots$$

where I^r are injective. It is easily constructed by first embedding A in an injective object I^0 then embedding $K = \mathrm{coker}(A \to I^0)$ in an injective object I^1 and continuing by induction. Verdier extended this notion to complexes:

A.16. Definition. An injective resolution of a complex $A^\bullet$ is a quasi-isomorphism $A^\bullet \to I^\bullet$ where each I^r is injective.

This fits beautifully with the classical definition, if we consider a single object A to be a complex

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \arrow[r] & \dots \\ 0 \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & \dots \end{tikzcd}$$

A.17. Proposition. *Suppose the Abelian category $\mathcal{C}$ has enough injectives.*

(1) every complex $A^\bullet$ has an injective resolution $A^\bullet \to I^\bullet$.
(2) Let $A^\bullet \to I^\bullet$ and $B^\bullet \to J^\bullet$ be injective resolutions of complexes $A^\bullet, B^\bullet$. Then any morphism $f : A^\bullet \to B^\bullet$ admits a lift $\tilde{f} : I^\bullet \to J^\bullet$ and any two such lifts are homtopic.
(3) Any two injective resolutions of $A^\bullet$ are homotopy equivalent.

Proof. For part (2), the existence of the lift $\tilde{f}$ is an exercise. The rest of Part (2), and all of part (3) is a restatement of Lemma A.14. For part (1), it is possible (see below) to find injective resolutions

$$A^j \overset{i}{\hookrightarrow} I^{j0} \underset{d_h}{\longrightarrow} I^{j1} \underset{d_h}{\longrightarrow} \cdots$$

of each A^j and to fit them together into a double complex $I^{\bullet\bullet}$ with the "vertical" differentials $d_v : I^{jr} \to I^{j+1,r}$ induced from the original $d_A : A^j \to A^{j+1}$, with $d_v d_v = d_h d_h = 0$:

$$\begin{tikzcd} 0 \arrow[r] & A^{r+1} \arrow[r, hook, "i"] & I_A^{r+1,0} \arrow[r] & I_A^{r+1,1} \arrow[r] & \cdots \\ 0 \arrow[r] & A^r \arrow[u, "d_A"'] \arrow[r, hook, "i"] & I_A^{r,0} \arrow[u] \arrow[r] & I_A^{r,1} \arrow[u] \arrow[r] & \cdots \\ 0 \arrow[r] & A^{r-1} \arrow[u, "d_A"'] \arrow[r, hook, "i"] & I_A^{r-1,0} \arrow[u] \arrow[r] & I_A^{r-1,1} \arrow[u] \arrow[r] & \cdots \end{tikzcd}$$

The cohomology of the resulting total complex $S^\bullet$ vanishes, by Lemma A.10. However $S^\bullet$ is precisely the cone of the morphism $i : A^\bullet \to T^\bullet$ where $T^\bullet$ is the total complex $T^r = \oplus_{p+q=r} I^{pq}$, which gives a magic triangle

$$\begin{tikzcd} A^\bullet \arrow[rr, hook, "i"'] & & T^\bullet \arrow[dl] \\ & S^\bullet \arrow[ul, "{[1]}"] & \end{tikzcd}$$

The long exact sequence on cohomology implies that $H^*(A^\bullet) \to H^*(T^\bullet)$ is an isomorphism. $\square$

A.18. Cartan-Eilenberg. It is tempting to construct the above double complex by injectively resolving each A^r and use the vertical arrows that are guaranteed by injectivity. But this procedure does not guarantee[16] that $d_v d_v = 0$. Instead, there is a Cartan-Eilenberg procedure [**?**] §5.7.2, [27]: Let $B^r \subset Z^r$ denote $\mathrm{Im}(d^{r-1}) \subset \ker(d^r) \subset A^r$ and choose corresponding injective resolutions

[16] If functorial injective resolutions exist then we will indeed obtain a double complex. In the case of sheaves, this can be accomplished with the Godement resolution, see §4.1.

$B^r \to I_B^{r\bullet}$ and $Z^r \to I_Z^{r\bullet}$ as well as $H^r(A^\bullet) \to I_H^{r\bullet}$ of $H^r(A^\bullet)$. The following diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & B^r \arrow[r] \arrow[d] & Z^r \arrow[r] \arrow[dl] \arrow[dr] & H^r \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I_B^{r\bullet} \arrow[r] & I_B^{r\bullet} \oplus I_H^{r\bullet} \arrow[r] & I_H^{r\bullet} \arrow[r] & 0
\end{tikzcd}$$

gives a resolution $Z^r \to I_Z^{r\bullet} = I_B^{r\bullet} \oplus I_H^{r\bullet}$ and again

$$\begin{tikzcd}
0 \arrow[r] & Z^r \arrow[r] \arrow[d] & A^r \arrow[r] \arrow[dl] \arrow[dr] & B^{r+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I_Z^{r\bullet} \arrow[r] & I_Z^{r\bullet} \oplus I_B^{r+1.\bullet} \arrow[r] & I_B^{r+1,\bullet} \arrow[r] & 0
\end{tikzcd}$$

defines the required resolution $A^r \to I_A^{r\bullet} = I_Z^{r\bullet} \oplus I_B^{r+1,\bullet}$ with vertical differential d_v given by the composition

$$I_A^{r,j} \to I_B^{r+1,j} \to I_Z^{r+1,j} \to I_A^{r+1,j}.$$

The result is a double complex with exact rows and $d_h d_h = d_v d_v = 0$.

A.19. Derived category: Preliminary model. Let $\mathcal{C}$ be an Abelian category. The derived category $D(\mathcal{C})$ is a category of complexes in which quasi-isomorphisms have become isomorphisms. There is no simple way to arrange this.

For a preliminary model, let us say the bounded derived category $\mathcal{I}^b(\mathcal{C})$ is the category whose objects are complexes $A^\bullet$ of injective objects in $\mathcal{C}$ whose cohomologyis bounded (meaning that $H^r(A^\bullet) = 0$ for sufficiently large r and for sufficiently small r). The morphisms are homotopy classes of morphisms (of complexes), so that $\mathcal{I}^b(\mathcal{C})$ is the *homotopy category* of (complexes of) injective objects.

Let $K^b(\mathcal{C})$ be the homotopy category of complexes $A^\bullet$ of objects in $\mathcal{C}$ with bounded cohomology: Choosing an injective resolution for each complex gives a functor from $K^b(\mathcal{D}) \to \mathcal{I}^b(\mathcal{C})$. According to Proposition **??** and Lemma A.14 it takes quasi-isomorphisms to isomorphisms. .

A.20. The derived category: second construction. The preliminary model has the disadvantage that it is adapted only for right derived functors, and besides, injective objects are usually intractable. A more balanced point of view constructs the derived category as a "quotient category" of the homotopy category $K^b(\mathcal{C})$ of complexes, by inverting quasi-isomorphisms. This gives a way of referring to elements of the derived category without having to injectively resolve. Moreover, each complex of sheaves is automatically an object in the derived category $D^b(\mathcal{C})$.

Let $D^b(\mathcal{C})$ be the category whose objects are complexes of elements of $\mathcal{C}$ and where a morphism $A^\bullet \to B^\bullet$ is an equivalence class of "roof" diagrams

$$\begin{tikzcd} & X^\bullet \arrow[dl, "qi"'] \arrow[dr] & \\ A^\bullet & & B^\bullet \end{tikzcd}$$

where $X^\bullet \to A^\bullet$ is a quasi-isomorphism, and where two such morphisms $A^\bullet \leftarrow X_1^\bullet \to B^\bullet$ and $A^\bullet \leftarrow X_2^\bullet \to B^\bullet$ are considered to be equivalent if there exists a diagram

$$\begin{tikzcd} & X_1^\bullet \arrow[dl] \arrow[dr] & \\ A^\bullet & X_3^\bullet \arrow[l, "qi"'] \arrow[r] \arrow[u] \arrow[d] & B^\bullet \\ & X_2^\bullet \arrow[ul] \arrow[ur] & \end{tikzcd}$$

that is commutative up to homotopy. The composition of two morphisms

$$\begin{tikzcd} & X^\bullet \arrow[dl] \arrow[dr, "f"] & & Y^\bullet \arrow[dl, "\phi"'] \arrow[dr] & \\ A^\bullet & & B^\bullet & & C^\bullet \end{tikzcd}$$

may be constructed using a further roof, $X^\bullet \leftarrow Z^\bullet \to Y^\bullet$ where $Z^\bullet$ is the cone of the composition $X^\bullet \xrightarrow{f} B^\bullet \to C(\phi)$. (Exercise: Show that these definitions give a category by checking that composition is well defined with respect to be above equivalence relation.)

A.21. Proposition. *The natural functor $\mathcal{I}^b(X) \to D^b(X)$ is an equivalence of categories.*

Proof. To show that a functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories it suffices to show (a) that it is essentially surjective, meaning that every object in $\mathcal{D}$ is isomorphic to an object $F(C)$ for some object C in $\mathcal{C}$, and (b) that F induces an isomorphism on Hom sets. The first part (a) is clear because we have injective resolutions. Part (b) follows immediately from the fact (Lemma A.14) that a quasi-isomorphism of injective complexes is a homotopy equivalence and has a homotopy inverse. □

A.22. Derived functors. Let $F : \mathcal{C} \to \mathcal{D}$ be a left exact functor between Abelian categories and assume that $\mathcal{C}$ has enough injectives. Its derived functor $RF : D^b(\mathcal{C}) \to D^b(\mathcal{D})$ is defined to be $RF(A^\bullet) = F(I^\bullet)$ where $A^\bullet \to I^\bullet$ is an[17] The value $RF(\phi)$ is defined on morphisms $\phi : A^\bullet \to B^\bullet$

[17]To avoid set-theoretic difficulties it is necessary to choose a fixed injective resolution for each complex. In the category of sheaves this can be done canonically using "the" Godement resolutioninjective resolution.

as above, because (Proposition A.17) ϕ extends to a morphism $I^\bullet \to J^\bullet$ on the corresponding injective resolutions of $A^\bullet, B^\bullet$.

If F is right exact and if $\mathcal{C}$ has enough projectives then the left derived functor $LF : D^b(\mathcal{C} \to \mathcal{D})$ is similarly defined to be $LF(A^\bullet) = F(P^\bullet)$ where $P^\bullet \to A^\bullet$ is a projective resolution.

Remark. Although $T(I^\bullet)$ is an object in $D^b(\mathcal{D})$ the individual terms $T(I^r)$ are not necessarily injective unless T also takes injectives to injectives. If $V : \mathcal{D} \to \mathcal{E}$ is another functor then $R(V \circ T)(A^\bullet) = (RV)(T(I^\bullet)) = V(J^\bullet)$ where $T(I^\bullet) \to J^\bullet$ is an injective resolution of $T(I^\bullet)$.

A.23. T-acyclic resolutions. Consider a functor $T : \mathcal{C} \to \mathcal{D}$ between abelian categories. An object $X \in \mathcal{C}$ is T-acyclic if $R^iT(X) = 0$ for all $i \neq 0$.

If T is exact then (exercise) every object X is T-acyclic. The great advantage of T-acyclic objects is that they may be often used in place of injective objects when computing the derived functors of T.

A.24. Lemma. *Let $T : \mathcal{C} \to \mathcal{D}$ be a left exact functor between Abelian categories with enough injectives. Let $A^\bullet$ be a complex of objects in $\mathcal{C}$ and let $A^\bullet \to X^\bullet$ be a quasi-isomorphism, where each of the objects X^r is T-acyclic. Then $R^rT(A^\bullet)$ is canonically isomorphic to the r-th cohomology object of the complex*

$$T(X^0) \to T(X^1) \to T(X^2) \to \cdots .$$

If T is exact then there is no need to take a resolution at all: $RT(A^\bullet) \cong T(A^\bullet)$.

Proof. An injective resolution of each X^r gives rise to a double complex $I^{\bullet\bullet}$. Apply T to the whole double complex. Since X^r is T-acyclic the rows of the augmented double complex (including the $T(X^r)$ column) are exact. Now follow the same the same double complex argument as in Proposition A.17. □

The same argument applies to left derived functors. If $A^\bullet, B^\bullet \in D^b(X)$ then $A^\bullet \overset{L}{\otimes} B^\bullet = \mathbf{P}^\bullet \otimes B^\bullet$ where $\mathbf{P}^\bullet \to A^\bullet$ is a projective resolution. The above argument proves that $P^\bullet$ may be replaced[18] by any flat[19] resolution of $A^\bullet$. The cohomology of the result is denoted

$$\mathrm{Tor}_j(A^\bullet, B^\bullet) = H^{-j}(A^\bullet \overset{L}{\otimes} B^\bullet).$$

[18]particularly useful because the category of sheaves does not always contain enough projectives.

[19]Recall that a module M over a commutative ring R is flat if the functor $\otimes_R M$ is exact.

A.25. Exercise. Verify the following.

(1) Injective objects are T-acyclic for any left exact functor T.
(2) If $f : X \to Y$ is a continuous map then f_* is left exact. (Hint: use the adjunction formula $\mathrm{Hom}_{\mathcal{S}(Y)}(B, f_*(I)) \cong \mathrm{Hom}_{\mathcal{S}(X)}(f^*(B), I)$.)
(3) Fine, flabby, and soft sheaves are Γ-acyclic, $f_!$ acyclic and f_* acyclic. In particular, for a complex of sheaves $A^\bullet$ on a topological space, the derived functors $H^*(A^\bullet) = R\Gamma(A^\bullet)$, $RHom(A^\bullet, B^\bullet)$, $Rf_*(A^\bullet)$ and $Rf_!(A^\bullet)$ may be computed using any injective or fine or flabby resolution of $A^\bullet$.

A.26. Examples.
1. The m-th derived functor of Hom is called Ext^m, (cf. Lemma A.12) i.e., it is the group

$$\mathrm{Ext}^m(A^\bullet, B^\bullet) = H^m(\mathrm{RHom}(A^\bullet, B^\bullet)) = H^m(\mathrm{Hom}^\bullet(A^\bullet, J^\bullet)) = H^0(\mathrm{Hom}^\bullet(A^\bullet, J^\bullet[m]))$$

where $B^\bullet \to J^\bullet$ is an injective resolution. (We consider $\mathrm{Hom}(A^\bullet, B^\bullet)$ to be a functor of the $B^\bullet$ variable and derive it by injectively resolving. The same result can be obtained by *projectively* resolving $A^\bullet$.) As in §1.9, there is a sheaf version of Hom, which also gives a sheaf version of Ext:

$$\underline{\underline{\mathrm{Ext}}}^m(A^\bullet, B^\bullet)) = \underline{\underline{H}}^m(\underline{\underline{\mathrm{RHom}}}(A^\bullet, B^\bullet))$$

2. The hypercohomology of RHom is exactly the group of homomorphisms in the derived category: $H^0(X, \mathrm{RHom}(A^\bullet, B^\bullet)) = \mathrm{Hom}_{D^b(X)}(A^\bullet, B^\bullet)$.
Exercise. Let G, H be abelian groups, as complexes in degree zero. Show that $\mathrm{Ext}^1(G, H)$ coincides with the usual definition of $\mathrm{Ext}_{\mathbb{Z}}(G, H)$,

A.27. The bad news. The derived category is not an abelian category. In fact, the homotopy category of complexes is not an abelian category. Kernels and cokernels do not make sense in these categories. The saving grace is that the cone operation still makes sense and in fact, it passes to the homotopy category.

A.28. Definition. A triangle of morphisms

$$\begin{tikzcd} A^\bullet \arrow[rr] & & B^\bullet \arrow[dl] \\ & C^\bullet \arrow[ul, "{[1]}"] & \end{tikzcd}$$

in $K^b(X)$ or in $D^b(X)$ is said to be a *distinguished triangle* or an *exact triangle* if it is homotopy equivalent to a triangle

$$\begin{tikzcd} X^\bullet \arrow[rr, "\phi"] & & Y^\bullet \arrow[dl] \\ & C(\phi) \arrow[ul, "{[1]}"] & \end{tikzcd}$$

where $C(\phi)$ denotes the cone on the morphism ϕ.

This means that there are morphisms between corresponding objects in the triangles such that the resulting squares commute up to homotopy. From §A.7 we have:

Every morphism $\phi : A^\bullet \to B^\bullet$ in $D^b(\mathcal{C})$ determines an object $C^\bullet$, unique up to canonical isomorphism, and a magic (or "distinguished") triangle

$$\begin{tikzcd} A^\bullet \arrow[rr, "\phi"] & & B^\bullet \arrow[dl] \\ & C^\bullet \arrow[ul, "{[1]}"] & \end{tikzcd}$$

with a long exact sequence $\cdots \to H^r(A^\bullet) \to H^r(B^\bullet) \to H^r(C^\bullet) \to H^{r+1}(A^\bullet) \to \cdots$. If $X^\bullet$ is a complex (bounded from below) then there are distinguished triangles

$$\begin{tikzcd} \mathrm{RHom}(X^\bullet,A^\bullet) \arrow[rr] & & \mathrm{RHom}(X^\bullet,B^\bullet) \arrow[dl] \\ & \mathrm{RHom}(X^\bullet,C^\bullet) \arrow[ul, "{[1]}"] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{RHom}(A^\bullet,X^\bullet) \arrow[dr] & & \mathrm{RHom}(B^\bullet,X^\bullet) \arrow[ll] \\ & \mathrm{RHom}(C^\bullet,X^\bullet) \arrow[ul, "{[1]}"] & \end{tikzcd}$$

The proof is the observation that Hom into a cone is equal to the cone of the Homs.

Appendix B. Transversality

Let $A, B \subset M$ be a smooth submanifolds of a smooth manifold M. Let $\{v_1, \cdots, v_r\}$ be vector fields on M which span the tangent space at each point[20] Let V be the r-dimensional vector space of formal linear combinations of these vector fields. Let Φ_v denote the time $= 1$ flow of the vector field v.

B.1. Theorem. *The set of elements $v \in V$ such that Φ_v fails to take A transversally to B has measure zero.*

Proof. The method of proof (using Sard's theorem, of course) is due originally to Marston Morse. Let $\Phi : V \times M \to M$ be the time $= 1$ flow mapping. The assumptions on the vector fields imply that Φ takes $V \times A$ transversally to B. (In fact for each $a \in A$ the partial map $\Phi_a : V \to M$ is a submersion.) So $\Phi^{-1}(B)$ is a submanifold of $V \times M$ and it is transverse to $V \times A$. Let $\pi : \Phi^{-1}(B) \cap (V \times A) \to V$ denote the projection. We claim the following are equivalent:

(1) $v \in V$ is a regular value of π
(2) $\{v\} \times A$ is transverse to $\Phi^{-1}(B)$ in $V \times M$
(3) Φ_v takes A transversally to B.

Equivalence of (2) and (3) is a short calculation. Equivalence of (1) and (2) is a dimension count. First note that $(V \times A)$ is transverse to $\phi^{-1}(B)$ in $V \times M$, by the assumption on the vector

[20]In fact we only need to assume that they span the normal space to B at each point in B.

fields. Fix $(w,a) \in (V \times A) \cap \Phi^{-1}(B)$. Let $\dot{A} = T_a A$. Let $\dot{M} = T_a M$, let $\dot{V} = T_w V$ and let $\dot{B} = T_{w,a}\Phi^{-1}(B)$. So we have a diagram of vector spaces

$$\begin{array}{c} \dot{V} \times \dot{M} \\ \uparrow \\ (\dot{V} \times \dot{A}) \cap \dot{B} \end{array}$$

Let r be the rank of the linear map $\pi : \dot{B} \cap (\dot{V} \times \dot{A}) \to \dot{V}$; its kernel is $\dot{B} \cap (\{0\} \times A$. Therefore

$$\dim(\dot{B} \cap (\dot{V} \times \dot{A})) = r + \dim(\dot{B} \cap (\{0\} \times \dot{A})$$

or

$$b + (v + a) - m = r + (b + a - \dim(\dot{B} + \{0\} \times \dot{A}$$

where $b = \dim(\dot{B})$, $a = \dim(\dot{A})$, $v = \dim(\dot{V})$. Therefore

$$v - r = m - \dim(\dot{B} + \{0\} \times \dot{A}).$$

Therefore the map π is surjective ($r = v$) if and only if $\dot{B} + \{0\} \times \dot{A}$ spans $\dot{M}$.

Now, by Sard's theorem the set of critical values $v \in V$ has measure zero. But this is exactly the set of elements such that Φ_v fails to take A transversally to B.

□

B.2. Exactly the same argument shows, for example, that two submanifolds of Euclidean space can be made transverse by an arbitrarily small *translation*. Similarly, two stratified subvarieties of projective space can be made transverse by an arbitrarily small projective transformation. The same method also applies to transversality of maps: if $F : M' \to M$ is a smooth mapping, if $A \subset M'$ and if $B \subset M$ then by composing F with the time $= 1$ flow of an arbitrarily small smooth vector field we can guarantee that the resulting map $\tilde{F} : M' \to M$ takes A transversally to B.

Appendix C. Proof of Theorem 18.7

The following lemma provides lifts of morphisms in the derived category, see [50]

C.1. Lemma. *Let $A^\bullet, B^\bullet$ be a objects in the derived category. Suppose $\underline{\underline{H}}^r(A^\bullet) = 0$ for all $r > p$ and suppose that $\underline{\underline{H}}^r(B^\bullet) = 0$ for all $r < p$. Then the natural map*

$$\mathrm{Hom}_{D^b_c(X)}(A^\bullet, B^\bullet) \to \mathrm{Hom}_{Sh(X)}(\underline{\underline{H}}^p(A^\bullet), \underline{\underline{H}}^p(B^\bullet))$$

is an isomorphism.

Proof. When we wrote IH II, Verdier (who was one of the referees) showed us how to replace our 4 page proof with the following simple proof. Up to quasi-isomorphism it is possible to replace the complexes $A^\bullet$, $B^\bullet$ with complexes

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & A^{p-1} & \xrightarrow{d_A} & A^p & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ \cdots & \longrightarrow & 0 & \longrightarrow & I^p & \xrightarrow[d_B]{} & I^{p+1} & \longrightarrow & I^{p+2} & \longrightarrow & \cdots \end{array}$$

where I^r are injective. This means that a morphism in the derived category is represented by an honest morphism between these complexes, that is, a mapping

$$\phi : \underline{\underline{H}}^p(A^\bullet) = \operatorname{coker}(d_A) \to \ker(d_b) = \underline{\underline{H}}^p(B^\bullet). \qquad \square$$

C.2. Proof of Theorem 18.7. We have a Whitney stratification of W and inclusions

$$U_2 \xrightarrow[j_2]{} U_3 \xrightarrow[j_3]{} \cdots \xrightarrow[j_{n-1}]{} U_n \xrightarrow[j_n]{} U_{n+1} = W$$

Let us suppose that $A^\bullet$ is constructible with respect to this stratification and that it satisfies the support (but not necessarily the co-support) conditions, that is

$$H^r(A^\bullet)_x = 0 \text{ for } r \geq p(c) + 1$$

whenever $x \in X^{n-c}$ lies in a stratum of codimension c. Fix $k \geq 2$ and consider the situation

$$U_k \xrightarrow[j_k]{} U_{k+1} \xleftarrow[i_k]{} X^{n-k}$$

where X^{n-k} is the union of the codimension k strata. Let $A_k^\bullet = A^\bullet | U_k$. Let $\bar{q}$ be the complementary perversity, $q(c) = c - 2 - p(c)$. The following proposition says that the vanishing of the stalk cohomology with compact supports $H^r(i_x^! A^\bullet)$ is equivalent to the condition that the attaching map is an isomorphism:

C.3. Proposition. *The following statements are equivalent.*

(1) $A_{k+1}^\bullet \cong \tau_{\leq p(k)} R j_{k*} A_k^\bullet$
(2) $\underline{\underline{H}}^r(A_{k+1}^\bullet)_x \to \underline{\underline{H}}^r(R j_{k*} A_k^\bullet)_x$ *is an isomorphism for all* $x \in X^{n-k}$
(3) $H^r(i_k^! A_{k+1}^\bullet) = 0$ *for all* $r \leq p(k) + 1$
(4) $H^r(i_x^! A_{k+1}^\bullet) = 0$ *for all* $r < n - q(k)$ *for all* $x \in X^{n-k}$

Since the sheaf $\underline{\underline{IC}}_{\bar{p}}^\bullet[-n]$ satisfies these conditions, this proposition (and induction) gives another proof of the isomorphism $\underline{\underline{IC}}_{\bar{p}}^\bullet[-n] \cong \underline{\underline{P}}_{\bar{p}}^\bullet$ of Theorem 18.3.

Proof. Items (1) and (2) are equivalent because there is a canonical morphism

$$A_{k+1}^\bullet \to R j_{k*} j_k^* A^\bullet = R j_{k*} A_k$$

truncation $\tau_{\leq p(k)}$ leaves an isomorphism in degrees $\leq p(k)$. Items (3) and (4) are equivalent because $i_x : \{x\} \to X^{n-k}$ is the inclusion into a manifold so $i_x^! = i_k^![n-k]$, and because $r <$

$p(k)+2+(n-k)=n-(k-2-p(k))=n-q(k)$. Items (2) and (3) are equivalent because there is a distinguished triangle,

$$\begin{tikzcd} Ri_{k*}i_k^!(A^\bullet_{k+1}) \arrow[rr] & & A^\bullet_{k+1} \arrow[dl, "\alpha"] \\ & Rj_{k*}j_k^*(A^\bullet_{k+1}) \arrow[ul] & \end{tikzcd}$$

and therefore an exact sequence on stalk cohomology as follows:

$$\begin{tikzcd} \underline{\underline{H}}^{p+2}(i_k^!A^\bullet_{k+1})_x \arrow[r] & \underline{\underline{H}}^{p+2}(A^\bullet_{k+1})_x \arrow[r] & \underline{\underline{H}}^{p+2}(Rj_{k*}A^\bullet_k)_x \\ \underline{\underline{H}}^{p+1}(i_k^!A^\bullet_{k+1})_x \arrow[r] & \underline{\underline{H}}^{p+1}(A^\bullet_{k+1})_x \arrow[r] & \underline{\underline{H}}^{p+1}(Rj_{k*}A^\bullet_k)_x \arrow[ull] \\ \underline{\underline{H}}^{p}(i_k^!A^\bullet_{k+1})_x \arrow[r] & \underline{\underline{H}}^{p}(A^\bullet_{k+1})_x \arrow[r, "\alpha"'] & \underline{\underline{H}}^{p}(Rj_{k*}A^\bullet_k)_x \arrow[ull] \\ \underline{\underline{H}}^{p-1}(i_k^!A^\bullet_{k+1})_x \arrow[r] & \underline{\underline{H}}^{p-1}(A^\bullet_{k+1})_x \arrow[r, "\alpha"'] & \underline{\underline{H}}^{p-1}(Rj_{k*}A^\bullet_k)_x \arrow[ull] \end{tikzcd}$$

Now use the fact that the yellow highlighted terms are zero and the green highlighted morphisms are isomorphisms to conclude the proof of the Proposition. □

C.4. Continuation of the proof of Theorem 18.7. Now let us show that if $\mathcal{E}_1, \mathcal{E}_2$ are local systems on U_2 and if $A^\bullet = \underline{\underline{P}}^\bullet_{\bar{p}}(\mathcal{E}_1)$ and if $B^\bullet = \underline{\underline{P}}^\bullet_{\bar{p}}(\mathcal{E}_2)$ then we have an isomorphism

$$\mathrm{Hom}_{Sh}(\mathcal{E}_1, \mathcal{E}_2) \cong \mathrm{Hom}_{D^b_c(X)}(A^\bullet, B^\bullet).$$

As before, let $A^\bullet_{k+1} = A^\bullet|U_{k+1} = \tau_{\leq p(k)} Rj_{k*}A^\bullet_k$. Assume by induction that we have established an isomorphism

$$\mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_2) \cong \mathrm{Hom}_{D^b_c(U_k)}(A^\bullet_k, B^\bullet_k).$$

Using the above triangle for $B^\bullet$ we get an exact triangle of $\underline{\underline{\mathrm{RHom}}}$ sheaves,

$$\begin{tikzcd} \underline{\underline{\mathrm{RHom}}}^\bullet(A^\bullet_{k+1}, Ri_{k*}i_k^!B^\bullet_{k+1}) \arrow[rr] & & \underline{\underline{\mathrm{RHom}}}^\bullet(A^\bullet_{k+1}, B^\bullet_{k+1}) \arrow[dl, "\alpha"] \\ & \underline{\underline{\mathrm{RHom}}}^\bullet(A^\bullet_{k+1}, Rj_{k*}B^\bullet_k) \arrow[ul, "{[1]}"] & \end{tikzcd}$$

By Lemma C.1 and the support conditions, we see that α is an isomorphism in degree zero,

$$\mathrm{Hom}_{D^b_cX}(A^\bullet_{k+1}, B^\bullet_{k+1}) = H^0(U_{k+1}; \underline{\underline{\mathrm{RHom}}}^\bullet(A^\bullet_{k+1}, B^\bullet_{k+1}) \cong H^0(U_{k+1}; \underline{\underline{\mathrm{RHom}}}^\bullet(A^\bullet_{k+1}, Rj_{k*}B^\bullet_k)).$$

Moreover,

$$Rj_{k*}\underline{\underline{\mathrm{RHom}}}^\bullet(A_k^\bullet, B_k^\bullet) \cong Rj_{k*}\underline{\underline{\mathrm{RHom}}}^\bullet(j_k^* A_{k+1}^\bullet, B_k^\bullet) \cong \underline{\underline{\mathrm{RHom}}}^\bullet(A_{k+1}^\bullet, Rj_{k*}B_k^\bullet)$$

by the standard identites (above), whose cohomology is

$$H^0(U_{k+1}; Rj_{k*}\underline{\underline{\mathrm{RHom}}}^\bullet(A_k^\bullet, B_k^\bullet)) \cong H^0(U_k; \underline{\underline{\mathrm{RHom}}}^\bullet(A_k^\bullet, B_k^\bullet)) \cong \mathrm{Hom}_{D_c^b(U_k)}(A_k^\bullet, B_k^\bullet).$$

So, putting these together we have a canonical isomorphism

$$\mathrm{Hom}_{D_c^b(U_k)}(A_k^\bullet, B_k^\bullet) \cong \mathrm{Hom}_{D_c^b(U_{k+1}}(A_{k+1}^\bullet, B_{k+1}^\bullet)$$

which was canonically isomorphic to $\mathrm{Hom}_{Sh}(\mathcal{E}_1, \mathcal{E}_2)$ by induction. This completes the proof of the theorem, but the main point is that the depth of the argument is the moment in which Lemma C.1 was used in order to lift a morphism $A_{k+1}^\bullet \to Rj_{k*}B_k^\bullet$ to a morphism $A_{k+1}^\bullet \to B_{k+1}^\bullet$. □

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