ON THE TOPOLOGY OF COMPLEX ALGEBRAIC MAPS

by

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In this largely expository note we give some homological properties of algebraic maps of complex algebraic varieties which are rather surprising from the topological point of view. These include a generalisation to higher dimension of the invariant cycle theorem for maps to curves.

These properties are all corollaries of a recent deep theorem of Deligne, Gabber, Beilinson, and Bernstein which is stated in §2 . This theorem involves intersection homology and the derived category. One of our objects here is to popularize it by giving corollaries involving only ordinary homology. For this reason some readers may wish to begin with §3.

§1. Intersection homology.

For any complex algebraic variety V , let $D_c^b(V)$ be the algebraically constructible bounded derived category of the category sheaves of $\mathbb{Q}$-module on V . (Objects of $D_c^b(V)$ are bounded complexes of sheaves of $\mathbb{Q}$-modules on V that are cohomologically locally constant on the strata for some stratification of V by complex algebraic submanifolds; see ([GM2], §1.11).

If $\underline{\underline{S}}^\cdot \in D_c^b(V)$ and $U \subset V$, $H^k(U,\underline{\underline{S}}^\cdot)$ (resp. $H_c^k(U,\underline{\underline{S}}^\cdot)$) denotes the hypercohomology (resp. hypercohomology with compact supports) of the restriction of $\underline{\underline{S}}^\cdot$ to U .

If $p \in V$, let $\mathcal{D}_p^o$ be the "open disk" of points at distance less than ε from p , where distance is the usual Euclidean distance using some local analytic embedding of a neighborhood of P in $\mathbb{C}^N$. For $\underline{\underline{S}}^\cdot \in D_c^b(V)$, $H^k(\mathcal{D}_p^o,\underline{\underline{S}}^\cdot)$ and $H_c^k(\mathcal{D}_p^o,\underline{\underline{S}}^\cdot)$ are independent of the choices for small enough ε .

A local system on a space U is a locally constant sheaf of $\mathbb{Q}$-module on U .

Definition - Proposition ([G M2], §4.1).

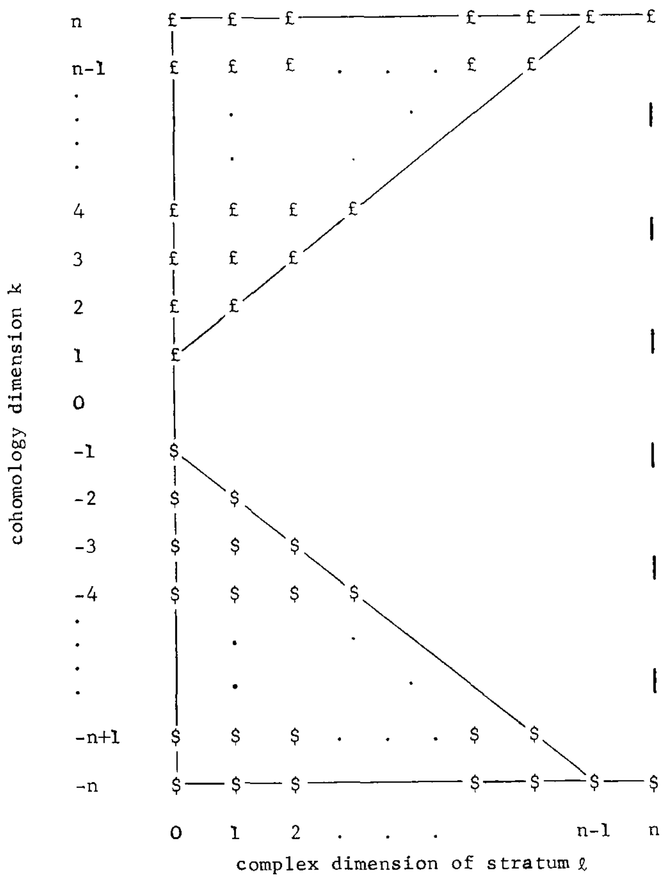
Let V be a complex algebraic variety of pure dimension n, U be a nonsingular Zariski open and dense subvariety, and L be a local system on U. Then there is an object $\underline{\underline{IC}}^\bullet(V,L)$ in $D^b_c(V)$ called the sheaf of intersection homology chains on V with coeficients in L, which is defined up to canonical isomorphism in $D^b_c(V)$ by the following properties :

1) $\underline{\underline{IC}}^\bullet(V,L)$ restricted to U is $L[n]$.

2) V can be stratified by strata $\{S_\ell\}$ where S_ℓ has dimension ℓ, so that if $p \in S_\ell$,

a) $H^k(\mathcal{D}^o_p, \underline{\underline{IC}}^\bullet(V,L)) = 0$ unless k is a dimension marked \$ in the figure below.

b) $H^k_c(\mathcal{D}^o_p, \underline{\underline{IC}}^\bullet(V,L)) = 0$ unless k is a dimension marked £ in the figure below.



Remarks.

1. The regions of marks and in the figure are sharp in the sense that if they are made smaller, existence fails and if they are made bigger, uniqueness fails.

2. $\underline{\underline{IC}}^{\cdot}(V,L)$ is independent of U in the sense that if L and L' agree where they are both defined, then $IC^{\cdot}(V,L)$ is equivalent in $D_c^b(V)$ to $\underline{\underline{IC}}^{\cdot}(V,L')$.

3. If L is the constant sheaf $\mathbb{Q}_U$, then $\underline{\underline{IC}}^{\cdot}(V,L)$ is denoted $\underline{\underline{IC}}^{\cdot}(V)$. It is a purely topological invariant of V .

Example. If V is nonsingular, $\underline{\underline{IC}}^{\cdot}(V)$ is $\mathbb{Q}_V[n]$.

§2. The decomposition theorem.

The following theorem was conjectured in [ΓM] §2.10 . It has been proved by P. Deligne, O. Gabber, A. Beilinson, and I. Bernstein. ([D4]) .

Theorem. Let $f : X \to Y$ be a proper projective map of complex algebraic varieties. Then there exist closed subvarieties $V_\alpha \subset Y$ and local systems L_α , and integers ℓ_α such that there is an equivalence in $D_c^b(Y)$

$$**)\qquad Rf_*\underline{\underline{IC}}^{\cdot}(X) \approx \bigoplus_\alpha i_*^\alpha\, \underline{\underline{IC}}^{\cdot}(V_\alpha,L_\alpha)\ [\ell_\alpha]$$

where $i^\alpha : V_\alpha \hookrightarrow Y$ is the inclusion.

Remarks.

1. A decomposition **) of $Rf_*\underline{\underline{IC}}^{\cdot}(X)$ can be found with the restriction that the varieties V_α are irreducible and the local systems L_α are indecomposable. Under this restriction the objects $i_*^\alpha\underline{\underline{IC}}^{\cdot}(V_\alpha,L_\alpha)[\ell_\alpha]$ are indecomposable in $D_c^b(Y)$ in the sense that whenever $i_*\underline{\underline{IC}}^{\cdot}(V_\alpha,L_\alpha)[\ell_\alpha] = \underline{\underline{S}}^{\cdot} \oplus \underline{\underline{T}}^{\cdot}$ then either $\underline{\underline{S}}^{\cdot}$ or $\underline{\underline{T}}^{\cdot}$ is equivalent to zero in $D_c^b(Y)$ ([GM2] , §4.1 , corollary 2) . Also with this restriction, the list of summands $i_*^\alpha\underline{\underline{IC}}\ (V_\alpha,L_\alpha)$ is uniquely determined. (We do not know if the category $D_c^b(Y)$ has such unique decompositions into indecomposables in general).

2. There are generalisations of the Poincaré duality theorem and the hard Lefschetz theorem relative to f . (They specialize to the classical theorems when X is nonsingular and Y is a point).

Let $Loc(V,\ell)$ be the direct sum of the L_α for those α such that $V = V_\alpha$ and $\ell = \ell_\alpha$. Then Poincaré duality ([GM2] , §5.3 and §1.6) says that there is an isomorphism.

$$\mathrm{Loc}(V,\ell) = \mathrm{Hom}(\mathrm{Loc}(V,-\ell)),\mathbb{Q}) \ .$$

Hard Lefschetz ([BB]) says that there exists a map $\Lambda : \mathrm{Loc}(V,\ell) \to \mathrm{Loc}(V,\ell+2)$ for all ℓ such that for $\ell > 0$

$$\Lambda^{\ell} : \mathrm{Loc}(V,-\ell) \xrightarrow{\simeq} \mathrm{Loc}(V,\ell)$$

is an isomorphism.

3. Although the theorem is a purely topological result about complex varieties, the proof uses characteristic p techniques. Such a decomposition exists for any complex which is pure in the sense of [D2] , §6.2. The complex $\underline{\underline{IC}}^{\cdot}(X)$ is pure by a result of Gabber [D3] and Rf_* preserves purity by [D2] , §6.2.

4. If X is nonsingular and of complex dimension n , then

$$H^k_c Rf_* \underline{\underline{IC}}^{\cdot}(X) \simeq H^k_c Rf_* \mathbb{Q}_X[n] \simeq H^{n+k}_c(X) \simeq H_{n-k}(X)$$

where $H_{n-k}(X)$ is the ordinary homology of X with rational coefficients. So the splitting **) gives rise to a decomposition of the homology of X . Using the numbering of dimensions of [GM1] and [CGM] ,

$$H_k(X) \approx \bigoplus_{\alpha} IH_{k_\alpha}(V_\alpha, L_\alpha)$$

where $k_\alpha = k - n + \dim V_\alpha + \ell_\alpha$. For an example where this decomposition is worked out in detail, see [BM] .

5. If $f : X \to Y$ is a resolution of singularities of Y , then one of the terms in the decomposition **) of $Rf_*\mathbb{Q}_X$ will be $\underline{\underline{IC}}^{\cdot}(X)$. Thus the intersection homology of X is contained in the ordinary homology of any resolution of X .

<u>Example</u>. Suppose $Y \subset \mathbb{C}P^{n+1}$ is the cone with vertex p over a nonsingular variety in $\mathbb{C}P^n$. Suppose $f : X \to Y$ is the blow-up of Y at p , D is the exceptional divisor, and $c_1(N) \in H^2(D)$ is the first Chern class of its normal bundle. Then

$$\mathrm{Loc}(Y,0) = \mathbb{Q}_{Y-P}$$

$$\mathrm{Loc}(p,\ell) = \mathbb{Q}_p \otimes (\mathrm{Image} \cap c_1(N))$$

and all of the other $\mathrm{Loc}(V,\ell)$ are zero.

The stalk at p of the cohomology sheaf $\underline{\underline{H}}^i(Rf_*\mathbb{Q}_X)$ is $H^i(D)$. It splits in pieces $H^i(\underline{\underline{IC}}^\cdot(Y,\mathbb{Q}))_p$ and $H^i(i_*^p\underline{\underline{IC}}^\cdot(p,\mathbb{Q}))_p$ as $H^i(D)$ splits into primitive and non-primitive cohomology.

§3. Resolutions.

Let X be a nonsingular complex algebraic variety and let $f : X \to Y$ be a proper projective algebraic map. For any point $p \in Y$, let $\mathcal{D}_p \subset Y$ be the set of points of distance at most ε from p and let $S \subset Y$ be the set of points of distance exactly ε from p . (Here "distance" means the usual Euclidean distance with respect to some local analytic embedding of a neighborhood of p in $\mathbb{C}^N$) . Let M be $f^{-1}(\mathcal{D}_p)$, B be $f^{-1}(S)$, and $\bar{f} : B \to S$ be the restriction of f .

$$\begin{array}{ccccc} B & \underset{i}{\subset} & M & \subset & X \\ \bar{f}\downarrow & & \downarrow & & \downarrow f \\ S & \subset & \mathcal{D}_p & \subset & Y \end{array}$$

It is well known from stratification theory that for small enough ε , M is a compact manifold with boundary B and the topological type of the pair (M,B) is independent of the choices.

Let $K \subset H_*(B)$ be the kernel of $i_* : H_*(B) \to H_*(M)$. In this section and the next, we address the following

<u>Question</u>. To what extent is K determined by the data $\bar{f} : B \to S$ (and the fact that f is algebraic) ?

<u>Remarks</u>. 1. Of the information in the long exact sequence in homology for the pair (M,B) , $K \subset H_*(B)$ is the only part that could be determined by these data since blowing up a point in $f^{-1}(p)$ will change $H_*(M)$.

2. Just from the topological fact that B is the boundary of the manifold M , we see that K is a maximal isotropic subspace for the intersection pairing on $H_*(B)$; i.e. $K = K^\perp$. (See [Do], prop. 9.6, p. 305). In particular $\dim K = \frac{1}{2} \dim H_*(B)$.

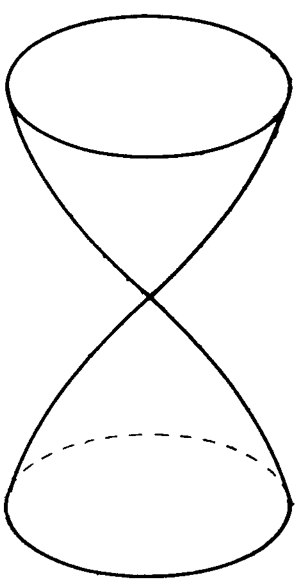
Corollary 1. If Y is an n dimensional variety with an isolated singular point at p, and $f : X \to Y$ is a resolution of singularities (so $\overline{f} : B \to S$ is a homeomorphism), then

$$K = H_n(B) \oplus H_{n+1}(B) \oplus \ldots \oplus H_{2n-1}(B)$$

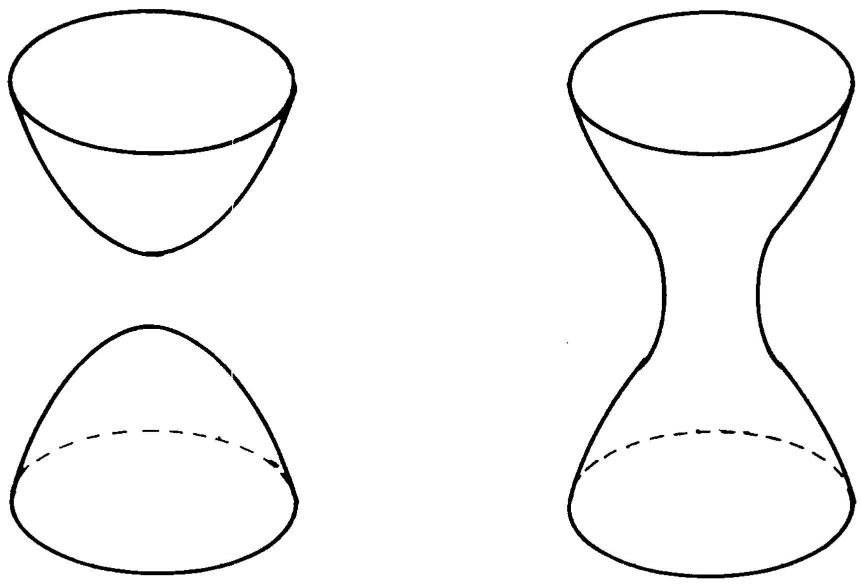
Proof. In $\mathcal{D}_p$ the decomposition **) of the theorem has the form $\underline{\underline{IC}}^{\cdot}(Y)$ plus terms concentrated at p. Only the term $\underline{\underline{IC}}^{\cdot}(Y)$ effects K. So the problem becomes the following : Which cycles in $H_*(S)$ are boundaries in $IH(\mathcal{D}_p)$? Here we use the interpretation of intersection homology of [GM1] and [CGM] (proved to be equivalent in [GM2]).

$\mathcal{D}_p$ is topologically a cone with base S and vertex p. Any cycle Z in S is the boundary of its cone to p. The cone is allowable as a chain in $\underline{\underline{IC}}^{\cdot}(B)$ if (and only if) the dimension of Z is at least n ([CGM], §2.1). So $H_n(S) \oplus H_{n+1}(S) \oplus \ldots$ is in K. Since it is a maximal isotropic subspace of $H_*(S)$, it is all of K.

Examples. The simplest example is a node (or normal crossing) of a curve. Topologically the picture is like this :



The corollary asserts that the resolution must be the figure on the left rather than the one on the right (as may be seen by several classical arguments).



For surfaces, the corollary follows from Grauert's blowing down criterion. For dimension three and more, it seems new.

§4. Generalized invariant cycle theorem.

Given $\overline{f} : B \to S$ as in the last section, we will construct a subspace $J \subset H_*(B)$. Choose a Whitney stratification of S by strata $\{U_\varphi\}$ of odd dimension such that $\overline{f}$ restricted to the inverse image of U_φ is a topological fibration onto U_φ for each φ . (This can be done : the U may be taken to be restrictions to S of a stratification of Y by complex manifolds with the similar fibration property. See [H] and [T] , p. 276). Choose a triangulation T of S so that each stratum U_α is a union of interiors of simplices. (See [G]). Let R be the union of all simplices Δ of the barycentric subdivision of T such that for all U_φ , $\dim (\Delta \cap U_\varphi) < \frac{1}{2} \dim U_\varphi$.

Definition . $J \subset H_*(B)$ is the image of the map

$$H_*(\overline{f}^{-1}(R)) \to H_*(B)$$

Lemma. J is independent of the choices $(\{U_\varphi\}$ and $T)$ in its construction. It is a maximal isotropic subspace of $H_*(B)$.

Example. If $\overline{f}$ is a topological fibration and S is a manifold of dimension $2m-1$, then $J = F_{m-1} H_*(B)$ where F_s denotes the fibration of $H_*(B)$ of the Leray spectral sequence for $\overline{f}$. (See [S], p. 473-4, theorem 1).

Remarks. For any "perversity" function p and any stratified map $B \to S$, we may similarly define a "perverse Leray filtered piece" $J_p \subset H_*(B)$ by using the inequality $\dim(\Delta \cap U_\varphi) \leq p(\dim U_\varphi)$. We conjecture that if $p(c)$ and $c-p(c)$ are both nondecreasing functions of c , then J_p is independent of the choices. If $p(c) = s$, then $J_p = F_s$.

Corollary 2. The subspace K is always a vector space complement to J in $H_*(B)$. (That is $J \cap K = \{0\}, J + K = H_*(B)$).

Proof. We decompose $H_*(B)$ as in §2 remark 4

$$H_*(B) = \bigoplus_\alpha IH_*(S \cap V_\alpha, L_\alpha)$$

Arguing as in the proof of corollary 1, we see that

$$K = \bigoplus_\alpha IH_{a_\alpha}(S \cap V_\alpha, L_\alpha) \oplus \dots \oplus IH_{2a_\alpha - 1}(S \cap V_\alpha, L_\alpha)$$

where $a_\alpha = \dim_{\mathbb{C}} V_\alpha$. We claim that

$$J = \bigoplus_\alpha IH_0(S \cap V_\alpha, L_\alpha) \oplus \dots \oplus IH_{a_\alpha - 1}(S \cap V_\alpha, L_\alpha)$$

From this, corollary 2 and the lemma clearly follow.

To establish the claim, let $R°$ be an open regular neighborhood of R . Then

$$J = \text{image}(H_*(\overline{f}^{-1}(R')) \to H_*(B)) = \bigoplus_\alpha \text{Image}(IH_*(R° \cap V_\alpha, L_\alpha) \to IH_*(S \cap V_\alpha, L_\alpha))$$

We will show that for all α .

$$IH_0(S \cap V_\alpha, L_\alpha) \oplus \dots \oplus IH_{a_\alpha - 1}(S \cap V_\alpha, L_\alpha) = \text{Image}(IH_*(R° \cap V_\alpha, L_\alpha) \to IH_*(S \cap V_\alpha, L_\alpha))$$

The inclusion $\subset$ follows from [GM1] §3.4 plus the fact that $R^{\overline{m}} \subset R \cap V_\alpha$ for $i \leq a_\alpha - 1$ since V_α is a union of strata U_φ. The inclusion $\supset$ then follows from the fact that J is self-annihilating under the intersection pairing. This fact may be seen by using stratified general position [M] to find a homeomorphism $h : V_\alpha \to V_\alpha$ isotopic to the identity such that $h(R \cap V_\alpha) \cap (R \cap V_\alpha)$ is empty.

Remark. For general f, corollary 2 is the best possible result on K from the data $\overline{f} : B \to S$, except for integrality considerations. For example, for $f : T \times T \to T$ where T is a curve of genus one, there is an automorphism of the topological fibration $\overline{f}$ taking any complement to $J \cap H_1(B, \mathbb{Z})$ in $H_1(B, \mathbb{Z})$ as a $\mathbb{Z}$-module to any other complement.

Example. If Y is a curve, corollary 2 is equivalent to the invariant cycle theorem:

> Let $\varphi : F \subset B$ be the inclusion of a fiber and let $\mu : H_*(F) \to H_*(F)$ be the monodromy map. Then the composed map
>
> $$H_{i+2}(M,B) \to H_{i+1}(B) \xrightarrow{\varphi^*} H_i(F)$$
>
> is a surjection to the kernel of $(1-\mu)$, i.e. to the invariant cycles (see [C], introduction).

This follows from the Wang exact sequence for the fibration $\overline{f} : B \to S$ over a circle ([S], p. 456, Cor. 6)

$$H_{i+1}(F) \xrightarrow{\varphi_*} H_{i+1}(B) \xrightarrow{\varphi^*} H_i(F) \xrightarrow{1-\mu} H_i(F)$$

and the fact that $J \cap H_{i+1}(B)$ is the image of φ_*

§5. Leray Spectral Sequence.

Corollary 3. Let $f : X \to Y$ and $f' : X' \to Y$ be two proper projective maps of non-singular complex algebraic varieties to Y. Then if $R^i f_* \mathbb{Q}_X$ and $R^i f_* \mathbb{Q}_{X'}$ are isomorphic for all i, then $H_*(X)$ and $H_*(X')$ are isomorphic. In fact the whole Leray spectral sequences of f and f' coincide.

Proof. By dévissage, the triples V_α, L_α, ℓ_α occuring in the decomposition **) may be determined from the $R^i f_* \mathbb{Q}_X$. Then $H_*(X)$ will be a direct sum of hypercohomology of the factors $\underline{\underline{IC}}^\cdot(V_\alpha, L_\alpha)$ with a dimension shift depending on ℓ_α. Similarly, the Leray spectral sequence for f will decompose into a sum of spectral sequences for the hypercohomology of each $\underline{\underline{IC}}^{\bullet}(V_{\alpha}, L_{\alpha})$ (see [GM2], §1.2).

<u>Examples 1</u>. If the $R^{i}f_{*}\mathbb{Q}_{X}$ are all locally constant sheaves, then all the V_{α} wil be Y, and the $\underline{\underline{IC}}^{\bullet}(V_{\alpha}, L_{\alpha})$ will be locally constant sheaves so their spectral sequences will degenerate at E^{2}. (This case was a result in [D1]).

2. For an example of a V such that the spectral sequence for $\underline{\underline{IC}}^{\bullet}(V)$ does not degenerate at E^{2}, take any surface S with a curve C such that $H_{1}(C) \to H_{1}(S)$ is not injective. Blow up enough points on C then blow down its reduced transform.

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