On the Topology of Algebraic Torus Actions

by

M. Goresky[1] and R. MacPherson[2]

To T. A. Springer

On his sixtieth birthday

§ 1. Introduction.

Suppose a compact complex algebraic variety X has an action of an algebraic torus $(\mathbb{C}^*)^n$. As a Lie group, the algebraic torus is the product of two topological subgroups: the compact torus $(S^1)^n \subset (\mathbb{C}^*)^n$ and $(\mathbb{R}^+)^n \subset (\mathbb{C}^*)^n$, where $\mathbb{R}^+$ is the positive reals. In this note, we determine the following information:

1. the topology of the orbit space $B = X/(S^1)^n$, and
2. the topological structure of the $(\mathbb{R}^+)^n$ action on B.

By the topological structure of the $(\mathbb{R}^+)^n$ action, we mean the orbits and the stabilizer subgroups $\mathrm{Stab}_{(\mathbb{R}^+)^n}(b) \subset (\mathbb{R}^+)^n$. Knowledge of this information goes a long way towards reconstructing X topologically, as explained in § 8.

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We express this information in terms of certain "torus action data" which can be associated to the $(\mathbb{C}^*)^n$ action on X. Torus action data is of two types: The first is a collection of polyhedra in Euclidean n-space. The second is a collection of algebraic varieties Z_F and some algebraic maps $\zeta_{FG}: Z_F \longrightarrow Z_G$ between them. If X had dimension k and the $(\mathbb{C}^*)^n$ action is effective, then the varieties Z_F have dimension at most $k-n$.

The association of polyhedra to the torus action is done by the moment map ([A1], [GS], [MW], [K]). The varieties Z_F are the various symplectic quotients associated to different points in the image of the moment map. These may also be identified with the geometric invariant theory quotients of various subvarieties of semistable points in X. All of this is standard. The new ingredient here is the collection of algebraic maps ζ_{FG}: $Z_F \longrightarrow Z_G$ and the role that they play in reconstructing the quotient space B.

This is a largely expository paper. The results are easy consequences of what are by now standard techniques. Our main contribution consists of an efficient presentation of the rather complicated picture of the orbit structure of a torus action.

§ 2. Definitions. In this section we give some elementary topological definitions which will be used throughout the paper.

A *piecification* of a topological space X is a partially ordered set $\mathcal{F}$ (with partial ordering denoted $\leq$) and a choice for each $F \in \mathcal{F}$ of a subset (or "piece") $X^F \subset X$ such that

(a) If $F \neq G$ then $X^F \cap X^G = \phi$

(b) $\cup \{X^F \mid F \in \mathcal{F}\} = X$

(c) If $X^G \cap \overline{X^F} \neq \phi$ then $G \leq F$

<u>Remarks</u>. A stratification is a piecification, however a piecification is more general: the pieces may be singular, and a piecification does not necessarily satisfy the axiom of the frontier (i.e. the closure of a piece is not necessarily a union of pieces). We allow the possibility that $X^F = \phi$. The partial ordering axiom (c) implies that each piece is locally closed.

<u>Definition</u>. A <u>space-valued cofunctor</u> $\mathcal{Z}$ on a partially ordered set $\mathcal{F}$ is a collection of topological spaces Z_F (indexed by the elements of $\mathcal{F}$) together with continuous maps

$$\zeta_{FG} : Z_F \longrightarrow Z_G$$

whenever $G \leq F$, with the property that if $H \leq G \leq F$ then $\zeta_{FH} = \zeta_{GH}\zeta_{FG}$ and ζ_{FF} is the identity.

<u>Definition</u>. Suppose $\mathcal{F}$ is a partially ordered set, X is a piecified space with pieces indexed by $\mathcal{F}$, and $\mathcal{Z}$ is a space-valued cofunctor on $\mathcal{F}$. The <u>realization</u> $R(\mathcal{Z})$ <u>over X</u> of the triple $(\mathcal{Z}, X, \mathcal{F})$ is the topological space

$$R(\mathcal{Z}) = \coprod_{F\in\mathcal{F}} Z_F \times \overline{X^F} / \sim$$

where $\sim$ identifies a point $(z,x) \in Z_F \times \overline{X^F}$ with $(\zeta_{FG}(z), x)$ whenever $x \in X^G \cap \overline{X^F}$.

<u>Example of a realization</u>: The mapping cylinder. Suppose the partially ordered set $\mathcal{F}$ consists of two elements $G \leq F$. Let $X = [0,1]$ with piecification $X^G = \{0\}$ and $X^F = (0,1]$. A cofunctor $\mathcal{Z}$ over $\mathcal{F}$ is a pair of spaces Z_F, Z_G, together with a continuous map $\zeta_{FG}: Z_F \longrightarrow Z_G$. The realization $R(\mathcal{Z})$ over X is the mapping cylinder of ζ_{FG}.

<u>Remarks</u>. The realization is canonically piecified with pieces

$$R(\mathscr{Z})^F = Z_F \times X^F$$

The realization comes with an obvious projection $\pi\colon R(\mathscr{Z}) \longrightarrow X$, which is proper if and only if each of the spaces Z_F is compact. $R(\mathscr{Z})$ is Hausdorff if X is Hausdorff and each Z_F is Hausdorff (this uses the commutation relations). $R(\mathscr{Z})$ is locally compact if X is locally compact, each Z_F is locally compact and each ζ_{FG} is proper.

§ 3. Torus Action Data

In this section we define a collection of data which can be obtained from any projective variety X with an action of an algebraic torus. In § 5 we will show how to reconstruct the topological space $X/(S^1)^n$ from this data.

Recall that a convex polyhedron $C \subset \mathbb{R}^n$ is the convex hull of a finite set of points. Its affine hull is the smallest affine subspace A containing C. The interior C^o of C is the topological interior of C, viewed as a subspace of A. The interior of a point is itself. The span of C is the Euclidean subspace span(C) which is obtained by translating the affine hull A so that it passes through the origin.

Definition. TA Data consists of the following four ingredients: TAD1, TAD2, TAD3, TAD4 :

TAD1 is a finite collection $\mathscr{C}$ of (closed) convex polyhedra (of various dimensions, possibly overlapping, possibly sharing interior points) in $\mathbb{R}^n$ such that

(a) If $C \in \mathscr{C}$ then each face D of C is also an element of $\mathscr{C}$.

(b) Each $C \in \mathscr{C}$ is rational, i.e. the Euclidean subspace span(C) $\subset \mathbb{R}^n$ has a basis consisting of integral points $b_1, \ldots, b_r \in \mathbb{Z}^n$.

Remarks: We obtain a partial order on $\mathscr{C}$ by defining

$$D \leq C \iff D \text{ is a face of } C$$

<u>Define</u> $P = \cup \mathscr{C}$

to be the compact subset of $\mathbb{R}^n$ which is the union of all these polyhedra. There is a natural (coarsest) piecification $\mathscr{F}$ of the topological space P with the property that each $C \in \mathscr{C}$ is a union of pieces: two points $x,y \in P$ are in the same piece of P if and only if they are contained in <u>exactly</u> the same convex polyhedra $C \in \mathscr{C}$. Thus $\mathscr{F}$ is the set of subsets of $\mathscr{C}$ and is partially ordered by inclusion. The pieces of P are then given by

$$P^F = \cap \{C \mid C \in F\} - \cup \{C \mid C \in \mathscr{C}-F\}$$

for each subset $F \subset \mathscr{C}$.

We remark that this piecification of P is in fact a Whitney stratification and in particular it satisfies the axiom of the frontier: $P^F \cap \overline{P^G} \neq \phi \iff P^F \subset \overline{P^G} \iff F \subset G$

<u>TAD 2</u> is a cofunctor $\mathscr{R}$ of complex (not necessarily compact) algebraic varieties over the partially ordered set $\mathscr{C}$, i.e. for each $C \in \mathscr{C}$ an algebraic variety R^C, and for each face $D \leq C$ an algebraic map $\rho_{CD}: R^C \longrightarrow R^D$.

<u>TAD 3</u> is a cofunctor $\mathscr{Z}$ of complex algebraic varieties over $\mathscr{F}$, i.e. for each $F \in \mathscr{F}$ an algebraic variety Z_F and for each relation $G \leq F$ an algebraic map $\varsigma_{FG}: Z_F \longrightarrow Z_G$.

<u>TAD 4</u> is a choice, for each $F \in \mathscr{F}$ and for each $C \in \mathscr{C}$ such that $P^F \subset C^o$, of an inclusion

$$i_F^C: R^C \longrightarrow Z_F$$

We shall denote the image $i_F^C(R^C)$ by Z_F^C . These data are furthermore assumed to satisfy the following axioms:

<u>Axiom 1</u>. Each Z_F is piecified by the images $Z_F^C = i_F^C(R^C)$, where C is allowed to vary over the partially ordered set

$$\mathscr{C}_F = \{ C \in \mathscr{C} \mid C^o \supset P^F \}$$

(which is partially ordered by containment, i.e.

$$D \leq C \Leftrightarrow D \subset C)$$

<u>Axiom 2</u>. If $G \leq F \in \mathcal{F}$, and if $C \in \mathcal{C}$, with $P^F \subset C^o$ and $P^G \subset C^o$ then

$$i_F^C \circ \varsigma_{FG} = i_G^C$$

<u>Axiom 3</u>. If $G \leq F \in \mathcal{F}$ and if $D \leq C \in \mathcal{C}$ with $P^G \subset D^o$ and $P^F \subset C^o$ then the following diagram commutes:

$$\begin{array}{ccc} R^C & \xrightarrow{\rho_{CD}} & R^D \\ {\scriptstyle i_F^C}\downarrow & & \downarrow{\scriptstyle i_G^D} \\ Z_F & \xrightarrow{\rho_{FG}} & Z_G \end{array}$$

<u>§ 4: A Torus Action Gives Rise to TA Data.</u>

Suppose X is a projective complex algebraic variety with an action of the algebraic torus $(\mathbb{C}^*)^n$. We assume the torus action extends to a linear action on the ambient projective space $\mathbb{P}^N$. Choose a Kaehler metric on $\mathbb{P}^N$ which is invariant under the compact torus $(S^1)^n \subset (\mathbb{C}^*)^n$ and let $\mu\colon X \longrightarrow \mathbb{R}^n$ be the (restriction to X of the) associated moment map ([K], [A1], [MS], [A2], [GS]). This map factors

$$X \xrightarrow{\alpha} B \xrightarrow{\beta} \mathbb{R}^n$$

through the quotient space $B = X/(S^1)^n$.

<u>TAD 1</u>: We define a collection $\mathcal{C}$ of convex polyhedra in $\mathbb{R}^n$ as follows: the closure in X

$$\overline{T} = \overline{(\mathbb{C}^*)^n \cdot x}$$

of each torus orbit projects to a polyhedron $C = \mu(\overline{T})$ and the torus orbit itself $(\mathbb{C}^*)^n \cdot x$ projects to the interior C^o of C ([A1], [GS], [K])

<u>Proposition.</u> The polyhedra obtained in this manner constitute TAD 1, i.e. they satisfy the following hypotheses:

1. Each $C = \mu(\bar{T})$ is a <u>rational</u> convex polyhedron

2. Only finitely many polyhedra ppear in the collection $\mathscr{C}$

3. If $C = \mu(\bar{T})$ is a polyhedron in this collection then each face of C is also the μ-image of a torus orbit closure.

This collection $\mathscr{C}$ of polyhedra indexes a canonical piecification of X as follows:

<u>Definition.</u> Let $C \in \mathscr{C}$. A point $x \in X$ is in the piece X^C if and only if the convex polyhedron corresponding to the orbit through x is equal to C, i.e.

$$X^C = \{x \in X \mid \mu(\overline{(\mathbb{C}^*)^n \cdot x}) = C\}$$

<u>TAD 2</u>: Define a space-valued cofunctor $\mathscr{R}$ of complex algebraic varieties over $\mathscr{C}$ as follows: for each convex polyhedron $C \in \mathscr{C}$ let

$$R^C = X^C / (\mathbb{C}^*)^n$$

If $D \leq C$ then there is a unique map $\rho_{CD} : R^C \longrightarrow R^D$ which can be characterized as follows: suppose $x \in X^C$ is a lift of $\bar{x} \in R^C$ and suppose $y \in X^D$ is a lift of $\bar{y} \in R^D$. Then $\rho_{CD}(\bar{x}) = \bar{y}$ if and only if $y \in \overline{(\mathbb{C}^*)^n \cdot x}$.

<u>Proposition</u>. Each map ρ_{CD} is well defined and algebraic.

Now let $P = \mu(X)$ denote the union of the convex polyhedra defined above and let $\mathscr{F}$ be the natural piecification of P as described in § 2.

<u>TAD 3</u>. Define a space-valued cofunctor $\mathscr{Z}$ over $\mathscr{F}$ as follows: given $F \in \mathscr{F}$ choose any $p \in P^F$ and set

$$Z_F = \beta^{-1}(p) = \mu^{-1}(p)/(S^1)^n$$

(This is the "symplectic quotient" which is identified with a particular "geometric invariant theory" algebro-geometric quotient [K], [M], [A2]. It is known that the symplectic quotient does not depend on the point $p \in P^F$).

If $G < F \in \mathcal{F}$ then we obtain a map

$$\zeta_{FG}: Z_F \longrightarrow Z_G$$

which is characterized in the following way: Let $p' \in Z_F$. This means that $p = \beta(p') \in P^F$. Choose a lift $\tilde{p} \in \alpha^{-1}(p')$. Choose $q \in P^G$. Then the closure of the torus orbit $\overline{(\mathbb{C}^*)^n.\tilde{p}}$ intersects $\mu^{-1}(q)$ in a single $(S^1)^n$ orbit, thus determining a single point

$$q' \in \mu^{-1}(q)/(S^1)^n = Z_G.$$

We define $\zeta_{FG}(p') = q'$.

Proposition. The preceding choices of Z_F and ζ_{FG} are well defined and satisfy the criteria of TAD 2, i.e. each Z_F is an algebraic variety and each ζ_{FG} is an algebraic map.

TAD 4. If $P^F \subset C^o$ then there is an inclusion ι_F^C: $R^C \longrightarrow Z_F$ because (by [K]) for any choice of point $p \in P^F \subset C^o$ there is a natural identification

$$R^C = X^C / (\mathbb{C}^*)^n \cong (\mu^{-1}(p) \cap X^C) / (S^1)^n$$

and it is clear that this second description of R^C is a subset of

$$Z_F = \mu^{-1}(p) / (S^1)^n.$$

Proposition. The data TAD1, TAD2, TAD3, TAD4 defined here satisfy the axioms AX1, AX2, AX3.

Sketch of proof. For each $F \in \mathcal{F}$ there is a $(\mathbb{C}^*)^n$ - invariant set of semistable points,

$$X_F^{ss} = \cup \{X^C \mid P^F \subset C\}$$

consisting of the union of those pieces X^C such that the closure of the μ-image of X^C contains the stratum P^F. Although the topological quotient $X_F^{ss}/(\mathbb{C}^*)^n$ may not even be Hausdorff, there is a categorical quotient ([M]), i.e. an algebraic variety (which we still denote by $X_F^{ss}/(\mathbb{C}^*)^n$) with the property that whenever $f: X_F^{ss} \longrightarrow Y$ is an algebraic map which takes each torus orbit to a single point, then f factors through an algebraic map

$$g: X_F^{ss}/(\mathbb{C}^*)^n \longrightarrow Y.$$

By [K], the categorical quotient can be identified with

$$Z_F = \mu^{-1}(p)/(S^1)^n$$

for any $p \in P^F$. (Neither Mumford nor Kirwan emphasize the fact that X^{ss} and Z vary with F. In Mumford's language, [M] p.148, a choice of lift of the action of $G = (\mathbb{C}^*)^n$ to the invertible sheaf L must be made, while Kirwan chooses a basepoint p = {0}, or equivalently, an embedding [K] p. 102 of G into PGL(n+1). Kirwan's choice of basepoint p does not necessarily correspond to Mumford's choice of X^{ss}.)

If $G \leq F$ then $X_F^{ss} \subset X_G^{ss}$ so we obtain an algebraic map

$$\zeta_{FG}: \ X_F^{ss}/(\mathbb{C}^*)^n \longrightarrow X_G^{ss}/(\mathbb{C}^*)^n$$

This agrees with the map ζ_{FG} as defined above because Mumford's categorical quotient is homeomorphic to the universal Hausdorff quotient.

§ 5: <u>Construction of the space B from TA Data</u>

Suppose we are given a collection of TA data, i.e.

<u>TAD1:</u> a finite collection $\mathscr{C}$ of convex polyhedra in $\mathbb{R}^n$ with union P = $\cup\, \mathscr{C}$ which is piecified by the decomposition $\mathscr{F}$,

<u>TAD2</u>: a space-valued cofunctor $\mathscr{X}$ of algebraic varieites defined over $\mathscr{C}$,

TAD3: a space-valued cofunctor $\mathcal{Z}$ of algebraic varieties defined over $\mathcal{F}$,

TAD4: a system of inclusions $i_F^C : R^C \longrightarrow Z_F$ which piecify Z_F into pieces indexed by the partially ordered set

$$\mathcal{C}_F = \{ C \in \mathcal{C} \mid P^F \subset C^o \}$$

Construction 1. Define a topological space B to be the realization (over P) $B = R(\mathcal{F})$ of the cofunctor triple $(\mathcal{Z}, P, \mathcal{F})$.

Construction 2. Construct a piecification of B indexed by $\mathcal{C}$ as follows: for each $C \in \mathcal{C}$ define a partially ordered set

$$\mathcal{F}_C = \{F \in \mathcal{F} \mid P^F \subset C^o\}.$$

This set indexes the pieces in the piecification of C^o and admits a cofunctor of spaces, $\mathcal{Z}_C$ which associates to any $F \in \mathcal{F}_C$ the algebraic subvariety

$$Z_F^C \subset Z_F$$

Definition. The piece B^C in the piecification of B is the realization $R(\mathcal{Z}_C)$ of the cofunctor triple $(\mathcal{Z}_C, C^o, \mathcal{F}_C)$

Remark. Since each Z_F^C is identified with R^C, and since C^o is a cell, there is a canonical homeomorphism

$$B^C \cong C^o \times R^C$$

and so B^C is foliated by subsets $C^o \times \{\text{point}\}$.

Construction 3. For each $C \in \mathcal{C}$ we associate a subgroup

$$St_C = \exp\ (\iota\ \mathrm{Ann}(\mathrm{span}(C))) \subset (\mathbb{R}^+)^n$$

as follows: Span(C) is a subspace of $\mathbb{R}^n$ which [A1], [GS] has been identified with the dual of the Lie algebra of $(S^1)^n$. Therefore its annihilator lies in the Lie algebra of $(S^1)^n$, and multiplication by $\iota = \sqrt{(-1)}$ identifies this with the Lie algebra of $(\mathbb{R}^+)^n$. The exponential map

$$\exp : \mathrm{Lie}\ (\mathbb{R}^+)^n \longrightarrow (\mathbb{R}^+)^n$$

is an isomorphism. This is summarized in the diagram

$$\begin{array}{ccccc} (\mathbb{R}^n)^* \cong \text{Lie } (S^1)^n & \xrightarrow[\iota]{\cong} & \text{Lie } (\mathbb{R}^+)^n & \xrightarrow[\exp]{\cong} & (\mathbb{R}^+)^n \\ \cup & & & & \cup \\ \text{Ann(span(C))} & & & & St_C \end{array}$$

__Theorem__. Suppose X is a projective algebraic variety with an action of the algebraic torus $(\mathbb{C}^*)^n$. Extract the corresponding TA Data, TAD1...TAD4. Let B be the space obtained from construction 1 as applied to this TAD, let B^C be the pieces obtained from construction 2 and let St_C be the subgroups obtained from construction 3. Then:

(1) there is a canonical homeomorphism $h : B \longrightarrow X / (S^1)^n$ such that, for each $C \in \mathscr{C}$ we have:

(2) h takes B^C homeomorphically to $X^C/ (S^1)^n$

(3) h takes each leaf $C^o \times \{\text{point}\} \subset B^C$ homeomorphically to a single $(\mathbb{R}^+)^n$ orbit in $X/(S^1)^n$

(4) for each $x \in X^C/(S^1)^n$, the isotropy subgroup $\text{Stab}_{(\mathbb{R}^+)^n}(x)$ is precisely the subgroup St_C.

__Example__. See [GGMS] for a family of examples where the polyhedra C are explicitly described and are in one to one correspondance with matroids of rank k on n elements, and where X is the Grassmann manifold $G_{n-k}(\mathbb{C}^n)$ with the usual action of the torus $(\mathbb{C}^*)^n$. The nonempty pieces in the piecification of the Grassmannian correspond to matroids which are representable over the complex numbers.

__§6 An example__.

Suppose that $(\mathbb{C}^*)^2$ acts on $X = \mathbb{CP}^3$, complex projective three space with homogeneous coordinates $(z_1: z_2: z_3: z_4)$, by the formula

$$(s, t) \bullet (z_1: z_2: z_3: z_4) = (z_1: sz_2: tz_3: stz_4)$$

The moment map is then given by

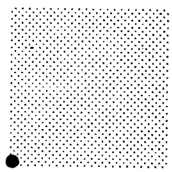
$$\mu([z_1:z_2:z_3:z_4]) = \frac{(|z_2|^2 + |z_4|^2 \ , \ |z_3|^2 + |z_4|^2)}{|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2}$$

The image P is the square

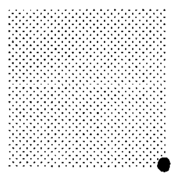
$$\{(x,y) \in \mathbb{R}^2 | \ 0 \le x \le 1, \ 0 \le y \le 1 \ \}$$

in $\mathbb{R}^2$. The various polygons C in $\mathcal{C}$ are listed below, along with the part of X which projects to their interior (where we make the convention that no coordinate typed z_i is zero):

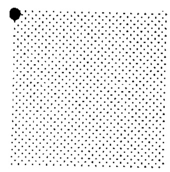
TYPE I:



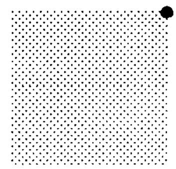
(1: 0: 0: 0)



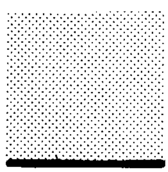
(0: 1: 0: 0)



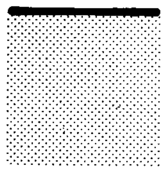
(0: 0: 1: 0)



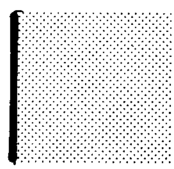
(0: 0: 0: 1)



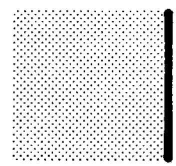
$(z_1:z_2:\ 0:\ 0)$



$(\ 0:\ 0:z_3:z_4)$

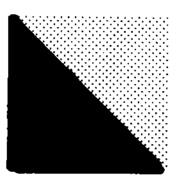


$(z_1:\ 0:z_3:\ 0)$

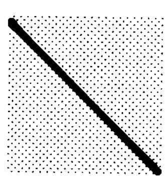


$(\ 0:z_2:\ 0:z_4)$

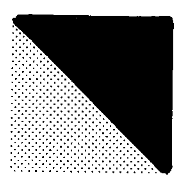
TYPE II



$(z_1:z_2:z_3:0)$

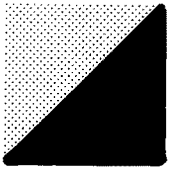


$(0:z_2:z_3:0)$

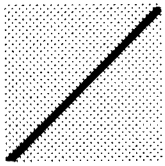


$(0:z_2:z_3:z_4)$

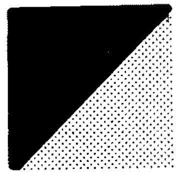
TYPE III



$(z_1:z_2:0:z_4)$



$(0:z_2:0:z_4)$

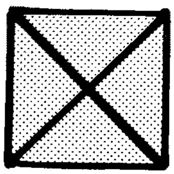


$(z_1:0:z_3:z_4)$

TYPE IV

$(z_1:\ z_2:\ z_3:\ z_4)$

The R^C is a point for C of TYPES I, II, or III, and is $\mathbb{C}^*$ for TYPE IV. The piecification $\mathcal{C}$ of P is like this:



Over each piece F on the edge of the square, Z_F is a point. For each piece F which is contained in the interior of the square, Z_F is a complex projective line, which we may identify with the standard complex projective line (with homogeneous coordinates $[y_1:y_2]$) and we may take each of the maps ζ_{FG} to be the identity. The inclusions $i_F^C\colon R^C \longrightarrow Z_F$ have as their image $(1 : 0)$ if C is of TYPE II, $(0 : 1)$ if C is of TYPE III, and the rest of $\mathbb{CP}^1$ for the C of TYPE IV.

It is easy to see that the realization of $\mathfrak{P}$ is the four sphere S^4, so it follows from the theorem that the orbit space B is S^4.

§7 Sketch of the proof. First we consider a lemma in pure topology. Suppose that we have:

1. a compact Hausdorff space B mapping to a piecewise linear subset of $\mathbb{R}^n$, $\beta: B \longrightarrow P$

2. a piecification of P (indexed by a partially ordered set $\mathcal{F}$) into finitely many piecewise linear subsets P^F, and which satisfies the axiom of the frontier: the closure $\overline{P^F}$ of any piece is a union of pieces,

3. a disjoint decomposition of B into (possibly uncountably many) topological ("open") balls of various dimensions,

such that:

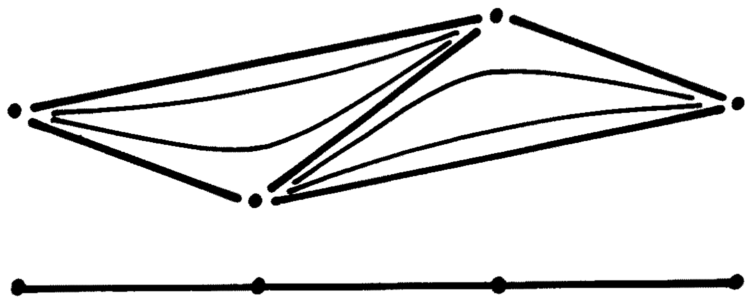
a. the map β takes each open ball homeomorphically onto a union of pieces F^o of P

b. the closure of each open ball is a "closed" ball which β takes homeomorphically to a union of pieces of P.

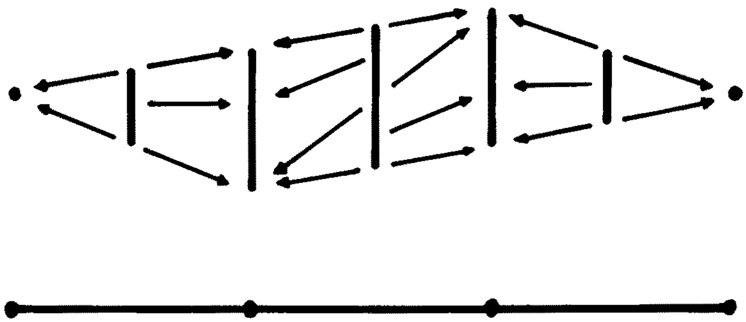
For each piece P^F of P, choose a point $p \in P^F$ and let Z_F be the fiber $\beta^{-1}(p)$ over p. Whenever G is a face of F let ζ_{FG}: $Z_F \longrightarrow Z_G$ be defined by the condition that $\zeta_{FG}(z)$ lies in the closure of the open ball through z. This forms space valued cofunctor on $\mathcal{F}$ which we call $\mathcal{Z}$.

<u>Lemma</u> Under these hypotheses, the space B is canonically homeomorphic to the realization $R(\mathcal{Z})$ over P of the cofunctor triple $(\mathcal{Z}, P, \mathcal{F})$.

For example, in the following picture, B is a subset of the plane, P is a subset of the line, β is vertical projection, and the open balls in B and pieces of P are sketched in.



In this example, the Z_F and the maps ζ_{FG} are as follows:



The proof of the lemma is straightforward: fix a stratum P^F of P and a point $p \in P^F$. By (3a) and (3c) there exists a unique homeomorphism

$$h_F\colon P^F \times \beta^{-1}(p) \longrightarrow \beta^{-1}(P^F) \subset B$$

which commutes with the projection to P^F and such that each $h_F(P^F \times \{\text{point}\})$ is a leaf of the foliation (i.e. lies in a single ball). Furthermore, by (3b), h_F extends to a continuous map

$$\overline{h_F}\colon \overline{P^F} \times \beta^{-1}(p) \longrightarrow \beta^{-1}(\overline{P^F}) \subset B$$

and it is easy to check that this is compatible with the relations defining the realization of $\mathcal{Z}$.

To apply this topological lemma to the theorem of §5, we take the decomposition of B into open balls to be the decomposition by $(\mathbb{R}^+)^n$ orbits. We claim that these satisfy the conditions of the topological lemma. This follows from the following facts about the composition

$$X \xrightarrow{\alpha} B \xrightarrow{\beta} \mathbb{R}^n$$

1. A single $(\mathbb{C}^*)^n$ orbit O in X projects to a topological open disk which is a single $(\mathbb{R}^+)^n$ orbit O in B, which projects homeomorphically to the interior C^o of a convex polyhedron C in P.

2. The closure of O in X consists of finitely many $(\mathbb{C}^*)^n$ orbits. It projects to the closure of $\bar{O}$ in B, which is a topological closed disk consisting of finitely many $(\mathbb{R}^+)^n$ orbits, each of which projects to the interior of a face of C in P.

To prove these facts, we observe that the moment map for the closure of O is the restriction to the closure of O of the moment map for X. However the closure of O is a toric variety, and these facts are standard for toric varieties.

§ 8. Reconstructing X.

In §5, we constructed the topology of $B = X/(S^1)^n$ and the stabilizer subgroups $\mathrm{Stab}_{(\mathbb{R}^+)^n}(b)$ from TAD. To what extent can the topology of X itself be reconstructed from this information?

The first remark is that the stabilizer subgroup $\mathrm{Stab}_{(S^1)^n}(x)$ of any point $x \in X$ projecting to b is determined by $\mathrm{Stab}_{(\mathbb{R}^+)^n}(b)$. This is because $\mathrm{Stab}_{(\mathbb{C}^*)^n}(x)$ is determined by $\mathrm{Stab}_{(\mathbb{R}^+)^n}(x) = \mathrm{Stab}_{(\mathbb{R}^+)^n}(b)$ since the $(\mathbb{C}^*)^n$ action is algebraic. In terms of TAD, the group $(S^1)^n$ identifies with $(\mathbb{R}^n)^*/(\mathbb{Z}^n)^*$, the space of linear functionals on $\mathbb{R}^n$ modulo those that take integral values on integral points. If x projects to $b \in B^C$, then $\mathrm{Stab}_{(S^1)^n}(x)$ is the subtorus

$$\mathrm{Stab}_{(S^1)^n}(x) = \mathrm{Ann}(\mathrm{span}(C))/\mathrm{Ann}(\mathrm{span}(C)) \quad (\mathbf{Z}^n)^*.$$

We call X a "piecified torus bundle" over B. The preimage of each piece B^C in B fibers over B^C with fiber the quotient torus $(S^1)^n/\mathrm{Stab}_{(S^1)^n}(x) = \mathrm{span}(C)^* / \Lambda_C$, where a covector in $\mathrm{span}(C)^*$ is in Λ iff it has some extension to $\mathbb{R}^n$ which takes integral values on $\mathbf{Z}^n$. The cohomology of X can be computed from the Leray spectral sequence for the projection from X to B. The above remarks imply that the E_2 term of this Leray spectral sequence can be computed from TAD alone.

The question of topologically reconstructing X from TAD reduces to the purely topological question of classifying "piecified torus bundles". For example, if the torus is a circle and all of the stabilizer subgroups are the identity, then X is a principal S^1 bundle over B, and its topology is determined by the first Chern class. It would be interesting to have a theory of first Chern classes classifying such "bundles" in general.

In case that the map from X to B admits a section, there is no twisting in the "piecified torus bundle" and the topology of X is determined by the TAD alone. This is the case, for example, when X is a toric variety.

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