

The topological trace formula

By *Mark Goresky* and *Robert MacPherson* at Princeton

Abstract. The topological trace formula is a computation of the Lefschetz number of a Hecke correspondence C acting on the weighted cohomology groups, defined in [GHM], of a locally symmetric space X . It expresses this Lefschetz number as a sum of contributions from fixed point components of C on the reductive Borel Serre compactification of X . The proof uses the Lefschetz fixed point formula of [GM2].

1. Introduction

1.1. The goal. Although this paper is self contained, it is actually the fourth in a series of five papers ([GM1], [GM2], [GHM], this paper, and [GKM]) in which we derive a formula for the Lefschetz number of a Hecke correspondence acting on the weighted cohomology groups of any locally symmetric space X . For various reasons, the publication of this paper was delayed for many years, and it is now appearing after [GKM], which logically depends on results from this paper.

In [GM1] the formula is described (without proof) for the special case of locally symmetric spaces associated to $\mathbf{Sp}(4, \mathbb{R})$.

In [GM2] we address the general topological problem of determining the contribution from a single fixed point component to the Lefschetz number of an arbitrary “weakly hyperbolic” correspondence acting on a complex of sheaves on a compact stratified space. (An error [GM2] is corrected in §11.7 of the present paper.)

In [GHM] we construct a family of (complexes of) sheaves $\mathbf{W}^v \mathbf{C}^\bullet(\bar{X}, \mathbf{E})$ on the reductive Borel Serre compactification \bar{X} of the locally symmetric space X , with coefficients in a local system \mathbf{E} . The (hyper) cohomology of this complex is the weighted cohomology $\mathbf{W}^v H^*(\bar{X}, \mathbf{E})$. For various choices of v the weighted cohomology may be identified with the ordinary cohomology $H^*(X, \mathbf{E})$, the compact support cohomology $H_c^*(X, \mathbf{E})$, the L^2 cohomology $H_{(2)}^*(X, \mathbf{E})$ (when it is finite dimensional), or with Franke’s weighted L^2 cohomology [F], [N]. The goal is (a) to apply the formula of [GM2] to the action of a Hecke correspondence $C[g]$ on the weighted cohomology $\mathbf{W}^v H^*(\bar{X}, \mathbf{E})$, and (b) to sum these contributions over all the fixed point components in \bar{X} , so as to obtain a formula for the Lefschetz trace of $C[g]$ on the weighted cohomology.

For part (a) one must verify that each Hecke correspondence $C[g]$ is “weakly hyperbolic” in the sense of [GM2], and this is the first main result of the present paper, Theorem 11.9. In the process, we describe some of the rich geometry of Hecke correspondences and their fixed points. In Theorem 13.6, the second main result of this paper, we complete part (a) by describing the local contribution to the Lefschetz number in terms of roots and weights.

For part (b), one may sum the contributions from the fixed point components of \bar{X} in either the adelic setting or the discrete group setting. This is accomplished in the adelic setting using orbital integrals, in [GKM], Theorem 7.14 (p. 535). Theorem 7.14 of [GKM] uses Theorem 13.6 of the present paper as its starting point: it occurs as the expression for $L_Q(\gamma)$ on page 534 of [GKM].

When the L^2 cohomology of X is finite dimensional (the equal rank case), the Lefschetz number of the Hecke correspondence $C[g]$ acting on the L^2 cohomology $H_{(2)}^*(X, \mathbf{E})$ was computed by J. Arthur in [Ar1], [Ar3] using the trace formula. In this case the L^2 cohomology coincides with the “middle” weighted cohomology (see [GHM]), so we obtain an independent computation of this Lefschetz number. In [GKM], it was shown that these two computations agree. Consequently, the present paper completes an independent proof of Arthur’s formula.

In the (slightly more general) discrete group setting, the fixed points can be explicitly “counted” using double cosets. This is accomplished in Proposition 8.4 which some geometers may find to be more accessible than the orbital integrals of [GKM] (although they are in fact equivalent). As a consequence, we obtain a Lefschetz formula, Theorem 1.5 (the third main result of this paper), in the discrete group setting. These matters will now be described in more detail.

1.2. Geometric overview. For the purposes of this introduction, a *locally symmetric space* X is a complete connected Riemannian manifold with finite volume and non-positive curvature, such that every point $p \in X$ has a neighborhood U_p with a *Cartan symmetry*: an isometry $U_p \rightarrow U_p$ that takes p to itself and induces minus the identity on the tangent space to X at p . As for any manifold, we have $X = \Gamma \backslash D$ where D is the simply connected covering space of X and Γ is the fundamental group of X . Because X has nonpositive curvature, it follows that D is a Riemannian symmetric space of noncompact type, that is, the metric product of a negatively curved symmetric space from Cartan’s classification, and a Euclidean factor \mathbb{R}^n . The discrete group Γ acts by Riemannian automorphisms of D . We assume that this action is “arithmetic” (see §1.3).

Correspondences. We are interested in the automorphisms of X . A *morphism* $f : X \rightarrow Y$ of locally symmetric spaces is a local isometry; i.e. a map f that restricts to an isometry $U_p \rightarrow U_{f(p)}$ for appropriate choices of neighborhoods. Topologically, a morphism is a covering map of finite degree. There are finitely many morphisms $X \rightarrow X$. Instead, we consider a correspondence on X , i.e. a locally symmetric space C together with two morphisms c_1 and c_2 from C to X . We can think of $(c_1, c_2) : C \rightrightarrows X$ as a multi-valued function, whose values at $p \in X$ are the points in the set $c_2(c_1^{-1}(p))$. There is a rich supply of correspondences. They include the *Hecke correspondence*, see §1.3.

Lefschetz numbers. Consider a cohomology theory of X , such as the L^2 cohomology

$H_{(2)}^i(X)$. A correspondence $(c_1, c_2) : C \rightrightarrows X$ acts on $H_{(2)}^i(X)$ by sending a differential form ω to $C^*\omega = (c_1)_*(c_2)^*\omega$. (The map $(c_1)_*$ adds the differential form over the sheets of the finite covering map c_1 .) It is believed that the induced maps $C^* : H_{(2)}^i(X) \rightarrow H_{(2)}^i(X)$ carry deep number theoretic significance. One would like to compute them. Unfortunately, this is too difficult. Even the rank of $H_{(2)}^i(X)$ is too hard to compute in most cases. As often happens, however, a more accessible goal is the computation of the Lefschetz number

$$L(C) = \sum_i (-1)^i \text{Tr}(C^* : H_{(2)}^i(X) \rightarrow H_{(2)}^i(X)).$$

Our goal is to use the Lefschetz fixed point theorem to express the Lefschetz number

$$L(C) = \sum_F L(F)$$

as a sum over fixed point components F of some local contribution $L(F)$.

Compactifying X . The first obstacle is the fact that X is, in general, non-compact. (When X is compact, the Lefschetz formula was described by M. Kuga and J. H. Sampson [Ku].) There is no hope for a Lefschetz fixed point formula on a noncompact space. To see this, consider the example where $X = \mathbb{C}$ is the complex line with the self map that sends $x \in X$ to $x + 1$. The Lefschetz number for ordinary cohomology is 1. But there are no fixed points, so the right hand side is 0 no matter how $L(F)$ is interpreted. There are similar examples for L^2 cohomology and locally symmetric spaces X .

The solution is to pass to a compactification \bar{X} of X . We need a compactification $X \subseteq \bar{X}$ satisfying:

- (1) The L^2 cohomology of X can be expressed locally on \bar{X} .
- (2) The correspondence $(c_1, c_2) : C \rightrightarrows X$ extends to a compactified correspondence $(c_1, c_2) : \bar{C} \rightrightarrows \bar{X}$.
- (3) The singularities of \bar{X} are simple enough to allow a calculation of $L(F)$.

Remarks on these properties. In (1), “expressed locally” means that the L^2 cohomology is the cohomology of a complex of sheaves on \bar{X} . For (2), we want a functorially constructed compactification \bar{C} of C . The map C^* on L^2 cohomology should be induced by a lift of the complex of sheaves to \bar{C} . Properties (1) and (2) together will imply that there exists an expression $L(C) = \sum L(F)$ for $L(C)$ as a sum of contributions $L(F)$ over fixed point components F of \bar{C} , by applying the fixed point theorem of Grothendieck and Illusie [GI].

A lot of effort has gone in to constructing various compactifications of X . Most of these fail some of the criteria, however. For example, the toroidal compactification [AMRT] for Hermitian X satisfies (3) but neither (1) nor (2). The Borel-Serre compactification [BS] satisfies (2) and (3) but not (1). (It does satisfy (1) in the case of ordinary cohomology rather than L^2 cohomology. In this case, U. Weselmann, following suggestions of G. Harder, has carried out the Lefschetz computations ([We]).)

It is likely that (for sufficiently high rank) any compactification satisfying (1) and (2) must be singular. A well known example is the Baily-Borel compactification for Hermitian X . This satisfies (1) because of the Zucker conjecture (i.e. the Looijenga [Lo], Saper-Stern [SaS] theorem) which expresses the L^2 cohomology of X on the Baily-Borel compactification as the intersection cohomology, which is the cohomology of a complex of sheaves (see [GM4]). It satisfies (2) because it is functorial. However, the singularities of the Baily-Borel compactification are as complex as a locally symmetric space only slightly smaller than X and are too complicated to allow a direct computation of $L(F)$.

The first miracle is that there is a compactification satisfying all three properties: the reductive Borel-Serre compactification \bar{X} (defined for all X , Hermitian or not). In the Hermitian case, it may be thought of as a (non algebraic) partial resolution of singularities of the Baily-Borel compactification. The reductive Borel-Serre compactification satisfies property (2) because it is functorial with respect to morphisms of locally symmetric spaces. So in the diagram $(c_1, c_2) : \bar{C} \rightrightarrows \bar{X}$, the space \bar{C} is the reductive Borel-Serre compactification of C . It satisfies property (1) because of the existence of weighted cohomology described below, and it satisfies property (3) because its singularities may be explicitly constructed from certain nilmanifolds (see §1.4).

Weighted cohomology. The Lefschetz fixed point formula of this paper is for the *weighted cohomology groups* $W^\nu H^i(\bar{X}, \mathbf{E})$ where X is any locally symmetric space and \mathbf{E} is a local system over it. These were introduced in [GHM]. The weighted cohomology is the cohomology of a complex of sheaves $\mathbf{W}^\nu \mathbf{C}^\bullet(\mathbf{E})$. Therefore it satisfies property (1) mentioned above.

The weighted cohomology groups $W^\nu H^i(\bar{X}, \mathbf{E})$ depend on an auxiliary parameter ν called a *weight profile*. When X has finite dimensional L^2 cohomology, then $W^\nu H^i(\bar{X}, \mathbf{E}) = H_{(2)}^i(X, \mathbf{E})$ provided the weight profile ν is chosen to be the “middle weight” ([GHM], [N]), so our formula includes the L^2 case. Another weight profile gives the ordinary cohomology of X . A. Nair [N] has shown that for any weight profile, the weighted cohomology $W^\nu H^i(X, \mathbf{E})$ is equal to J. Franke’s “weighted L^2 cohomology” [F] for a particular weight function. For a leisurely account of the properties of weighted cohomology, see the introduction to [GHM].

The Lefschetz formula. Even on a compact space with mild singularities, the fixed point contribution $L(F)$ is usually too difficult to compute. The second miracle is that on the reductive Borel-Serre compactification \bar{X} , each correspondence is *hyperbolic*. A formula for the contributions $L(F)$ for hyperbolic correspondences was developed in [GM1], [GM2] expressly for the application in this paper to Hecke correspondences. A related result in a different language, which applies to functions (rather than correspondences) was discovered in [KS], Prop. 9.6.12.

The rest of the introduction proceeds as follows. The next section enumerates the fixed point components and determines their topology. Section 1.4 describes the local contribution from each fixed point component, and §1.5 adds them up to give the Lefschetz number $L(C)$.

1.3. The structure of a correspondence on \bar{X} . The theory of correspondences on \bar{X} is very self-referential. The reductive Borel-Serre compactification \bar{X} is a stratified space

whose strata are themselves locally symmetric spaces. The closure of such a stratum is its reductive Borel-Serre compactification. The fixed point components of a correspondence on X are (almost) locally symmetric. A correspondence restricted to a stratum of \bar{X} is itself a correspondence.

Obtaining X from G . In order to be precise, we need the language of algebraic groups. We use boldface symbols for linear algebraic groups, and Roman symbols for their Lie group of real points, for example, $G = \mathbf{G}(\mathbb{R})$. Throughout this paper we fix a reductive linear algebraic group \mathbf{G} defined over the rational numbers \mathbb{Q} . The *symmetric space* D for \mathbf{G} is defined to be $G/S_G K$. Here K is a maximal compact subgroup of G and S_G is the greatest \mathbb{Q} -split torus in the center of \mathbf{G} . The group G acts on D by Riemannian automorphisms. Let $X = \Gamma \backslash D = \Gamma \backslash G/S_G K$, where $\Gamma \subset \mathbf{G}(\mathbb{Q}) \subset G$ is assumed to be a neat arithmetic subgroup. This is the arithmeticity assumption on the locally symmetric space X of §1.2. By results of Margulis, in most cases the arithmeticity assumption is automatically satisfied. (The space D may have Euclidean factors because G may have a part of its center that is split over \mathbb{R} but not over \mathbb{Q} . After dividing by Γ , these Euclidean factors will become wound into tori.)

Rational parabolic subgroups \mathbf{P} of \mathbf{G} . If \mathbf{P} is a rational parabolic subgroup, let \mathbf{L}_P be its Levi quotient; let $\nu_P : \mathbf{P} \rightarrow \mathbf{L}_P$ be the projection; let \mathcal{U}_P be the unipotent radical; let $\mathbf{S}_P \subseteq \mathbf{L}_P$ be the maximal \mathbb{Q} -split torus in the center of \mathbf{L}_P ; and let Δ_P be the set of simple roots occurring in $\mathfrak{A}_P = \text{Lie}(\mathcal{U}_P)$. Let $K_P = K \cap L_P$ be the maximal compact subgroup of L_P which corresponds to K ; set $\Gamma_P = \Gamma \cap P$; and $\Gamma_L = \nu_P(\Gamma_P)$.

The reductive Borel-Serre compactification \bar{X} (§2.10). The strata of \bar{X} are indexed by Γ -conjugacy classes of rational parabolic subgroups \mathbf{P} of \mathbf{G} . The stratum $X_P \subseteq \bar{X}$ corresponding to the conjugacy class containing a parabolic \mathbf{P} is the locally symmetric space $\Gamma_L \backslash D_P$, where $D_P = L_P/S_P K_P$ is the symmetric space of the Levi factor \mathbf{L}_P . If $\mathbf{P} \subseteq \mathbf{Q}$ then the stratum X_P is contained in the closure \bar{X}_Q of X_Q (which is the reductive Borel-Serre compactification of X_Q).

Hecke correspondences. Let $g \in \mathbf{G}(\mathbb{Q})$. Let $\Gamma' \subset \Gamma \cap g^{-1}\Gamma g$ be a subgroup of finite index. This data determines a correspondence $(c_1, c_2) : C[g, \Gamma'] \rightrightarrows X$ as follows. Let $C[g, \Gamma'] = \Gamma' \backslash D$. The mapping c_1 is obtained by factoring the projection $d_1 : D \rightarrow \Gamma \backslash D = X$ through $C[g, \Gamma']$ which may be done since $\Gamma' \subset \Gamma$. The mapping c_2 is obtained by factoring the projection $d_2 : D \rightarrow \Gamma \backslash D$ through $C[g, \Gamma']$, where $d_2(x) = d_1(gx)$. Such a factoring exists because $\Gamma' \subset g^{-1}\Gamma g$.

It is a fact (Proposition 6.9) that every correspondence may be obtained in this way. For the maximal choice $\Gamma' = \Gamma \cap g^{-1}\Gamma g$, the resulting correspondence is called a *Hecke correspondence* and is denoted $C[g] \rightrightarrows X$. Up to isomorphism, this correspondence depends only on the double coset $\Gamma g \Gamma \in \Gamma \backslash \mathbf{G}(\mathbb{Q}) / \Gamma$ (cf. Lemma 6.6).

The correspondence $C[g, \Gamma'] \rightrightarrows X$ is a covering of the correspondence $C[g] \rightrightarrows X$ of degree $d = [\Gamma : \Gamma']$. The action of $C[g, \Gamma']$ on weighted cohomology is simply d times the action of $C[g]$. So, without loss of generality, we may concentrate on computing the Lefschetz number of the Hecke correspondence $C[g]$ for a fixed double coset $\Gamma g \Gamma \in \Gamma \backslash \mathbf{G}(\mathbb{Q}) / \Gamma$.

The correspondence on a stratum of \bar{X} . Each Hecke correspondence $C[g] \rightrightarrows X$ has a

unique continuous extension to the reductive Borel-Serre compactification $\bar{C}[g] \rightrightarrows \bar{X}$. Every boundary stratum C_Q of $\bar{C}[g]$ will be a correspondence taking a boundary stratum of \bar{X} to another one. Since we are interested in fixed points, we want to classify those C_Q which take a stratum $X_P \subseteq \bar{X}$ to itself. There is one of these for every double coset $\Gamma_P g_i \Gamma_P$ in the intersection $P \cap \Gamma g \Gamma$ (Proposition 7.3). It is isomorphic to a correspondence of the form $C[\bar{g}_i, \Gamma'_L]$ as described above, but with G replaced by L_P . (Here $\bar{g}_i = v_P(g_i) \in L_P$ and $\Gamma'_L = v_P(\Gamma_P \cap g_i^{-1} \Gamma g_i) \subset L_P$.)

Fixed point components. The fixed point set (sometimes called the *coincidence set*) of a correspondence $(c_1, c_2) : C \rightrightarrows X$ is by definition the set of points in C on which the two maps c_1 and c_2 agree.

The fixed point set of the correspondence $C[g, \Gamma'] \rightrightarrows X$ (before compactification) breaks up into constituent pieces $F(e)$ indexed by Γ conjugacy classes of elliptic (modulo S_G) elements $e \in \Gamma g \Gamma$ (§8.2). The piece $F(e)$ corresponding to the conjugacy class containing e is the space $\Gamma'_e \backslash G_e / K'_e$, where G_e is the centralizer of e in G , $\Gamma'_e = \Gamma' \cap G_e$, and $K'_e = G_e \cap (z(S_G K)z^{-1})$ where $zA_G K \in D = G/A_G K$ is a fixed point of the action of e on D . (Such a point exists since e is elliptic (§8.4).) The constituent $F(e)$ is a finite union of spaces, each of which is almost a locally symmetric space for the group G_e . (It may have infinite volume because it may have Euclidean factors that are not wound into tori.)

Applying this result to the boundary stratum X_P , in conjunction with the calculation (§1.3) of the part of the correspondence $\bar{C}[g]$ which preserves X_P , we get a group theoretic enumeration of all the fixed points of $\bar{C}[g]$ which lie over points in X_P : For each choice of a double coset $\Gamma_P g_i \Gamma_P \subset \Gamma g \Gamma \cap P$, and for each conjugacy class of elliptic (modulo S_P) elements e in $\Gamma_L \bar{g}_i \Gamma_L$, there is a fixed point constituent $F_P(e)$. (It is a smooth submanifold of the stratum C_Q of $\bar{C}[g]$ which is determined by the double coset $\Gamma_P g_i \Gamma_P$ as in §1.3.) Summing over Γ conjugacy classes of rational parabolics P gives the complete enumeration of fixed points of $\bar{C}[g]$.

The topology of the fixed point set. There are finitely many constituents $F_P(e)$ of the fixed point set and they are disjoint. Unfortunately however they may not be topologically isolated from each other. If $X_{P'} \subseteq \bar{X}_P$, then the closure of $F_P(e)$ can contain points in some $F_{P'}(e')$. So a single connected component of the fixed point set may have a very complicated structure. This phenomenon is the main source of technical difficulty of this paper. (The only real limit we have found on the possible complexity of a connected component of the fixed point set is Proposition 10.4.) We get around this problem by composing the correspondence with a mapping, very close to the identity, which shrinks a neighborhood of the singularity set $\bar{X} - X$ into the singularity set, and which does something similar on the closure of each stratum of \bar{X} . This has the effect of “truncating” each connected component of the fixed point set into pieces each of which is contained in a single stratum of \bar{X} (a process which may be considered as a sort of topological analog to Arthur’s truncation procedure). The Lefschetz number of this “modified” Hecke correspondence is equal to that of the original one. We prove that the modified Hecke correspondence is hyperbolic. The resulting formula (Theorem 1.5) would be the same if no truncation were used, however the proof would be more technical.

1.4. Calculating the local contribution $L(F)$. Let E be a finite dimensional representation of \mathbf{G} , and let \mathbf{E} be the associated local system over X . Denote by \mathbf{P}_0 a fixed min-

imal (“standard”) rational parabolic subgroup of G and by \mathbf{S}_0 a maximal \mathbb{Q} -split torus in the center of its Levi factor. Then $\mathbf{S}_G \subseteq \mathbf{S}_0$. A weight profile $\nu \in \chi_{\mathbb{Q}}^*(\mathbf{S}_0)$ (§12.2) is a (quasi-) character of \mathbf{S}_0 whose restriction to \mathbf{S}_G coincides with the character by which \mathbf{S}_G acts on E . The Hecke correspondence $\bar{C}[g] \rightrightarrows \bar{X}$ has a canonical lift (§13.1) to the weighted cohomology sheaf $\mathbf{W}^{\nu}\mathbf{C}^{\bullet}(\mathbf{E})$, so it induces a homomorphism on weighted cohomology whose Lefschetz number

$$(1.4.1) \quad L(C[g]) = \sum_{i \geq 0} (-1)^i \operatorname{Tr}(C[g]; W^{\nu} H^i(\bar{X}; \mathbf{E}))$$

is what we want to compute.

Hyperbolicity of the correspondence. Let us assume for the moment that the fixed point set is topologically the disjoint union of the constituent pieces $F_P(e)$. This is not always the case, but the formula we obtain is nevertheless always valid, as explained in §1.3.

We focus on a single stratum X_P which is preserved by the correspondence, and on a single stratum of the correspondence \bar{C} corresponding to a single double coset $\Gamma_P g_i \Gamma_P \subset P \cap \Gamma g \Gamma$. Within this stratum, we focus on a single constituent $F_P(e)$ of the fixed point set. Each stratum X_Q which contains X_P in its closure corresponds to a rational parabolic subgroup \mathbf{Q} containing \mathbf{P} , and therefore to a unique subset $I \subset \Delta_P$. The empty subset corresponds to X_P itself and the largest subset $\Delta_P \subseteq \Delta_P$ corresponds to X . Let a_e be the projection of e to the identity component A_P of $\mathbf{S}_P(\mathbb{R})$. The elements of Δ_P are rational characters of \mathbf{S}_P so we may define

$$\Delta_P^+(e) = \{\alpha \in \Delta_P \mid \alpha(a_e) < 1\}.$$

Let X_Q be the stratum containing X_P which corresponds to the subset $\Delta_P^+(e) \subseteq \Delta_P$. The correspondence $C[g]$ is hyperbolic near $F_P(e)$ (11.7), with “expanding” (or “unstable”) set X_Q (Theorem 11.9, §13.10). In other words, near $F_P(e)$ the Hecke correspondence is “expanding” in those directions normal to X_P which point into X_Q .

Let $F' = c_1(F_P(e)) = c_2(F_P(e)) \subset X_P$ and let L_e be the centralizer of e in L_P . There are diffeomorphisms (Proposition 8.4),

$$F_P(e) \cong \Gamma'_e \backslash L_e / K'_e \quad \text{and} \quad F' \cong \Gamma_e \backslash L_e / K'_e$$

where $K'_e = L_e \cap z(K_P A_P) z^{-1}$ (for appropriately chosen $z \in L_P$), $\Gamma_e = L_e \cap \Gamma_L$, and $\Gamma'_e = L_e \cap \Gamma'_L$. The projection $F_P(e) \rightarrow F'$ is a covering of degree $d_e = [\Gamma_e : \Gamma'_e]$ (cf. §8.6).

It follows from the Lefschetz fixed point theorem of [GM2] that the local contribution is given by

$$L(F_P(e)) = \chi_c(F') \sum_{i \geq 0} (-1)^i \operatorname{Tr}(C[g]^* : H_x^i(\mathbf{A}^{\bullet}) \rightarrow H_x^i(\mathbf{A}^{\bullet}))$$

(see Theorem 13.2). Here $\mathbf{A}^{\bullet} = h^! j^* \mathbf{W}^{\nu} \mathbf{C}^{\bullet}(E)$ where h is the inclusion $F' \hookrightarrow X_P \hookrightarrow \bar{X}_Q$ and j is the inclusion $\bar{X}_Q \hookrightarrow \bar{X}$; $H_x^i(\mathbf{A}^{\bullet})$ denotes its stalk cohomology at $x \in F'$; and χ_c denotes

the Euler characteristic with compact supports. See the introduction to [GM2] for a geometric account of hyperbolic correspondences.

The stalk cohomology. Let c denote the codimension of F' in X_P and let \mathcal{O} be the top exterior power of the normal bundle of F' in X_P . Let r denote the index $[\Gamma_P \cap \mathcal{U}_P : \Gamma'_P \cap \mathcal{U}_P]$ where $\Gamma'_P = \Gamma_P \cap g_i^{-1} \Gamma_P g_i$.

The stalk cohomology of the sheaf \mathbf{A}^\bullet at a fixed point $x \in F' \subset X_P$ is given by (13.10.4) and Proposition 12.8:

$$(1.4.2) \quad H_x^\bullet(\mathbf{A}^\bullet) \cong \bigoplus_{\substack{w \in W_P^1 \\ I_v(w) = \Delta_P^+(e)}} V_{w(\lambda_B + \rho_B) - \rho_B}^L [-\ell(w) - |\Delta_P^+(e)| - c] \otimes \mathcal{O}_x.$$

The Hecke correspondence $\bar{C}[g]$ acts on the first factor by rd_e times the action of e^{-1} and it acts on the second factor by $(-1)^c$, cf. (13.10.5). We now describe the other symbols in this formula.

Let \mathbf{T} be a maximal torus (over \mathbb{C}) in \mathbf{G} , and let \mathbf{B} be a Borel subgroup (over \mathbb{C}) of \mathbf{G} containing \mathbf{T} . These may be chosen as in §12.7 so that $\mathbf{S}_0(\mathbb{C}) \subset \mathbf{T}(\mathbb{C})$ and so that $\mathbf{B} \subset \mathbf{P}_0$. Let $W_G = W(\mathbf{G}(\mathbb{C}), \mathbf{T}(\mathbb{C}))$ denote the Weyl group of \mathbf{G} . The choice of \mathbf{B} determines positive roots $\Phi_G^+ = \Phi^+(\mathbf{G}(\mathbb{C}), \mathbf{T}(\mathbb{C}))$, and a length function ℓ on W_G . Let $W_P^1 \subset W_G$ denote the set of Kostant representatives: the unique elements of minimal length from each of the cosets $W_P x \in W_P \backslash W_G$, where $W_P = W(\mathbf{L}_P(\mathbb{C}), \mathbf{T}(\mathbb{C}))$, (§12.7). The sum in (1.4.2) is over those $v \in W_P^1$ such that the set

$$I_v(w) = \{ \alpha \in \Delta_P \mid \langle w(\lambda_B + \rho_B) - \rho_B - v, t_\alpha \rangle < 0 \}$$

coincides with the set $\Delta_P^+(e)$ defined above (after conjugating \mathbf{P} so as to contain \mathbf{B}). Here, as in §12.7, λ_B denotes the highest weight of the representation E , and $\{t_\alpha\}$ form the basis of the cocharacter group $\chi_*^{\mathbb{Q}}(\mathbf{S}_P/\mathbf{S}_G)$ which is dual to the basis Δ_P of the simple roots. Also, ρ_B denotes the half-sum of the positive roots Φ_G^+ . The product $\langle v(\lambda_B + \rho_B) - \rho_B - v, t_\alpha \rangle$ makes sense: the restriction $(v(\lambda_B + \rho_B) - \rho_B - v) |_{\mathbf{S}_P}$ is trivial on \mathbf{S}_G and hence defines an element of $\chi^*(\mathbf{S}'_P) \otimes \mathbb{Q}$ which can then be paired with t_α . For any \mathbf{B} -dominant weight β , the symbol V_β^L denotes the irreducible \mathbf{L}_P -module with highest weight $\beta \in \chi^*(\mathbf{T}(\mathbb{C}))$ and $V_\beta^L[-m]$ means that the module V_β^L is placed in degree m .

The geometry behind this formula is roughly this: Consider the intersection of a small neighborhood of x in \bar{X} with the largest stratum X . This intersection will deformation retract to the nilmanifold $(\Gamma \cap \mathcal{U}_P) \backslash \mathcal{U}_P$. The cohomology of this intersection with coefficients in \mathbf{E} coincides with the \mathfrak{R}_P cohomology by Van Est's theorem, which is computed by Kostant's theorem to be $\bigoplus_{w \in W_P^1} V_{w(\lambda_B + \rho_B) - \rho_B}^L [-\ell(w)]$. The cut-off $(I_v(w) = \Delta_P^+(e))$ and the degree shift (by $\ell(w) + |\Delta_P^+(e)|$) come from the computation of $h^! j^* \mathbf{W}^v \mathbf{C}^\bullet(E)$ in §12. The integer r is the ramification index: the degree of the mapping c_1 when it is restricted to this nilmanifold (§13.9).

By adding the contributions $L(F_P(e))$ over all the fixed point constituents $F_P(e)$ we arrive at the final result in this paper. It is proven in §14.

1.5. Theorem. *Let $g \in \mathbf{G}(\mathbb{Q})$. Let $\bar{C}[g] \rightrightarrows \bar{X}$ be the resulting Hecke correspondence. Fix a weight profile $\nu \in \chi_{\mathbb{Q}}^*(\mathbf{S}_0)$. The Lefschetz number $L(C[g])$ (1.4.1) is given by*

$$\sum_{\mathbf{P}} \sum_i \sum_{\{e\}} r\chi_c(\Gamma_e' \backslash L_e / K_e') (-1)^{|\Delta_P^+(e)|} \sum_{\substack{w \in W_P^1 \\ I_\nu(w) = \Delta_P^+(e)}} (-1)^{\ell(w)} \mathrm{Tr}(e^{-1}; V_{w(\lambda_B + \rho_B) - \rho_B}^L).$$

The first sum is over a choice of representative \mathbf{P} , one from each Γ -conjugacy class of rational parabolic subgroups of \mathbf{G} . For such a \mathbf{P} , set $\Gamma g \Gamma \cap \mathbf{P}(\mathbb{Q}) = \coprod_i \Gamma_P g_i \Gamma_P$ (where $\Gamma_P = \Gamma \cap \mathbf{P}(\mathbb{Q})$ and where $g_i \in \mathbf{P}(\mathbb{Q})$). The second sum is over these finitely many double cosets. Set $\bar{g}_i = \nu_P(g_i) \in L_P$ and $\Gamma_L = \nu_P(\Gamma_P)$. The third sum is over a choice of representatives e , one from each Γ_L -conjugacy class of elliptic (modulo S_P) elements $e \in \Gamma_L \bar{g}_i \Gamma_L$. The rest of the notations are explained above.

There are various ways to rewrite the Lefschetz formula; see §14.4, §15.8 and §15.9.

1.6. Adelic formulation. One of the main goals of the series of papers [GM1], [GM2], [GHM], [GKM], and the present paper is Theorem 7.14.B (p. 535) of [GKM], an expression for the Lefschetz number $L(C[g])$ in the adelic setting. If the weight profile ν is the “middle” weight (and if the rank of \mathbf{G} equals the rank of \mathbf{K}) then the weighted cohomology coincides with the L^2 cohomology, and this formula coincides with Arthur’s formula [Ar1] (Theorem 6.1). If the weight profile $\nu = -\infty$ then the weighted cohomology coincides with the ordinary “full cohomology” $H^*(X, E)$ and this formula coincides with Franke’s formula [F] (thm. 21, p. 273). The paper [GKM] uses the above Theorem 1.5 as its starting point (see the remarks following Theorem 13.6), then modifies it using three main steps.

(1) The quantity $r\chi_c(\Gamma_e' \backslash L_e / K_e')$ which appears in Theorem 1.5, and the sum \sum_i over double cosets $\Gamma_P g_i \Gamma_P \subset \Gamma g \Gamma \cap P$ (which precedes it) are replaced by an orbital integral.

(2) If L_P / A_P does not contain a compact maximal torus, then the stratum C_P makes no contribution to Arthur’s formula or to Franke’s formula. The same holds for the general formula in Theorem 7.14.B of [GKM]. However fixed points in such a stratum may make a nonzero contribution to the formula in Theorem 1.5 above. In [GKM], §7.14 the method of descent is used to re-attribute such a contribution to smaller strata C_Q for which L_Q / A_Q does admit a compact maximal torus. See also §15.8 of this paper.

(3) Theorem 1.5 above involves a sum over parabolic subgroups, while Theorem 7.14.B of [GKM] involves a sum over Levi subgroups. This is achieved in [GKM] (p. 529) by grouping together the contributions from those parabolic subgroups with a given Levi factor. (This has the remarkable effect of grouping together fixed points with different contracting-expanding behavior.) In [GKM] it is shown that the resulting contribution from a single Levi subgroup may be interpreted in terms of the (Harish-Chandra) character of a certain admissible representation. In the case of the middle weight, this fact gives rise to a combinatorial formula for the stable discrete series characters, which is the second main result of [GKM]. (Although this discrete series character formula was discovered by comparing Arthur’s formula to Theorem 1.5, the statement and proof of the character formula in [GKM] is independent of the part of the paper dealing with Lefschetz numbers.)

1.7. Related literature. Besides the articles listed above, and an extensive literature on the co-compact case, we mention several other closely related papers. The Lefschetz formula in the rank one case was studied by Moscovici [Mo] and Barbasch-Moscovici [BaM], also by Bewersdorff [Be] and Rapoport [R]. In [St], M. Stern gave a general Lefschetz formula for Hecke correspondences. We do not easily see how to compare his formula with ours. In [Sh] S. Shokranian, following the outline in [GKM], describes a formula for the Lefschetz numbers of Hecke operators on twisted groups. We wish to draw attention to Langlands' article [L1], in which the expanding and contracting nature of the fixed points on the boundary was first isolated (see especially Proposition 7.12, p. 485).

1.8. Acknowledgments. We would like to thank J. Arthur, W. Casselman and R. Langlands for encouraging us to work on this question. We would especially like to thank R. Kottwitz for patiently explaining Arthur's formula to us and for helping to interpret our early results in this direction. Some of the results in this paper appeared earlier in the adelic setting in our joint paper [GKM] with R. Kottwitz. We have profited from useful conversations with A. Nair, A. Borel, W. Casselman, P. Deligne, G. Harder, E. Looijenga, S. Morel, A. Nicas, M. Rapoport, L. Saper, J. Steenbrink, M. Stern, and S. Zucker. The first author is grateful to the Institute for Advanced Study for its support while much of this paper was written. This research was begun and partially completed when the authors were at Northeastern University and the Massachusetts Institute of Technology, respectively. We are also grateful to the following institutions for their hospitality and support during various phases of this project: the Centre de Recherches Mathématiques at the Université de Montréal, the Max Planck Institut für Mathematik in Bonn, the Department of Mathematics at the University of Chicago, the Università di Roma la Sapienza. This research was partially supported by the National Science Foundation under grants number DMS-8802638, DMS-9001941, DMS-9303550, DMS-9626616, DMS-9900324, DMS-0139986(Goresky) and DMS-8803083, DMS-9106522(MacPherson).

1.9. List of symbols.

- §2. 2.1: $\mathbf{G}, \mathbf{S}_{\mathbf{G}}, A_{\mathbf{G}}, \mathbf{0}\mathbf{G}, K, D, \mathbf{G}^{(1)}, K^{(1)}, A_0, T_g, K(x), \psi_x, K',$ elliptic, θ, Γ, X, τ
- 2.2: $\mathbf{P}, \mathcal{U}_P, R_dP, L_P, v_P, M_P, \mathbf{S}_{\mathbf{P}}, A_P, \Gamma_P, K_P, \mathbf{S}'_{\mathbf{P}}, A'_P, i_{x_0},$ Langlands' decomposition, $a_g,$ geodesic action, torus factor
- 2.3: $\mathbf{P}_0, \mathbf{S}_0, \Phi, \mathfrak{R}_0, \Delta, \mathbf{P}_0(I), \chi_Q(\mathbf{S}'_{\mathbf{P}})$
- 2.4: $\bar{\mathbf{P}}, i(\Delta_Q),$ complementary decomposition, orthogonal decomposition
- 2.5: boundary component, boundary stratum, $e_P, Y_P, D_P, X_P, \mu, F_P,$ canonical cross section
- 2.6: \mathcal{M}_x
- 2.9: $\bar{A}'_P, A'_P (>1), \bar{A}'_P (\geq 1), \bar{D}, D(P), \tilde{X}$
- 2.10: $\pi_P,$ geodesic projection, $\bar{D}, \mu, D[P], \bar{X}, \tau, X(P), X[P]$
- §3. 3.1: α, β, Γ -parabolic
- 3.2: root function, f_{α}^P, π_P

- §4. 4.1: \mathcal{P}_1 , \mathbf{b} , parameter, \mathcal{B} , tiling, D^P , $\partial^P D^0$, D_Q^0
 4.2: $T(\bar{D}_Q)$, $\partial T(\bar{D}_Q)$, r_α^Q , partial distance function
 4.5: X^P , $T(\bar{X}_P)$, $\partial T(\bar{X}_P)$
 4.6: R , retraction, W , exhaustion, R_Q , W_Q
- §5. 5.2: $D\{Q\}$, ρ , $Sh(Q, t)$, $Sh_{\mathbf{Q}}(t)$
 5.4: $Sh(\mathbf{t})$
- §6. 6.2: morphism, $\text{Mor}(X', X)$
 6.3: \tilde{f} , \bar{f}
 6.5: correspondence, $\Gamma[g]$, $C[g]$
 6.11: narrow
- §7. 7.1: parabolic correspondence, $\Gamma_P[y]$, modeled
 7.3: Ξ
- §8. 8.2: fixed point, characteristic element, e , $F_P(e)$, elliptic, L_e
- §9. 9.1: Δ_P^+ , Δ_P^- , Δ_P^0
 9.2: neutral, $\mathbf{P} \prec \mathbf{Q}$
 9.4: P^\dagger
- §11. 11.4: E , d_E
 11.7: hyperbolic
- §12. 12.1: t_α , $\chi_{\mathbb{Q}}^*(\mathbf{S}_P)_{[v_P, J]}$, $\chi_{\mathbb{Q}}^*(\mathbf{S}_P)_{\geq v_P(J)}$
 12.2: \mathbf{E} , v , weight profile, $\mathbf{W}^v \mathbf{C}^\bullet(\mathbf{E})$
 12.3: $\mathcal{O}_{X/Y}$
 12.6: N_y , \mathcal{L}_y , δ , \triangleright^{s-1} , \triangleright_J
 12.7: Φ_G^+ , Φ_L^+ , ρ_B , W_G , W_P , W_P^1 , V_β^L , λ_B , $I_v(w)$
- §13. 13.1: \mathbf{A}^\bullet
 13.2: χ_c
 13.5: r
 13.7: N_P
 13.9: $\mathbf{C}^\bullet(N_P, \mathbf{E})$, $\mathbf{C}^\bullet(\mathfrak{R}_P, E)$, $\Omega_{\text{inv}}(\mathcal{U}_P, E)$

2. Notation and terminology

2.1. Locally symmetric spaces. Linear algebraic groups will be represented by bold-face symbols (e.g., \mathbf{G} , \mathbf{S}) and their real points will be in Roman type (e.g., $G = \mathbf{G}(\mathbb{R})$, $S = \mathbf{S}(\mathbb{R})$). Throughout this paper we fix a connected reductive linear algebraic group \mathbf{G} defined over \mathbb{Q} . Denote by $\mathbf{S}_{\mathbf{G}}$ the greatest \mathbb{Q} -split torus in the center of G , and let $A_G = \mathbf{S}_{\mathbf{G}}(\mathbb{R})^0$ denote the identity component of the group of real points of $\mathbf{S}_{\mathbf{G}}$. Following [BS], §1.1 let

$${}^0\mathbf{G} = \bigcap_{\chi} \ker(\chi^2)$$

be the intersection of the kernels of the squares of all the rationally defined characters $\chi : \mathbf{G} \rightarrow \mathbf{GL}_1$. Then 0G is normal in G ; it contains every compact subgroup and every arithmetic subgroup of G , and $G = A_G \times {}^0G$. Let $K \subset \mathbf{G}(\mathbb{R})$ be a maximal compact subgroup and define $D = G/KA_G$. We refer to D as the “symmetric space” associated to \mathbf{G} . The derived group $\mathbf{G}^{(1)}$ is semisimple and $K^{(1)} = G^{(1)} \cap K$ is a maximal compact subgroup. The space D is diffeomorphic to the Cartesian product of the Riemannian symmetric space $D^{(1)} = G^{(1)}/K^{(1)}$ with A_0/A_G where A_0 is the identity component of the greatest \mathbb{R} -split torus in the center of G . Both $\mathbf{G}(\mathbb{R})$ and 0G act transitively on D , an action which we usually denote by $(g, x) \mapsto gx$ or $g.x$ but occasionally it will be necessary to refer to this action as a mapping, in which case we write

$$(2.1.1) \quad T_g : D \rightarrow D$$

for $g \in G$. (For most geometric questions involving the symmetric space D , one could replace G by 0G , however there are Hecke correspondences for G which do not necessarily come from 0G .) For each $x \in D$ the stabilizer $K(x)$ of x in 0G is a maximal compact subgroup of 0G so we obtain a G -equivariant diffeomorphism

$$\psi_x : G/A_GK(x) \rightarrow D.$$

The choice of $K \subset G$ corresponds to a “standard” basepoint $x_0 \in D$. We write $K = K(x_0)$ and $K' = A_GK(x_0)$. An element $x \in G$ is *elliptic* mod A_G (often shortened to “elliptic”) if it is $\mathbf{G}(\mathbb{R})$ -conjugate to an element of K' . There is a unique “algebraic” Cartan involution $\theta = \theta_{x_0} : G \rightarrow G$ whose fixed point set is K . If $x_1 \in D$ is another basepoint with $x_1 = gx_0$ then the Cartan involution for the new basepoint is given by

$$(2.1.2) \quad \theta_{x_1}(y) = g\theta_{x_0}(g^{-1}yg)g^{-1}$$

and the composition $\psi_{x_1}^{-1}\psi_{x_0} : G/A_GK(x_0) \rightarrow G/A_GK(x_1)$ is given by

$$(2.1.3) \quad yA_GK(x_0) \mapsto yg^{-1}A_GK(x_1).$$

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the ± 1 eigenspace decomposition of θ in $\text{Lie}(G)$. The Cartan involution θ preserves 0G and determines a decomposition of its Lie algebra, ${}^0\mathfrak{g} = \mathfrak{k} \oplus {}^0\mathfrak{p}$. Then ${}^0\mathfrak{p}$ may be canonically identified with the tangent space $T_{x_0}D$. Any choice of K -invariant inner product on ${}^0\mathfrak{p}$ induces a G -invariant Riemannian metric on X .

Throughout this paper we also fix an arithmetic subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ and denote by $\tau : D \rightarrow X = \Gamma \backslash D$ the projection to the locally symmetric space X .

2.2. Parabolic subgroups. Fix a rationally defined parabolic subgroup $\mathbf{P} \subset \mathbf{G}$. We have the following groups:

- (1) \mathcal{U}_P = the unipotent radical of \mathbf{P} ; $\mathfrak{N}_P = \text{Lie}(\mathcal{U}_P)$ its Lie algebra.
- (2) $\mathbf{R}_d\mathbf{P}$ = the \mathbb{Q} split radical of \mathbf{P} .
- (3) \mathbf{L}_P = the Levi quotient; $v_P : \mathbf{P} \rightarrow \mathbf{L}_P$ the projection.
- (4) $\mathbf{M}_P = {}^0\mathbf{L}_P = \bigcap_{\chi} \ker(\chi^2)$.
- (5) $\mathbf{S}_P = \mathbf{R}_d\mathbf{P}/\mathcal{U}_P$.
- (6) $A_P = \mathbf{S}_P(\mathbb{R})^0$ the identity component of the set of real points.
- (7) $\Gamma_P = \Gamma \cap P$, $\Gamma_L = \Gamma_{L(P)} = v_P(\Gamma_P) \subset M_P$.
- (8) $K_P = K \cap P$, $K_L = K_{L(P)} = v_P(K_P) \subset M_P$, $K'_P = K' \cap P = K_P A_G$.

The torus \mathbf{S}_P may also be identified as the greatest \mathbb{Q} -split torus in the center of \mathbf{L}_P . It contains \mathbf{S}_G and we denote the quotient by $\mathbf{S}'_P = \mathbf{S}_P/\mathbf{S}_G$, with corresponding identity component $A'_P = \mathbf{S}'_P(\mathbb{R})^0 = A_P/A_G$. We identify A'_P with the subgroup $A_P \cap {}^0G$ to obtain a canonical decomposition $A_P = A'_P A_G$.

The group of real points of the Levi quotient is the direct product, $L_P = M_P \times A_P$. For any $x \in P$ write $v_P(x) = v_M(x)v_A(x)$ for its M_P and its A_P components and write $v_{A'}(x)$ for the further projection of $v_A(x)$ to the quotient $A'_P = A_P/A_G$. The group P acts transitively on D with isotropy $K'_P = A_G K_P = \text{Stab}_P(x_0)$.

The choice of standard basepoint $x_0 \in D$ with associated Cartan involution $\theta : \mathbf{G} \rightarrow \mathbf{G}$ determines a unique θ -stable lifting [BS], §1.9, $i = i_{x_0} : L_P \rightarrow P$. Denote the image by $L_P(x_0) = i(L_P)$. We obtain liftings of subgroups, $A_P(x_0) = i(A_P)$ and $M_P(x_0) = i(M_P)$. Thus the choice $x_0 \in D$ of basepoint determines a canonical *Langlands' decomposition*

$$(2.2.1) \quad P = \mathcal{U}_P A_P(x_0) M_P(x_0)$$

and we write

$$(2.2.2) \quad g = u_g a_g m_g$$

where $u_g = giv_P(g^{-1})$, $a_g = iv_A(g)$, and $m_g = iv_M(g)$ for any $g \in P$. The groups $K_P \subset P$ and $K_{L(P)} = v_P(K_P)$ are canonically isomorphic, in fact,

$$(2.2.3) \quad K_P = i(K_{L(P)}) \subset M_P(x_0) \subset L_P(x_0).$$

By abuse of notation we will usually write $K_P \subset L_P$. If $x_1 \in D$ is another basepoint with associated Cartan involution $\theta_{x_1} : \mathbf{G} \rightarrow \mathbf{G}$ then, by (2.1.2), the associated θ_{x_1} -stable lifting $i_{x_1} : L_P \rightarrow P$ is given by

$$(2.2.4) \quad i_{x_1}(y) = gi(y)g^{-1}$$

where $g \in P$ is any element such that $g \cdot x_0 = x_1 \in D$. The *geodesic action* of Borel and Serre [BS], §3 is the right action of A_P on D which is given by

$$(2.2.5) \quad (zK'_P) \cdot a = zi(a)K'_P \in D = P/K'_P$$

for any $a \in A_P$ and $z \in P$. It is well defined since $i(A_P)$ commutes with K'_P , and it passes to an action of $A'_P = A_P/A_G$. The geodesic action commutes with the (left) action of P , and it is independent of the choice of basepoint, by (2.1.3). It is *not* an action by isometries.

For $g = u_g a_g m_g \in P$ as in (2.2.2), the element $a_g \in A_P$ is called the *torus factor* of g . We will often use without mention the following fact: if $\gamma = u_\gamma a_\gamma m_\gamma \in \Gamma \cap P$ then $a_\gamma = 1$.

2.3. Roots. Fix once and for all a minimal rational parabolic subgroup $\mathbf{P}_0 \subset \mathbf{G}$. The parabolic subgroups $\mathbf{P} \supseteq \mathbf{P}_0$ are called *standard*. Let $\mathbf{S}_0 = i(\mathbf{S}_{\mathbf{P}_0})$ be the lift of $\mathbf{S}_{\mathbf{P}_0}$. Let $\Phi = {}_{\mathbb{Q}}\Phi(\mathbf{S}_0, \mathfrak{g})$ denote the rational relative roots of \mathfrak{g} with respect to \mathbf{S}_0 . The unipotent radical $\mathcal{U}_{\mathbf{P}_0}$ determines a linear order on the root system ${}_{\mathbb{Q}}\Phi$ such that the positive roots are those occurring in $\mathfrak{N}_0 = \text{Lie}(\mathcal{U}_{\mathbf{P}_0})$. Let Δ denote the resulting collection of simple roots. Each subset $I \subset \Delta$ corresponds to a unique standard parabolic subgroup $\mathbf{P}_0(I) \supset \mathbf{P}_0$ ([BS], §4, [Bo3], §14.17, §21.11) such that $\mathbf{S}_{\mathbf{P}_0(I)} \subset \ker(\alpha)$ for all $\alpha \in I$.

Suppose $\mathbf{P} \subset \mathbf{G}$ is any rational parabolic subgroup. Then \mathbf{P} is $\mathbf{G}(\mathbb{Q})$ -conjugate to a unique standard parabolic subgroup $\mathbf{P}_0(I)$. Any choice of conjugating element $\mathbf{P} = g\mathbf{P}_0(I)g^{-1}$ gives rise to the same (canonical) isomorphism $\mathbf{S}_{\mathbf{P}} \cong \mathbf{S}_{\mathbf{P}_0(I)}$. The elements of $\Delta - I$ give rise (by conjugation and restriction to $\mathbf{S}_{\mathbf{P}}$) to the set $\Delta_{\mathbf{P}}$ of simple roots of $\mathbf{S}_{\mathbf{P}}$ occurring in $\mathfrak{N}_{\mathbf{P}}$. The roots $\alpha \in \Delta_{\mathbf{P}}$ are trivial on $\mathbf{S}_{\mathbf{G}}$ and form a basis for the character module $\chi_{\mathbb{Q}}(\mathbf{S}'_{\mathbf{P}}) = \chi^*(\mathbf{S}_{\mathbf{P}}/\mathbf{S}_{\mathbf{G}}) \otimes_{\mathbb{Z}} \mathbb{Q}$. Rather than follow the common practice of identifying $\Delta_{\mathbf{P}}$ with $\Delta - I$ we will, for any $\alpha \in \Delta_{\mathbf{P}}$ denote by $\alpha_0 \in \Delta$ the unique simple root which agrees with α after conjugation and restriction to $\mathbf{S}_{\mathbf{P}}$.

2.4. Two parabolic subgroups. If $\mathbf{P} \subset \mathbf{Q}$ are rational parabolic subgroups then $\bar{\mathbf{P}} = v_{\mathbf{Q}}(\mathbf{P})$ is a rational parabolic subgroup of $\mathbf{L}_{\mathbf{Q}}$, with unipotent radical $\mathcal{U}_{\bar{\mathbf{P}}} = \mathcal{U}_{\mathbf{P}}/\mathcal{U}_{\mathbf{Q}}$. The θ -stable lifts of the Levi quotients satisfy $L_{\mathbf{P}}(x_0) \subset L_{\mathbf{Q}}(x_0)$ and we have a diagram

$$\begin{array}{ccccccc} \mathcal{U}_{\mathbf{Q}} & \subset & \mathcal{U}_{\mathbf{P}} & \subset & \mathbf{P} & \subset & \mathbf{Q} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow v_{\mathbf{Q}} \\ 1 & \subset & \mathcal{U}_{\mathbf{P}}/\mathcal{U}_{\mathbf{Q}} & \subset & \bar{\mathbf{P}} & \subset & L_{\mathbf{Q}} \\ \downarrow & & \downarrow & & \downarrow v_{\bar{\mathbf{P}}} & & \\ 1 & \subset & 1 & \subset & L_{\mathbf{P}} & & \end{array}$$

with $v_{\mathbf{Q}}v_{\bar{\mathbf{P}}} = v_{\mathbf{P}}$. The inclusion $R_d\mathbf{Q} \subset R_d\mathbf{P}$ induces an injection $\mathbf{S}_{\mathbf{Q}} \hookrightarrow \mathbf{S}_{\mathbf{P}}$ which agrees with the inclusion $S_{\mathbf{P}}(x_0) \supset S_{\mathbf{Q}}(x_0)$. It follows from (2.4.4) below that the geodesic action of $A'_{\mathbf{Q}}$ on D agrees with the restriction (to $A'_{\mathbf{Q}} \subset A'_{\mathbf{P}}$) of the geodesic action of $A'_{\mathbf{P}}$ on D (cf. [BS], prop. 3.11). Each $\alpha \in \Delta_{\mathbf{Q}}$ is the restriction to $\mathbf{S}_{\mathbf{Q}}$ of a unique simple root $i(\alpha) \in \Delta_{\mathbf{P}}$. The association $i : \Delta_{\mathbf{Q}} \rightarrow \Delta_{\mathbf{P}}$ is injective, so $\Delta_{\mathbf{P}}$ is the disjoint union

$$(2.4.1) \quad \Delta_{\mathbf{P}} = i(\Delta_{\mathbf{Q}}) \amalg J \quad \text{with } \mathbf{S}_{\mathbf{Q}} = \left(\bigcap_{\alpha \in J} \ker(\alpha) \right)^0.$$

Among rational parabolic subgroups containing \mathbf{P} , the group \mathbf{Q} is determined by the set J , and we will write $\mathbf{Q} = \mathbf{P}(J)$. The subset $J \subset \Delta_{\mathbf{P}}$ of simple roots may be identified with the set

$$(2.4.2) \quad J = \Delta_{\bar{P}}$$

of simple roots $\Delta_{\bar{P}}$ of $\mathbf{S}'_{\bar{P}} = \mathbf{S}_{\bar{P}}/\mathbf{S}_{\mathbf{Q}}$ occurring in $\mathfrak{N}_{\bar{P}} = \text{Lie}(\mathcal{U}_{\bar{P}})$. (Although the projection ν_P induces a canonical isomorphism $\mathbf{S}_{\mathbf{P}} \cong \mathbf{S}_{\bar{P}}$, the torus $\mathbf{S}_{\mathbf{P}}$ corresponds to the parabolic subgroup $\mathbf{P} \subset \mathbf{G}$ so $\mathbf{S}'_{\mathbf{P}} = \mathbf{S}_{\mathbf{P}}/\mathbf{S}_{\mathbf{G}}$ while $\mathbf{S}_{\bar{P}}$ corresponds to the parabolic subgroup $\bar{\mathbf{P}} \subset \mathbf{L}_{\mathbf{Q}}$ so $\mathbf{S}'_{\bar{P}} = \mathbf{S}_{\bar{P}}/\mathbf{S}_{\mathbf{Q}}$.)

A certain amount of confusion arises from the fact that A'_Q has two natural complements in A'_P . One is the identity component $A'_{Q'}$ of the group of real points of the torus $\mathbf{S}'_{Q'} = \mathbf{S}_{Q'}/\mathbf{S}_{\mathbf{G}}$ where

$$\mathbf{S}_{Q'} = \left(\bigcap_{\alpha \in i(\Delta_Q)} \ker(\alpha) \right)^0 \subset \mathbf{S}_{\mathbf{P}}.$$

Then $\mathbf{S}_{Q'}$ is the (identity component of) the center of the Levi quotient of the largest parabolic subgroup $\mathbf{Q}' \supset \mathbf{P}$ such that $\mathbf{Q} \cap \mathbf{Q}' = \mathbf{P}$, which we refer to as the parabolic subgroup containing \mathbf{P} that is *complementary to \mathbf{Q}* . We therefore refer to the *complementary decomposition* $A'_P = A'_Q A'_{Q'}$. The other complement is

$$A_P^Q(x_0) = A'_P \cap M_Q(x_0)$$

whose Lie algebra \mathfrak{a}_P^Q is the orthogonal complement to \mathfrak{a}'_Q in \mathfrak{a}'_P with respect to any Weyl-invariant inner product on \mathfrak{a}'_P . We will usually identify the quotient $A_P/A_Q = A'_P/A'_Q$ with this second complement, A_P^Q , and we will refer to the *orthogonal decompositions* $A'_P = A'_Q A_P^Q$ and $A_P = A_Q A_P^Q$.

The canonical Langlands decompositions of P and Q are related as follows: Set $\mathcal{U}_{\bar{P}}(x_0) = i_{x_0}(\mathcal{U}_P/\mathcal{U}_Q)$. Note that $M_P(x_0) \subset M_Q(x_0)$ and $A_Q = A'_Q A_G$. If

$$(2.4.3) \quad \bar{P} = \mathcal{U}_{\bar{P}}[A_P^Q(x_0)A_G]M_P(x_0)$$

is the canonical Langlands decomposition of \bar{P} , then

$$(2.4.4) \quad P = [\mathcal{U}_Q \mathcal{U}_{\bar{P}}(x_0)][A_Q(x_0)A_P^Q(x_0)]M_P(x_0)$$

$$(2.4.5) \quad = \mathcal{U}_Q A_Q(x_0)[\mathcal{U}_{\bar{P}}(x_0)A_P^Q(x_0)M_P(x_0)].$$

The first is the canonical Langlands decomposition of P while the second is the decomposition of P which is induced from the canonical Langlands decomposition of Q .

2.5. Boundary strata. Fix a rational parabolic subgroup $\mathbf{P} \subset \mathbf{G}$. Define

- (1) the Borel-Serre boundary *component* $e_P = D/A'_P$ (quotient under geodesic action),
- (2) the Borel-Serre boundary *stratum* $Y_P = \Gamma_P \backslash e_P$,
- (3) the reductive Borel-Serre boundary *component*

$$D_P = \mathcal{U}_P \backslash e_P = P/K_P A_P \mathcal{U}_P = L_P/K_P A_P = M_P/K_P A_G,$$

- (4) the reductive Borel-Serre boundary *stratum* $X_P = \Gamma_P \backslash D_P = \Gamma_{L(P)} \backslash D_P$.

The projection $\nu_P : \mathbf{P} \rightarrow \mathbf{L}_P$ induces a projection $\mu : e_P \rightarrow D_P$ which passes to a projection

$\mu : Y_P \rightarrow X_P$. Writing $Y_P = \Gamma_P \backslash P / K_P A_P$ and $X_P = \Gamma_{L(P)} \backslash L_P / K_P A_G$, the mapping μ is just $\mu(\Gamma_P x K_P A_P) = \Gamma_{L(P)} v_P(x) K_P A_G$.

As in [Bo4], §4.2, the Langlands' decomposition (2.2.1) determines a (basepoint-dependent) diffeomorphism,

$$(2.5.1) \quad F = F_P : \mathcal{U}_P \times A'_P \times D_P \rightarrow D = P / K'_P$$

by

$$(2.5.2) \quad F(u, a, mK_L) = ui_{x_0}(a)i_{x_0}(m)K_P$$

where $u \in \mathcal{U}_P$, $a \in A'_P$, and $m \in M_P$. With respect to the coordinates defined by the diffeomorphism F , the mapping μ is the projection to the third factor. The (left) action of $g \in P$ and the (right) geodesic action of $b \in A'_P$ on D are given by

$$(2.5.3) \quad g.(u, a, mK_L) \cdot b = (gui_{x_0}v_P(g^{-1}), v_A(g).ab, v_M(g).mK_L)$$

(where $u \in \mathcal{U}_P$, $a, b \in A'_P$, and $m \in M_P$), as may be seen by applying the function F to both sides of this equation. For any fixed $b \in A'_P$ the set $F(\mathcal{U}_P \times \{b\} \times D_P) \subset D$ is called a *canonical cross section*; it is a single orbit of the group

$${}^0P = \bigcap_{\chi} \ker(\chi^2) = \mathcal{U}_P M_P,$$

the intersection being taken over all rationally defined characters $\chi : \mathbf{P} \rightarrow \mathbf{GL}_1$. The pull-back by F of the canonical Riemannian metric on D is given ([Bo4], §4.3) by the orthogonal sum,

$$(2.5.4) \quad F^*(ds^2) = \sum_{\beta \in \Phi} a^{-2\beta} h_{\beta}(z) \oplus da^2 \oplus ds_M^2$$

where ds_M^2 is the canonical Riemannian metric on D_P as determined by the Killing form for M_P , where Φ denotes the set of roots of \mathcal{U}_P with respect to A_P , and $h_{\beta}(z)$ is a smoothly varying metric on the root space \mathfrak{u}_{β} .

2.6. The flat connection ([GHM], §7.10). For any point $x = gK'_P \in D$ with $g = uam$ decomposed according to (2.2.1), define the submanifold

$$(2.6.1) \quad \mathcal{M}_x = F_P(\{u\} \times \{a\} \times D_P) = u.i_{x_0}(a).i_{x_0}(M_P)K_P \subset D.$$

2.7. Lemma. *The manifold \mathcal{M}_x is perpendicular to the fibers of the mapping $v_P : D \rightarrow D_P$. The restriction $v_P|_{\mathcal{M}_x}$ is an isometry. The submanifolds \mathcal{M}_x form the horizontal submanifolds of a flat connection on the fiber bundle $\mu : e_P \rightarrow D_P$, which is independent of the choice of basepoint and is invariant under the action of Γ_P and which therefore passes to a flat connection on $\mu : Y_P \rightarrow X_P$.*

2.8. Proof. Perpendicularity follows from (2.5.4). Also, by (2.5.4), the mapping v_P is an isometry. Finally the flat connection is Γ_P -invariant because by (2.5.3) the action of $\gamma \in \Gamma_P$ on D is given by

$$(2.8.1) \quad \gamma \cdot (u, a, mK_L) = (\gamma ui_{x_0}v_P(\gamma^{-1}), a, v_M(\gamma).mK_L)$$

which does not mix the factors. \square

2.9. Borel-Serre compactification. In this section we recall basic facts from [BS]. Let $\mathbf{P} \subset \mathbf{G}$ be a rational parabolic subgroup. The elements of Δ_P determine a canonical isomorphism ([BS], §4.2) $A'_P \cong (0, \infty)^{\Delta_P}$ which extends to a unique partial compactification,

$$\bar{A}'_P \cong (0, \infty]^{\Delta_P}.$$

So each $\alpha \in \Delta_P$ extends to a homomorphism of semigroups $\alpha : \bar{A}'_P \rightarrow (0, \infty]$. Denote by

$$(2.9.1) \quad A'_P(>1) = \{a \in A'_P \mid \alpha(a) > 1 \text{ for all } \alpha \in \Delta_P\},$$

$$(2.9.2) \quad A'_P(\geq 1) = \{a \in A'_P \mid \alpha(a) \geq 1 \text{ for all } \alpha \in \Delta_P\}$$

and similarly for $\bar{A}'_P(>1)$ and $\bar{A}'_P(\geq 1)$. The Borel-Serre partial compactification \tilde{D} of D is obtained by adjoining, for each rational parabolic subgroup $\mathbf{P} \subset \mathbf{G}$ the rational boundary component $e_P = D/A'_P$ as the set of limits of the A'_P geodesic orbits in D , together with the Satake topology [Sat], §2 (p. 562), [BS], §7.1, [Z3], §3.7. It is covered by “corners”; the corner associated to P is

$$(2.9.3) \quad D(P) = D \times_{A'_P} \bar{A}'_P = \coprod_{Q \cong P} e_Q.$$

Then $D(P)$ is an open $\mathbf{P}(\mathbb{Q})$ -invariant neighborhood (in \tilde{D}) of the boundary component e_P , on which $\mathbf{P}(\mathbb{Q})$ acts in a continuous and component-preserving way. The diffeomorphism F of equation (2.5.2) extends to a diffeomorphism of manifolds with corners,

$$(2.9.4) \quad \bar{F} : \mathcal{U} \times \bar{A}_P \times D_P \cong D(P).$$

The action $T_g : D \rightarrow D$ of any $g \in \mathbf{G}(\mathbb{Q})$ extends continuously to a mapping

$$(2.9.5) \quad \tilde{T}_g : \tilde{D} \rightarrow \tilde{D}$$

which takes the neighborhood $D(P)$ of e_P isomorphically to the neighborhood $D({}^gP)$ of $e_{{}^gP}$ (where ${}^gP = gPg^{-1}$). (The proof of this fact is recalled in §6.3, §6.4.) It follows that the Borel-Serre compactification $\tilde{X} = \Gamma \backslash \tilde{D}$ is a (compact) manifold with corners, stratified with one stratum $Y_P = \Gamma_P \backslash e_P$ for each Γ -conjugacy class of rational parabolic subgroups \mathbf{P} . The real analytic structure on D extends to a semi-analytic structure on \tilde{D} and passes to a sub-analytic structure on \tilde{X} . Denote by $\tilde{\tau} : \tilde{D} \rightarrow \tilde{X}$ the natural projection.

2.10. Reductive Borel-Serre compactification. The reductive Borel-Serre partial compactification \bar{D} of D was first described in [Z1], §4.2, p. 190; see also [GHM], §8. It is the topological space obtained by collapsing each boundary component e_P in \tilde{D} to its reductive quotient D_P (§2.5), together with the quotient topology. (See also [Z3], §3.7.) The *geodesic projection*

$$(2.10.1) \quad \pi_P : D \rightarrow D_P$$

is the composition $D \rightarrow e_P \rightarrow D_P$. The closure \bar{D}_P of D_P in \bar{D} is the reductive Borel-Serre partial compactification of D_P . Let $\mu : \tilde{D} \rightarrow \bar{D}$ denote the quotient mapping: it is continuous, its restriction to D is the identity, and its restriction to each boundary stratum agrees with the projection $\mu : Y_P \rightarrow X_P$ of §2.5. Define

$$(2.10.2) \quad D[P] = \mu(D(P)) = \bigcup_{Q \cong P} D_Q$$

to be the image of the corner associated to P : it is an open $\mathbf{P}(\mathbb{Q})$ -invariant neighborhood of D_P in \bar{D} on which $\mathbf{P}(\mathbb{Q})$ acts in a component-preserving way. The action $T_g : \tilde{D} \rightarrow \tilde{D}$ (2.9.5) of any $g \in \mathbf{G}(\mathbb{Q})$ passes to a mapping $\bar{T}_g : \bar{D} \rightarrow \bar{D}$ which takes the neighborhood $D[P]$ of D_P isomorphically to the neighborhood $D[{}^gP]$ of $D_{{}^gP}$. It follows that the *reductive Borel-Serre compactification*

$$\bar{X} = \Gamma \backslash \bar{D}$$

is a compact singular space, canonically stratified with one boundary stratum $X_P = \Gamma_P \backslash D_P$ for each Γ -conjugacy class of rational parabolic subgroups $\mathbf{P} \subset \mathbf{G}$. The closure \bar{X}_P of X_P in \bar{X} is the reductive Borel-Serre compactification of X_P . There are $|\Delta_P|$ maximal boundary strata X_Q such that $\bar{X}_Q \supset \bar{X}_P$, each corresponding to a maximal (rational) parabolic subgroup $\mathbf{Q} = \mathbf{P}(\Delta_P - \{\alpha\})$ for $\alpha \in \Delta_P$ (cf. §2.4). Then \bar{X}_P is the intersection

$$(2.10.3) \quad \bar{X}_P = \bigcap_Q \bar{X}_Q$$

of these $|\Delta_P|$ maximal boundary strata.

It is not difficult to see ([Bo3], §11.7 (iii)) that if \mathbf{P} and \mathbf{P}' are $\mathbf{G}(\mathbb{Q})$ -conjugate but are not Γ -conjugate, then

$$(2.10.4) \quad \bar{X}_P \cap \bar{X}_{P'} = \emptyset.$$

The identity mapping $X \rightarrow X$ extends uniquely to a continuous surjection $\mu : \tilde{X} \rightarrow \bar{X}$ and the subanalytic structure on \tilde{X} passes to a subanalytic structure on \bar{X} . Denote by $\bar{\tau} : \bar{D} \rightarrow \bar{X}$ the projection. Define $X(P) = \bar{\tau}(D(P))$ and $X[P] = \bar{\tau}(D[P])$. The following diagram may be useful in helping to sort out these spaces:

$$(2.10.5) \quad \begin{array}{ccccccc} e_P & \subset & D(P) & \subset & \tilde{D} & \xrightarrow{\bar{\tau}} & \tilde{X} & \supset & X(P) & \supset & Y_P \\ \downarrow & & \downarrow & & \downarrow \mu & & \downarrow \mu & & \downarrow & & \downarrow \mu \\ D_P & \subset & D[P] & \subset & \bar{D} & \xrightarrow{\bar{\tau}} & \bar{X} & \supset & X[P] & \supset & X_P. \end{array}$$

3. Parabolic neighborhoods and root functions

As in §2, \mathbf{G} denotes a connected linear reductive algebraic group defined over \mathbb{Q} , D denotes the associated symmetric space, $K' = A_G K(x_0)$ is the stabilizer in G of a fixed base-point $x_0 \in D$, $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is an arithmetic group and $X = \Gamma \backslash D$. Although the constructions in this section refer to the reductive Borel-Serre compactification \bar{X} of X (and the reductive Borel-Serre partial compactification \bar{D} of D), they may just as well be applied to the Borel-Serre compactification \tilde{X} (and the Borel-Serre partial compactification \tilde{D} of D). Rather than repeat each statement for both compactifications, we will present the RBS case only.

3.1. Parabolic neighborhoods. Let $\mathbf{P} \subset \mathbf{G}$ be a rational parabolic subgroup. Let $\alpha : \bar{D} \rightarrow \Gamma_P \backslash \bar{D}$ and $\beta : \Gamma_P \backslash \bar{D} \rightarrow \Gamma \backslash \bar{D} = \bar{X}$ be the projections. We say that an open set $V \subset \bar{D}$ is Γ -parabolic (with respect to \mathbf{P}) if

(1) it is invariant under the geodesic action of the semigroup $A_P (\geq 1)$ (2.9.2) and

(2) if $\gamma \in \Gamma$ and $\gamma V \cap V \neq \emptyset$ then $\gamma \in \Gamma \cap P$.

Item (2) means that the covering $\beta : \Gamma_P \backslash \bar{D} \rightarrow \bar{X}$ is one to one on the set $\alpha(V)$ so it takes $\alpha(V)$ homeomorphically to its image $\bar{\tau}(V) \subset \bar{X}$. In this case we will also refer to $\alpha(V) \subset \Gamma_P \backslash \bar{D}$ (resp. $\bar{\tau}(V) \subset \bar{X}$) as Γ -parabolic open sets.

$$\begin{array}{ccccc}
 \bar{D} & \supset & V & \supset & D_P \\
 \alpha \downarrow & & \downarrow & & \downarrow \\
 \Gamma_P \backslash \bar{D} & \supset & \alpha(V) & \supset & \Gamma_P \backslash D_P \\
 \beta \downarrow & & \downarrow \cong & & \downarrow \cong \\
 \bar{X} & \supset & \bar{\tau}(V) & \supset & X_P.
 \end{array}$$

Every stratum X_P admits a fundamental system of Γ -parabolic neighborhoods. In section 4.1 we will review a theorem of Saper [Sa] (thm. 8.1) which states the stronger fact that *the closure \bar{X}_P of each stratum $X_P \subset \bar{X}$ admits a fundamental system of Γ -parabolic neighborhoods.*

3.2. Root functions. Let $\mathbf{P} \subset \mathbf{G}$ be a rational parabolic subgroup. Each character $\alpha \in \chi_{\mathbb{Q}}^*(\mathbf{S}'_{\mathbf{P}})$ determines a mapping

$$(3.2.1) \quad f_{\alpha}^P : D \rightarrow \mathbb{R}_{>0}$$

by $f_{\alpha}^P(F(u, a, mK_L)) = \alpha(a)$ using (2.5.2). The mapping f_{α}^P is independent of the choice of basepoint. For any $g' = u'a'm' \in P$, any $b' \in A'_P$, any $\gamma \in \Gamma \cap P$, and any $x \in D$ we have

$$(3.2.2) \quad f_{\alpha}^P(\gamma g'x \cdot b') = \alpha(a'b')f_{\alpha}^P(x).$$

If $\alpha \in \Delta_P$ is a simple root, we say f_{α}^P is a *root function*. If $\gamma \in \Gamma$, $\mathbf{P}' = \gamma\mathbf{P}\gamma^{-1}$ and if $\alpha' \in \Delta_{P'}$ is the root corresponding to $\alpha \in \Delta_P$ then, for all $x \in D$,

$$(3.2.3) \quad f_{\alpha'}^{P'}(\gamma x) = f_{\alpha}^P(x).$$

The root function $f_{\alpha}^P : D \rightarrow (0, \infty)$ extends to a continuous function $D[P] \rightarrow (0, \infty]$ (cf. §3.5 below) which passes to a function $\Gamma_P \backslash D[P] \rightarrow (0, \infty]$ whose restriction to any Γ -parabolic neighborhood $U \subset \bar{X}$ of X_P we also denote by

$$f_{\alpha}^P : U \rightarrow (0, \infty].$$

Similarly the geodesic projection $\pi_P : D \rightarrow D_P$ (cf. (2.10.1)) extends continuously to a projection $D(P) \rightarrow D_P$ and passes to projections $D[P] \rightarrow D_P$ and $\Gamma_P \backslash D[P] \rightarrow \Gamma_P \backslash D$ whose restriction to any parabolic neighborhood $U \subset \bar{X}$ we denote by

$$(3.2.4) \quad \pi_P : U \rightarrow X_P.$$

The following lemma is a straightforward consequence of the definitions.

3.3. Lemma. *Let $U \subset \bar{X}$ be a parabolic neighborhood of the stratum X_P . Let $\{x_n\} \subset U$ be a sequence of points and let $y \in X_P$. The sequence $\{x_n\}$ converges to y in \bar{X} if and only if the following hold:*

- (1) $\pi_P(x_n) \rightarrow y$ in X_P and
- (2) $f_\alpha^P(x_n) \rightarrow \infty$ for all $\alpha \in \Delta_P$. \square

3.4. Suppose $\mathbf{Q} \supset \mathbf{P}$ is another rational parabolic subgroup of \mathbf{G} , corresponding, say, to a subset $J \subset \Delta_P$ with $\mathbf{S}_Q \subset \ker(\alpha)$ for all $\alpha \in J$, so that $\Delta_P = i(\Delta_Q) \amalg J$ as in (2.4.1). Let $\bar{P} = v_Q(P) \subset L_Q$ be the resulting parabolic subgroup of L_Q . It acts transitively on the boundary component D_Q .

Let $x \in D$, say $x = u_Q a'_Q a_{G\bar{P}} a_{\bar{P}}^Q m_P K'_P$ is decomposed according to (2.4.5) with $a_Q = a'_Q a_G$. Then $\pi_Q(x) = u_{\bar{P}} a_{\bar{P}}^Q a_{Gm_P} K'_P \in \bar{P}/K'_P = D_Q$ so the following equations hold:

$$(3.4.1) \quad f_\alpha^P(x) = \alpha(a'_Q a_{\bar{P}}^Q) \quad \text{for all } \alpha \in \Delta_P,$$

$$(3.4.2) \quad f_\beta^Q(x) = i(\beta)(a'_Q) \quad \text{for all } \beta \in \Delta_Q,$$

$$(3.4.3) \quad f_\alpha^{\bar{P}}(\pi_Q(x)) = f_\alpha^P(x) = \alpha(a_{\bar{P}}^Q) \quad \text{for all } \alpha \in J = \Delta_{\bar{P}},$$

since $\alpha(a'_Q) = 1$ for all $\alpha \in J$. From this we may conclude:

3.5. Proposition. *For all $\alpha \in \Delta_P$, the root function f_α^P extends continuously to a function $f_\alpha^P : D[P] \rightarrow (0, \infty]$ such that, for all $x \in D$ we have*

$$(3.5.1) \quad f_\alpha^P(\pi_Q(x)) = \begin{cases} f_\alpha^P(x) & \text{for } \alpha \in J, \\ \infty & \text{for } \alpha \in \Delta_P - J. \end{cases}$$

The boundary component $D_Q \subset D[P]$ is the set of $x \in D[P]$ such that:

$$(3.5.2) \quad \begin{cases} f_\alpha^P(x) = \infty & \text{for all } \alpha \in \Delta_P - J, \\ f_\alpha^P(x) < \infty & \text{for all } \alpha \in J. \end{cases}$$

3.6. Remarks. Of course similar statements apply to the root function $f_\alpha^P : U \rightarrow (0, \infty]$ for any parabolic neighborhood $U \subset \bar{X}$ of X_P . We think of the “negative gradient” of the root functions f_α^P as pointing in the “normal directions” to X_P . For $\alpha \in J$, $-\text{grad} f_\alpha^P$ points from X_P “into” X_Q .

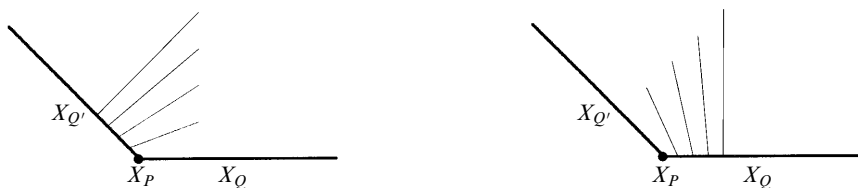


Figure 1. Level curves of f_α^P for $\alpha \in i(\Delta_Q)$ and $\alpha \in J$ respectively.

Zucker’s vexatious point ([Z1], §3.19) is that for $\mathbf{P} \subset \mathbf{Q}$ and for $\beta \in \Delta_Q$, the root functions f_β^Q and $f_{i(\beta)}^P$ do not necessarily agree: see (3.4.1) and (3.4.2) above. (In fact, they agree precisely if $i(\beta)(a_P^Q) = 1$, which is to say, if $A'_{Q'}$ and A'_Q are orthogonal, where $\mathbf{Q}' \supset \mathbf{P}$ is the parabolic subgroup complementary to \mathbf{Q} .) The nature of the level sets of f_β^Q are depicted in Figure 2.

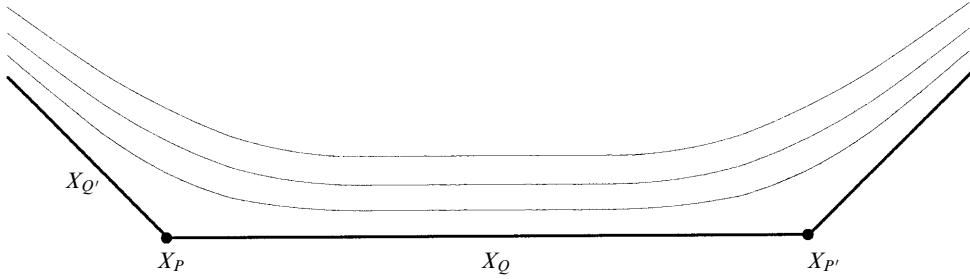


Figure 2. Level sets of f_β^Q .

This shortcoming will be circumvented by replacing the root function f_α^P with Saper’s partial distance function r_α^P (associated to a tiling), which is patched together from the various relevant root functions; cf. (4.3.2).

4. Tilings

In this section we recall a construction of Saper [Sa]. An equivalent construction of Leuzinger [Le1] could be used instead. See also [Ar2] and [L2].

4.1. Tilings of \bar{D} . As in §2, \mathbf{G} denotes a connected linear reductive algebraic group defined over \mathbb{Q} , D denotes the associated symmetric space, $K' = A_G K(x_0)$ is the stabilizer in G of a fixed basepoint $x_0 \in D$, $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is an arithmetic group and $X = \Gamma \backslash D$. Let \mathcal{P}_1 denote the set of proper *maximal* rational parabolic subgroups of \mathbf{G} . For each $\mathbf{Q} \in \mathcal{P}_1$ choose $b_Q \in A'_Q$. The collection $\mathbf{b} = \{b_Q\}$ of such choices is called a *parameter*, the set of which we denote by \mathcal{B} . The parameters are partially ordered with $\mathbf{b} \leq \mathbf{c}$ iff $\alpha_Q(b_Q) \leq \alpha_Q(c_Q)$ for all $\mathbf{Q} \in \mathcal{P}_1$, where $\Delta_Q = \{\alpha_Q\}$ is the simple root associated with the maximal parabolic subgroup \mathbf{Q} . A choice $\mathbf{b} \in \mathcal{B}$ of parameter determines, for any rational parabolic subgroup $\mathbf{P} \subset \mathbf{G}$ a unique element $b_P \in A'_P$ such that, for each rational maximal parabolic subgroup $\mathbf{Q} \supset \mathbf{P}$, the element $b_P b_Q^{-1}$ lies in A'_P (cf. §2.4). In other words, $\log(b_Q)$ is the orthogonal projection of $\log(b_P) \in \mathfrak{a}'_P$ with respect to any Weyl-invariant inner product on \mathfrak{a}'_P .

Recall from [Sa] that a *tiling with parameter $\mathbf{b} \in \mathcal{B}$* is a cover of the reductive Borel-Serre partial compactification

$$(4.1.1) \quad \bar{D} = \coprod_{\mathbf{P} \in \mathcal{P}} D^{\mathbf{P}}$$

by disjoint sets (called tiles) such that:

- (1) The central tile $D^0 = D^G$ is a closed, codimension 0 submanifold with

corners contained in D . Its closed boundary faces $\{\partial^P D^0\}$ are indexed by $\mathbf{P} \in \mathcal{P}$ with $\mathbf{P} \subset \mathbf{Q} \Leftrightarrow \partial^P D^0 \subset \partial^Q D^0$.

(2) Each boundary face $\partial^P D^0$ is contained in the ‘‘cross section’’ $F(\mathcal{U}_P \times \{b_P\} \times D_P)$ where F is defined in (2.5.2).

(3) Each tile $D^P = \partial^P D^0 \cdot \bar{A}_P (>1)$ is obtained from $\partial^P D^0$ by flowing out under the geodesic action of the cone $\bar{A}_P (>1)$ (cf. (2.9.1)).

For any rational parabolic subgroup \mathbf{Q} , the intersections $\{D^P \cap \bar{D}_Q\}$ (over all rational parabolic subgroups $\mathbf{P} \subseteq \mathbf{Q}$) form a tiling of the reductive Borel-Serre partial compactification \bar{D}_Q , whose central tile we denote by

$$(4.1.2) \quad D_Q^0 = D^Q \cap D_Q.$$

Then the tile D^Q is given by

$$(4.1.3) \quad D^Q = \{x \in D[Q] \subset \bar{D} \mid \pi_Q(x) \in D_Q^0 \text{ and } f_x^Q(x) > \alpha(b_Q) \forall \alpha \in \Delta_Q\}$$

and the boundary face $\partial^Q D^0$ is

$$(4.1.4) \quad \partial^Q D^0 = \{x \in D \mid \pi_Q(x) \in D_Q^0 \text{ and } f_x^Q(x) = \alpha(b_Q) \forall \alpha \in \Delta_Q\}.$$

A tiling, if it exists, is uniquely determined by its parameter $\mathbf{b} \in \mathcal{B}$, in which case we say that the parameter is *regular*. The parameter \mathbf{b} is Γ -invariant if, for all $\gamma \in \Gamma$, we have $b_{\gamma Q \gamma^{-1}} = a_{\gamma Q \gamma^{-1}}(\gamma x_0) b_Q$. The tiling $\{D^Q\}$ is Γ -invariant if $\gamma D^P = D^{\gamma P \gamma^{-1}}$ for all $\gamma \in \Gamma$. A tiling is Γ -invariant if and only if its parameter \mathbf{b} is Γ -invariant ([Sa], Corollary 2.7). In [Sa], Thm. 10.1, Saper proves the following.

4.2. Theorem. *If the tiling parameter $\mathbf{b} \in \mathcal{B}$ is chosen sufficiently large (with respect to the above partial ordering) and Γ -invariant, then there exists a unique tiling with parameter $\mathbf{b} \in \mathcal{B}$, and it is Γ -invariant. Moreover, for any $\mathbf{Q} \in \mathcal{P}$ the union*

$$(4.2.1) \quad T(\bar{D}_Q) = \coprod_{P \subseteq Q} D^P$$

is an open Γ_Q -invariant parabolic neighborhood of \bar{D}_Q in \bar{D} which may be made arbitrarily small by choosing the parameter \mathbf{b} sufficiently large.

Henceforth we shall refer to such a parameter as *regular and sufficiently large*. Fix such a parameter $\mathbf{b} = \{b_Q\}$. Denote the closure of $T(\bar{D}_Q)$ by $\bar{T}(\bar{D}_Q)$, and the boundary by $\partial T(\bar{D}_Q) = \bar{T}(\bar{D}_Q) - T(\bar{D}_Q)$. Following [Sa], Thm. 8.1 (ii), for each $\alpha \in \Delta_Q$, define the *partial distance function* $r_\alpha^Q : \bar{T}(\bar{D}_Q) \rightarrow [0, 1]$ by

$$(4.2.2) \quad r_\alpha^Q(x) = \begin{cases} f_\alpha^Q(x)^{-1} \alpha(b_Q) & \text{for } x \in \bar{D}^Q, \\ f_{i(\alpha)}^P(x)^{-1} i(\alpha)(b_P) & \text{for } x \in \bar{D}^P, \end{cases}$$

whenever $\mathbf{P} \subset \mathbf{Q}$. Here, $i : \Delta_Q \hookrightarrow \Delta_P$ is the inclusion (2.4.1) and \bar{D}^Q (resp. \bar{D}^P) is the closure of the tile D^Q (resp. D^P).

4.3. Lemma. *The following statements hold.*

(1) *The mapping $r_\alpha^Q : \bar{T}(\bar{D}_Q) \rightarrow [0, 1]$ is well-defined, continuous, and piecewise analytic.*

(2) *For all $\alpha \in \Delta_Q$, the geodesic action by $t \in A_Q (\geq 1)$ satisfies*

$$(4.3.1) \quad r_\alpha^Q(x \cdot t) = r_\alpha^Q(x)\alpha(t)^{-1}$$

whenever $x \in \bar{T}(\bar{D}_Q)$.

(3) *If $x \in \bar{T}(\bar{D}_Q)$ then*

$$\begin{aligned} x \in \bar{D}_Q &\Leftrightarrow r_\alpha^Q(x) = 0 \text{ for all } \alpha \in \Delta_Q, \\ x \in \partial T(\bar{D}_Q) &\Leftrightarrow r_\alpha^Q(x) = 1 \text{ for some } \alpha \in \Delta_Q. \end{aligned}$$

(4) *If $\gamma \in \Gamma \cap Q$ then $r_\alpha^Q(\gamma x) = r_\alpha^Q(x)$.*

(5) *If $\gamma \in \Gamma$ and $\mathbf{Q}' = \gamma\mathbf{Q}\gamma^{-1}$ and if $\alpha' \in \Delta_{\mathbf{Q}'}$ is the simple root corresponding to $\alpha \in \Delta_{\mathbf{Q}}$ then*

$$r_{\alpha'}^{\mathbf{Q}'}(\gamma x) = r_\alpha^Q(x).$$

(6) *If $\mathbf{P} \subset \mathbf{Q}$ and if $\Delta_P = i(\Delta_Q) \amalg J$ as in (2.4.1) then, for all $\alpha \in \Delta_Q$ and for all $x \in T(\bar{D}_P) \subset T(\bar{D}_Q)$ we have*

$$(4.3.2) \quad r_{i(\alpha)}^P(x) = r_\alpha^Q(x).$$

4.4. Proof. As in §2.4, write $A'_P = A'_Q A'_P{}^Q$ with \mathfrak{a}'_Q the orthogonal complement to \mathfrak{a}_P^Q in \mathfrak{a}'_P . So the elements b_P and b_Q determined by the parameter \mathbf{b} satisfy $b_P = b_Q b_P^Q$ for some $b_P^Q \in A_P^Q$. Now suppose that $x = u_Q a'_Q a_G u_{\bar{P}} a'_P m_P K_P \in P/K_P = D$ is decomposed according to (2.4.5). Set $a'_P = a'_Q a'_P{}^Q \in A'_P$. If $x \in \partial^P D^0$ then by property (2), $a'_P = b_P$, that is, $a'_Q = b_Q$ and $a'_P{}^Q = b_P^Q$. Flowing out under the geodesic action of A'_Q we see that $x \in \bar{D}^Q \cap \bar{D}^P \cap D$ implies that $a'_P{}^Q = b_P^Q$. For such a point x and for each $\alpha \in \Delta_Q$, we have

$$\begin{aligned} f_{i(\alpha)}^P(x)^{-1} i(\alpha)(b_P) &= i(\alpha)(a_Q a'_P{}^Q)^{-1} i(\alpha)(b_Q b_P^Q) \\ &= \alpha(a_Q^{-1} b_Q) = f_\alpha^Q(x)^{-1} \alpha(b_Q) \end{aligned}$$

so both equations (4.2.2) agree on their common domain of definition, proving (1).

By continuity, it suffices to prove (2) for points $x \in T(\bar{D}_Q)$. The geodesic action by $t \in A_Q (\geq 1)$ preserves the tiles in $T(\bar{D}_Q)$ so (2) may be checked tile by tile. If $x \in D^Q$ then $f_\alpha^Q(x \cdot t) = f_\alpha^Q(x)\alpha(t)$ by (3.2.2). If $x \in D^P$ for some $\mathbf{P} \subset \mathbf{Q}$, write $\Delta_P = i(\Delta_Q) \amalg J$ as in (2.4.1), note that $A_Q (\geq 1) \subset A_P (\geq 1)$ and compute, for $\alpha \in \Delta_Q$,

$$f_{i(\alpha)}^P(x \cdot t) = f_{i(\alpha)}^P(x) i(\alpha)(t) = f_{i(\alpha)}^P(x) \alpha(t)$$

which proves the second statement. Part (3) follows from (3.2.2) for points $x \in \bar{T}_Q$ and from (3.2.3) for points $x \in \bar{T}_P$. Part (4) follows from Lemma 3.3 for points $x \in \bar{T}_Q$ and from (3.5.1) for points $x \in \bar{T}_P$. Part (5) follows from (3.2.3) and part (6) is an immediate consequence of the definition. \square

4.5. Tiling of \bar{X} . Suppose $\mathbf{b} \in \mathcal{B}$ is a sufficiently large regular parameter and (4.1.1) is the associated Γ -invariant tiling. Let $\bar{\tau} : \bar{D} \rightarrow \bar{X}$ denote the projection to the reductive Borel-Serre compactification of $X = \Gamma \backslash D$. If \mathbf{P}, \mathbf{P}' are rational parabolic subgroups of \mathbf{G} then either $\bar{\tau}(D^{\mathbf{P}}) \cap \bar{\tau}(D^{\mathbf{P}'}) = \emptyset$ or else they coincide. Hence the collection of images

$$X^P = \bar{\tau}(D^P)$$

forms a decomposition of \bar{X} whose “tiles” are indexed by the set of Γ -conjugacy classes of rational parabolic subgroups of \mathbf{G} . Let $X_P^0 = X^G = \bar{\tau}(D_P^0)$ be the “central tile.” Denote by

$$(4.5.1) \quad T(\bar{X}_P) = \bar{\tau}(T(\bar{D}_P)) = \coprod_{\{R\} \subseteq P} X^R$$

the resulting neighborhood of \bar{X}_P in \bar{X} , and by $\partial T(\bar{X}_P) = \bar{T}(\bar{X}_P) - T(\bar{X}_P)$ its boundary. (Here, R runs through a set of representatives, one from each Γ -conjugacy class $\{R\}$ of parabolic subgroups contained in P .) For all $\alpha \in \Delta_P$ the functions r_α^P pass to piecewise analytic functions on $T(\bar{X}_P)$, which we also denote by r_α^P .

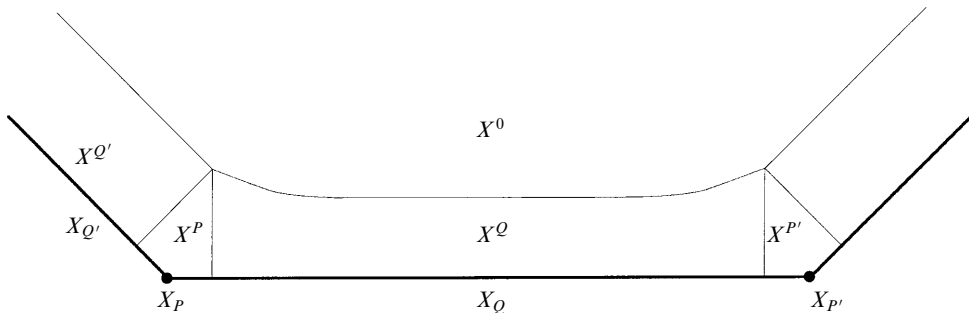


Figure 3. Tiles.

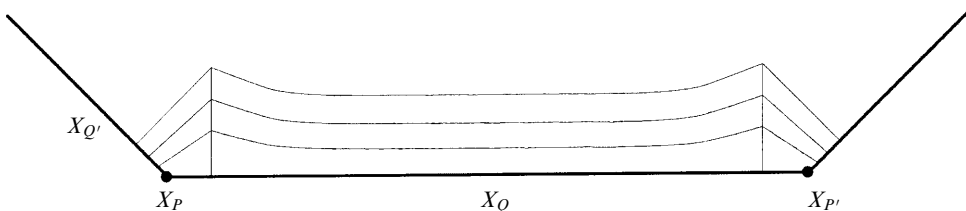


Figure 4. Level sets of r_α^Q .

4.6. Retraction and exhaustion. Saper proves ([Sa], §6.1) that there exists a unique Γ -equivariant continuous and piecewise analytic “geodesic” retraction $R : D \rightarrow D^0$ which is the identity on D^0 such that, for all $y \in D^P$ and for all $t \in A_P (\geq 1)$ the following holds:

$$(4.6.1) \quad R(y \cdot t) = R(y).$$

Then R preserves tiles and it passes to a retraction which we also denote by $R : X \rightarrow X^0$; it has the same property (4.6.1). Define $W : \bar{X} \rightarrow [0, 1]$ by

$$W(x) = \begin{cases} 1 & \text{if } x \in X^0, \\ \sup_{\alpha \in \Delta_Q} \{1 - r_\alpha^Q(x)\} & \text{if } x \in X^Q. \end{cases}$$

We refer to W as an *exhaustion function* because

$$W^{-1}(0) = X^0 \quad \text{and} \quad W^{-1}(1) = \bar{X} - X.$$

The function W is continuous (and piecewise analytic): If $\mathbf{P} \subset \mathbf{Q}$, write $\Delta_P = i(\Delta_Q) \amalg J$ as in (2.4.1). Let $\bar{P} = v_Q(P) \subset L_Q$ be the resulting parabolic subgroup of L_Q ; then $\Delta_{\bar{P}} = J$. If $x \in \bar{X}^P \cap \bar{X}^Q$ then $\pi_Q(x) \in \bar{X}_{\bar{P}}^P \cap \bar{X}_Q^0$ so $r_\alpha^P(x) = r_\alpha^{\bar{P}}(\pi_Q(x)) = 1$ for all $\alpha \in J$ as in §3.4. Hence

$$\sup_{\alpha \in \Delta_P} \{1 - r_\alpha^P(x)\} = \sup_{\alpha \in \Delta_Q} \{1 - r_\alpha^Q(x)\}.$$

For each boundary stratum X_Q the same constructions define a tile-preserving retraction

$$(4.6.2) \quad R_Q : X_Q \rightarrow X_Q^0$$

and an exhaustion function $W_Q : \bar{X}_Q \rightarrow [0, 1]$ with $W_Q^{-1}(0) = X_Q^0$ and $W_Q^{-1}(1) = \bar{X}_Q - X_Q$. In fact, the stratum closure \bar{X}_Q is tiled by the collection of intersections $\bar{X}_Q^P = \bar{X}_Q \cap X^P$ for $\mathbf{P} \subseteq \mathbf{Q}$ and

$$(4.6.3) \quad W_Q(x) = \begin{cases} 1 & \text{if } x \in X_Q^0, \\ \sup_{\alpha \in J} \{1 - r_\alpha^P(x)\} & \text{if } x \in X_Q^P, \end{cases}$$

where $\Delta_P = i(\Delta_Q) \amalg J$.

4.7. Remarks. We risk a certain amount of confusion by having defined $r_\alpha^P(x)$ so as to decrease as $x \rightarrow X_P$ whereas the root function $f_\alpha^P(x)$ increases as $x \rightarrow X_P$. Although Saper [Sa] actually constructs a tiling of the Borel-Serre compactification \bar{X} the same approach gives a tiling of the reductive Borel-Serre compactification \bar{X} . The collection $\{T(\bar{X}_P), \pi_P, r_P = \max_{\alpha \in \Delta_P} \{r_\alpha^P\}\}$ of tubular neighborhoods, tubular projections, and tubular distance functions are very much like a “system of control data” [Mat], [Gi], [GM3] for the stratified space \bar{X} , but there are several important differences. The functions r_P are continuous and piecewise analytic but are not smooth. Whenever $\mathbf{Q} \subseteq \mathbf{P}$ we have $\pi_Q \pi_P = \pi_Q$ however we do not have $r_Q \pi_P = r_Q$. For this price we gain an especially strong form of “local triviality” for the stratification of \bar{X} : the neighborhood $T(\bar{X}_P)$ is (homeomorphic to) a mapping cylinder neighborhood of the closure of the stratum X_P . In fact, it is possible to use the various geodesic actions to construct a (piecewise analytic) homeomorphism between $T(\bar{X}_P)$ and the (open) mapping cylinder of the projection $\pi_P : \partial T(\bar{X}_P) \rightarrow \bar{X}_P$. (The open mapping cylinder of a mapping $\pi : A \rightarrow B$ is the quotient $(A \times [0, 1] \amalg B) / \sim$ under the relation $(a, 0) \sim \pi(a)$.) Analogous statements for other Satake compactifications (such as the Baily-Borel compactification) are false.

5. A little shrink

5.1. As in §2, \mathbf{G} denotes a connected linear reductive algebraic group defined over \mathbb{Q} , D denotes the associated symmetric space, $K' = A_G K(x_0)$ is the stabilizer in G of a fixed basepoint $x_0 \in D$, $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is an arithmetic group and $X = \Gamma \backslash D$. In this section we construct a homeomorphism $\bar{X} \rightarrow \bar{X}$ which moves a neighborhood of the boundary towards the boundary. When composed with a Hecke correspondence, this will have the effect of chopping the fixed point set into pieces, each of which is contained in a single stratum of \bar{X} . The resulting behavior is much easier to analyze. This “shrink” homeomorphism may be considered to be a topological analog to Arthur’s truncation procedure.

5.2. Let $\mathbf{Q} \in \mathcal{P}_1$ be a standard proper maximal rational parabolic subgroup of \mathbf{G} . Fix $t \in A'_Q (>1)$ so $\alpha(t) > 1$, where $\alpha \in \Delta_Q$ is the unique simple root. The geodesic action of t on D extends continuously to the neighborhood $D[Q]$ (2.10.2) of D_Q in the reductive Borel-Serre partial compactification \bar{D} of D . This geodesic action even extends continuously to the neighborhood

$$D\{Q\} = \bigcup_{\mathbf{P} \subset \mathbf{Q}} D[P]$$

of the closure \bar{D}_Q , where the union is taken over all rational parabolic subgroups $\mathbf{P} \subset \mathbf{G}$ which are contained in \mathbf{Q} . (For if $\mathbf{P} \subset \mathbf{Q}$, let $i : A_Q \rightarrow A_P$ be the canonical inclusion. It follows from (2.4.4) (or [BS], Prop. 3.11) that the geodesic action of the image $i(t) \in A_P$ agrees with the geodesic action of $t \in A_Q$, so it extends continuously to $D[P]$.) We continue to denote this action by $x \mapsto x \cdot t$ for $x \in D\{Q\}$.

Now fix a sufficiently large Γ -invariant regular parameter $\mathbf{b} \in \mathcal{B}$ with its resulting tiling (4.1.1) and partial distance functions (4.2.2) satisfying Lemma 4.3. Let $T(\bar{D}_Q) \subset D\{Q\}$ denote the neighborhood of \bar{D}_Q (4.2.1) consisting of the union of all tiles which intersect \bar{D}_Q nontrivially, and let $\bar{T}(\bar{D}_Q)$ denote its closure. The geodesic action by $t \in A'_Q (>1)$ preserves $\bar{T}(\bar{D}_Q)$ since $\alpha(t) > 1$.

Fix once and for all a smooth non-increasing function $\rho : [0, 1] \rightarrow [0, 1]$ with $\rho(r) = 1 \Leftrightarrow r \leq 1/2$ and with $\rho(r) = 0 \Leftrightarrow r = 1$.

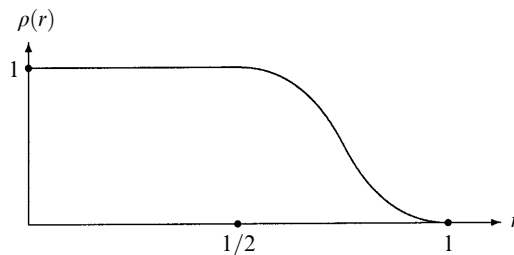


Figure 5. The function ρ .

Let $r_\alpha^Q : T(\bar{D}_Q) \rightarrow [0, 1)$ be the partial distance function (4.2.2) which corresponds to the unique simple root $\alpha \in \Delta_Q$. For $t \in A'_Q (>1)$ define $Sh(Q, t) : \bar{T}(\bar{D}_Q) \rightarrow \bar{T}(\bar{D}_Q)$ by

$$(5.2.1) \quad Sh(Q, t)(x) = x \cdot t^{\rho(r_x^Q(x))}.$$

Then by (3.2.2) and (4.3.1),

$$(5.2.2) \quad r_x^Q(Sh(Q, t)(x)) = r_x^Q(x)\alpha(t)^{-\rho(r_x^Q(x))},$$

$$(5.2.3) \quad f_x^Q(Sh(Q, t)(x)) = f_x^Q(x)\alpha(t)^{\rho(r_x^Q(x))}$$

for all $x \in \bar{T}(\bar{D}_Q)$. The quantity $\alpha(t)^{\rho(r_x^Q(x))}$ is bounded between 1 and $\alpha(t)$. It equals 1 if and only if $x \in \partial T(\bar{D}_Q)$, that is, if and only if $r_x^Q(x) = 1$.

If Q' is another maximal rational parabolic subgroup of G , let us write $Q' \sim Q$ if Q' is $G(\mathbb{Q})$ conjugate to Q . In this case, any choice $g \in G(\mathbb{Q})$ of conjugating element induces the same isomorphism $S_Q \cong S_{Q'}$ so we obtain a corresponding $t' \in A'_{Q'}$ and a corresponding mapping $Sh(Q', t') : \bar{T}(\bar{D}_{Q'}) \rightarrow \bar{T}(\bar{D}_Q)$. Define the *shrink* $Sh_Q(t) : \bar{D} \rightarrow \bar{D}$ corresponding to conjugates of the standard parabolic subgroup Q by

$$Sh_Q(t)(x) = \begin{cases} Sh(Q', t')x & \text{if } x \in \bar{T}(\bar{D}_{Q'}) \text{ for some } Q' \sim Q, \\ x & \text{otherwise.} \end{cases}$$

Then $Sh_Q(t)$ is well defined and continuous because $T(\bar{D}_{Q'}) \cap T(\bar{D}_Q) = \emptyset$ whenever $Q' \sim Q$ (and $Q' \neq Q$), cf. [Bo3], §11.17 (iii). Moreover, if $\gamma \in \Gamma$ then

$$Sh_Q(t)(\gamma x) = \gamma Sh_Q(t)(x)$$

by Lemma 4.3. So (dividing by Γ), the homeomorphism $Sh_Q(t)$ passes to a homeomorphism which we denote in the same way, $Sh_Q(t) : \bar{X} \rightarrow \bar{X}$.

5.3. Suppose $P \subseteq Q$ is a rational parabolic subgroup of G ; set $\Delta_P = i(\Delta_Q) \amalg J$ as in (2.4.1). It follows from (4.3.1) that for all $\beta \in J$ and for all $x \in \bar{T}(\bar{D}_P)$ we have

$$(5.3.1) \quad r_\beta^P Sh(Q, t)(x) = r_\beta^P(x)$$

since $\beta(t) = 1$ for any $\beta \in J$. Now suppose Q_1, Q_2 are two standard maximal rational parabolic subgroups of G whose intersection $P = Q_1 \cap Q_2$ is parabolic. Let $\alpha_i \in \Delta(Q_i)$ be the unique nonzero roots. Choose $t_i \in A'_i = A'_{Q_i}$ with $\alpha_i(t_i) > 1$ and let

$$Sh_i = Sh(Q_i, t_i) : \bar{T}(\bar{D}_{Q_i}) \rightarrow \bar{T}(\bar{D}_{Q_i})$$

denote the resulting two shrinks. It follows by taking $P = Q_1 \cap Q_2$ in (5.3.1) that the mappings Sh_1 and Sh_2 commute on their common domain of definition,

$$\bar{T}(\bar{D}_P) = \bar{T}(\bar{D}_{Q_1}) \cap \bar{T}(\bar{D}_{Q_2}).$$

5.4. Let $P_0 \subset G$ be the standard minimal rational parabolic subgroup with $S'_0 = S_{P_0}/S_G$ and with simple roots $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$ numbered in any order. Each j (with $1 \leq j \leq r$) corresponds to a standard maximal proper rational parabolic subgroup Q_j with split torus $S'_j = S_{P_j}/S_G$ and identity component $A'_j = S'_j(\mathbb{R})^0$. Choose $t \in A'_0$ to be

dominant and regular. In other words, with respect to the canonical *complementary* decomposition (cf. 2.4) $A'_0 = \prod_j A'_j$ (with $\ker(\alpha_j) = \prod_{i \neq j} S'_i$) we may write $\mathbf{t} = t_1 t_2 \dots t_r$ where $t_j \in A'_j$ and $\alpha_j(t_j) = \alpha_j(\mathbf{t}) > 1$. Define $Sh(\mathbf{t}) : \bar{D} \rightarrow \bar{D}$ to be the composition

$$Sh(\mathbf{t}) = Sh_{\mathbf{Q}_1}(t_1) \circ Sh_{\mathbf{Q}_2}(t_2) \circ \dots \circ Sh_{\mathbf{Q}_r}(t_r)$$

where each $Sh_{\mathbf{Q}_j}(t_j) : \bar{D} \rightarrow \bar{D}$ is the shrink defined above, corresponding to conjugates of the standard parabolic subgroup \mathbf{Q}_j .

5.5. Proposition. *The mapping $Sh(\mathbf{t}) : \bar{D} \rightarrow \bar{D}$ is independent of the ordering $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$ of the simple roots. It is a Γ -equivariant homeomorphism and passes to a homeomorphism $Sh(\mathbf{t}) : \bar{X} \rightarrow \bar{X}$ with the following properties:*

(1) *It preserves the tiles and the strata, that is, for each rational parabolic subgroup $\mathbf{P} \subseteq \mathbf{G}$, we have $Sh(\mathbf{t})(X^{\mathbf{P}}) = X^{\mathbf{P}}$ and $Sh(\mathbf{t})(X_{\mathbf{P}}) = X_{\mathbf{P}}$.*

(2) *Within each tile, it is given by a geodesic action: for each $x \in X^{\mathbf{P}}$ there exists $b = b_x \in A_{\mathbf{P}} (\geq 1)$ so that $Sh(\mathbf{t})(x) = x \cdot b$.*

(3) *It is the identity on each central tile $X_{\mathbf{P}}^0$ and $\pi_{\mathbf{P}}(Sh(\mathbf{t})(x)) = \pi_{\mathbf{P}}(x) \in X_{\mathbf{P}}^0$ for all $x \in X^{\mathbf{P}}$.*

(4) *It commutes with the geodesic projection, that is, for any rational parabolic subgroup $\mathbf{P} \subset \mathbf{G}$ and for each $x \in \bar{T}(\bar{X}_{\mathbf{P}})$ we have $\pi_{\mathbf{P}}(Sh(\mathbf{t})(x)) = Sh(\mathbf{t})(\pi_{\mathbf{P}}(x))$.*

(5) *It is (globally) homotopic to the identity.*

(6) *For any rational parabolic subgroup $\mathbf{P} \subset \mathbf{G}$ and for each $\alpha \in \Delta_{\mathbf{P}}$ and for each $x \in \bar{T}(\bar{X}_{\mathbf{P}})$, by equation (5.2.2) we have:*

$$(5.5.1) \quad r_{\alpha}^{\mathbf{P}}(Sh(\mathbf{t})(x)) = r_{\alpha}^{\mathbf{P}}(x) \alpha_0(\mathbf{t})^{-\rho(r_{\alpha}^{\mathbf{P}}(x))}$$

where $\alpha_0 \in \Delta$ is the unique root which agrees with α after conjugation and restriction to $\mathbf{S}_{\mathbf{P}}$. If $x \in X^{\mathbf{P}}$ is constrained to lie in the single tile $X^{\mathbf{P}}$ then also

$$(5.5.2) \quad f_{\alpha}^{\mathbf{P}}(Sh(\mathbf{t})(x)) = f_{\alpha}^{\mathbf{P}}(x) \alpha_0(\mathbf{t})^{\rho(r_{\alpha}^{\mathbf{P}}(x))}. \quad \square$$

We remark that the mapping $Sh(\mathbf{t})$ depends on the choice of regular parameter (which determines the size of the tiles).

6. Morphisms and Hecke correspondences

6.1. As in §2, \mathbf{G} denotes a connected linear reductive algebraic group defined over \mathbb{Q} , D denotes the associated symmetric space, and $K' = A_G K(x_0)$ is the stabilizer in G of a fixed basepoint $x_0 \in D$. Let $\Gamma, \Gamma' \subset \mathbf{G}(\mathbb{Q})$ be arithmetic subgroups and set $X = \Gamma \backslash D$ and $X' = \Gamma' \backslash D$.

6.2. Definition. A mapping $f : X' \rightarrow X$ is a *morphism* if there exists $g \in \mathbf{G}(\mathbb{Q})$ such that

- (1) $g\Gamma'g^{-1} \subset \Gamma$,
- (2) $[\Gamma : g\Gamma'g^{-1}] < \infty$,
- (3) $f(\Gamma'xK') = \Gamma gxK'$ for any $x \in G$.

The morphism f is determined by the pair (Γ', g) by (3); it is well defined by (1). For any $\gamma \in \Gamma$, $\gamma' \in \Gamma'$ the pair $(\Gamma', \gamma g \gamma')$ determines the same morphism. If Γ is torsion-free then f is an unramified covering of degree $[\Gamma : g\Gamma'g^{-1}]$ and it is locally an isometry with respect to the invariant Riemannian metrics on X' and X induced from any K invariant inner product on ${}^0\mathfrak{p}$ (§2.1). Denote by $\text{Mor}(X', X)$ the set of morphisms $X' \rightarrow X$.

6.3. Lemma. *Each morphism $f \in \text{Mor}(X', X)$ admits unique continuous extensions $\tilde{f} : \tilde{X}' \rightarrow \tilde{X}$ to the Borel-Serre compactification and $\bar{f} : \bar{X}' \rightarrow \bar{X}$ to the reductive Borel-Serre compactification. The mappings \tilde{f} and \bar{f} are finite, and they restrict to morphisms on each boundary stratum. If $\tilde{f}(X'_p) = X_Q$, if U' and U are Γ' and Γ -parabolic neighborhoods of X'_p and X_Q in \tilde{X}' and \tilde{X} respectively, then*

$$(6.3.1) \quad \bar{f}(\pi_P(x)) = \pi_Q(\tilde{f}(x))$$

for all $x \in U' \cap \tilde{f}^{-1}(U)$.

6.4. Proof. Suppose the morphism $f : X' \rightarrow X$ is given by the pair (Γ', g) . Let $T_g : D \rightarrow D$ denote the action of g on D . It moves the basepoint x_0 to a new basepoint $x_1 = gx_0$ with stabilizer $K'(x_1) = {}^gK' = gK'g^{-1}$. If \mathbf{P} is a rational parabolic subgroup and if $\mathbf{Q} = {}^g\mathbf{P} = g\mathbf{P}g^{-1}$ set $K'_p(x_0) = K' \cap P$ and $K'_Q(x_1) = K'(x_1) \cap Q$. Then T_g may also be described as the mapping

$$(6.4.1) \quad D = P/K'_p(x_0) \rightarrow Q/K'_Q(x_1) = D$$

which is given by $xK'_p(x_0) \mapsto gxg^{-1}K'_Q(x_1)$ by (2.1.3). This intertwines the geodesic actions of A'_p and A'_Q , that is,

$$(6.4.2) \quad T_g(x \cdot a) = gxg^{-1}gi_{x_0}(a)g^{-1}K'_Q(x_1) = gxg^{-1}i_{x_1}(\hat{a})K'_Q(x_1) = T_g(x) \cdot \hat{a}$$

where $a \mapsto \hat{a}$ is the canonical identification $A'_p \cong A'_Q$ of §2.3. It follows that T_g extends to a mapping $\tilde{T}_g : \tilde{D} \rightarrow \tilde{D}$ on the Borel-Serre partial compactification, which takes the boundary component $e_p = P/K'_pA'_p$ to $e_Q = Q/K'_QA'_Q$ and satisfies

$$(6.4.3) \quad \pi_Q\tilde{T}_g(x) = \tilde{T}_g(\pi_P(x)).$$

The mapping \tilde{T}_g passes to a mapping $\tilde{f} : \Gamma' \backslash \tilde{D} \rightarrow \Gamma \backslash \tilde{D}$ which is the desired extension. It maps $Y'_p = \Gamma'_p \backslash e_p$ to $Y_Q = \Gamma_Q \backslash e_Q$ by $\tilde{f}(\Gamma'_p x K'_p(x_0) A'_p) = \Gamma_Q g x g^{-1} K'_Q(x_1) A'_Q$, which is a mapping of degree $[\Gamma_Q : g\Gamma'_p g^{-1}] < \infty$. (Here, $\mathbf{Q} = {}^g\mathbf{P}$.) The extension \tilde{f} may map several

strata of \tilde{X}' to a single stratum of \tilde{X} : let $\mathbf{Q}_1, \mathbf{Q}_2, \dots, \mathbf{Q}_m$ be a set of representatives for the ${}^g\Gamma'$ -conjugacy classes of rational parabolic subgroups which are Γ -conjugate to \mathbf{Q} , and set $\mathbf{P}_j = g^{-1}\mathbf{Q}_jg$. Then \tilde{f} maps each stratum $Y'_{\mathbf{P}_j} \subset \tilde{X}'$ to the stratum $Y_{\mathbf{Q}} \subset \tilde{X}$ by a morphism which may be described in a manner similar to (6.4.1). This shows that \tilde{f} is finite and that its restriction to each boundary stratum is a morphism.

Similarly the mapping \tilde{T}_g passes to a mapping $\bar{T}_g : \bar{D} \rightarrow \bar{D}$ on the reductive Borel-Serre compactification of D , which further passes to a mapping $\bar{f} : \Gamma' \backslash \bar{D} \rightarrow \Gamma \backslash \bar{D}$. Then \bar{f} maps $X'_P = \Gamma'_P \backslash P / K_P A_P \mathcal{U}_P$ to $X_Q = \Gamma_Q \backslash Q / K_Q A_Q \mathcal{U}_Q$ by

$$\bar{f}(\Gamma'_P x K_P(x_0) A_P \mathcal{U}_P) = \Gamma_Q g x g^{-1} K_Q(x_1) A_Q \mathcal{U}_Q.$$

The degree of this mapping is not obviously finite because the intersection $\Gamma'_P \cap \mathcal{U}_P$ is nontrivial. By (2.1.2) conjugation by g takes $L_P(x_0)$ to $L_Q(x_1)$. Let $K_{L(P)}(x_0) A_P$ be the stabilizer in L_P of the basepoint $\pi_P(x_0) \in D_P$ and let $K_{L(Q)}(x_1) A_Q$ be the stabilizer in L_Q of the basepoint $\pi_Q(x_1) \in D_Q$. Set $\Gamma'_{L(P)} = v_P(\Gamma'_P) \subset L_P$ and $\Gamma_{L(Q)} = v_Q(\Gamma_Q) \subset L_Q$. Then $X'_P = \Gamma'_{L(P)} \backslash L_P / K_{L(P)}(x_0) A_P$ and $X_Q = \Gamma_{L(Q)} \backslash L_Q / K_{L(Q)}(x_1) A_Q$ with respect to which we may express \bar{f} as follows:

$$\bar{f}(\Gamma'_{L(P)} x K_{L(P)}(x_0) A_P) = \Gamma_{L(Q)} g x g^{-1} K_{L(Q)}(x_1) A_Q$$

which has degree $[\Gamma_{L(Q)} : g\Gamma'_{L(P)}g^{-1}] < \infty$. As in the preceding paragraph, the mapping \bar{f} will take each of the finitely many strata X'_j to the stratum X_Q (for $1 \leq j \leq m$) by a similarly defined finite morphism. \square

6.5. Definition. A *correspondence* on $X = \Gamma \backslash G / K'$ is an arithmetic subgroup $\Gamma' \subset \mathbf{G}(\mathbb{Q})$ together with two morphisms $c_1, c_2 \in \text{Mor}(C, X)$, where $C = \Gamma' \backslash D$. A point $x \in C$ is *fixed* if $c_1(x) = c_2(x)$. Two correspondences $(c_1, c_2) : C \rightrightarrows X$ and $(c'_1, c'_2) : C' \rightrightarrows X$ are said to be *isomorphic* if there is an invertible morphism $\alpha : C \rightarrow C'$ such that $c'_j \circ \alpha = c_j$ (for $j = 1, 2$).

Each $g \in \mathbf{G}(\mathbb{Q})$ gives rise to a *Hecke correspondence* $C = C[g] \rightrightarrows X$ as follows: set $\Gamma[g] = \Gamma \cap g^{-1}\Gamma g$, $C = \Gamma[g] \backslash G / K'$, and define

$$(6.5.1) \quad (c_1, c_2)(\Gamma[g]xK') = (\Gamma xK', \Gamma g x K').$$

Modifying g by an element of $\mathbf{S}_{\mathbf{G}}(\mathbb{Q})$ does not change the Hecke correspondence. By Lemma 6.3 each correspondence $C \rightrightarrows X$ has a unique continuous extension $\bar{C} \rightrightarrows \bar{X}$ to the reductive Borel-Serre compactification, and an isomorphism $\alpha : C \rightarrow C'$ of correspondences $C \rightrightarrows X$, $C' \rightrightarrows X$ extends uniquely to an isomorphism $\bar{C} \rightarrow \bar{C}'$ of the extended correspondences.

6.6. Lemma. *Let $X = \Gamma \backslash D$ and let $g \in \mathbf{G}(\mathbb{Q})$. The isomorphism class of the resulting Hecke correspondence $\bar{C} = \bar{C}[g] \rightrightarrows \bar{X}$ depends only on the double coset $\Gamma g \Gamma \in \Gamma \backslash \mathbf{G}(\mathbb{Q}) / \Gamma$.*

6.7. Proof. It suffices to verify the statement for the correspondence $C \rightrightarrows X$ since the extension to the reductive Borel-Serre compactification exists uniquely. Let $\gamma_1, \gamma_2 \in \Gamma$ and

let $g' = \gamma_1 g \gamma_2$ be another element in the same double coset $\Gamma g \Gamma$. Set $\Gamma[g'] = \Gamma \cap g'^{-1} \Gamma g'$, $C' = \Gamma[g'] \backslash G/K'$, and define $(c'_1, c'_2) : C' \rightrightarrows X$ by

$$c'_1(\Gamma[g']xK') = \Gamma xK', \quad c'_2(\Gamma[g']xK') = \Gamma g'xK'.$$

One verifies by direct calculation that the morphism $f : C' \rightarrow C$ which is given by

$$(6.7.1) \quad f(\Gamma[g']xK') = \Gamma[g]\gamma_2 xK'$$

is a well-defined isomorphism of correspondences, with inverse given by $f^{-1}(\Gamma[g]xK') = \Gamma[g']\gamma_2^{-1}xK'$.

6.8. Remark. It can be shown that the mapping $(c_1, c_2) : C \rightarrow X \times X$ is generically one-to-one. In the event that every element of $g^{-1}\Gamma g$ is neat, then this mapping is globally an embedding. The following proposition says every correspondence is a covering of a Hecke correspondence.

6.9. Proposition. *Let $\Gamma' \subset \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup, $C' = \Gamma' \backslash D$ and $(c'_1, c'_2) : C' \rightrightarrows X$ be a correspondence. Then there is a Hecke correspondence $(c_1, c_2) : C[g] \rightrightarrows X$ and a subgroup $\Gamma'' \subset \Gamma[g]$ such that the correspondence $C' \rightrightarrows X$ is isomorphic to the correspondence*

$$(6.9.1) \quad C'' \xrightarrow{h} C[g] \rightrightarrows X$$

where $C'' = \Gamma'' \backslash G/K$ and $h(\Gamma''xK) = \Gamma[g]xK$.

6.10. Proof. Suppose $c'_1(\Gamma'xK') = \Gamma g_1 xK'$ and $c'_2(\Gamma'xK') = \Gamma g_2 xK'$ where $g_j \Gamma' g_j^{-1} \subset \Gamma$ are subgroups of finite index. Then $g = g_2 g_1^{-1}$ determines a Hecke correspondence $C[g] = \Gamma[g] \backslash D \rightrightarrows X$. Define $\Gamma'' = g_1 \Gamma' g_1^{-1}$ and $C'' = \Gamma'' \backslash D$. Since $\Gamma'' \subset \Gamma[g]$ we obtain a correspondence in “standard form”,

$$(6.10.1) \quad C'' \xrightarrow{h} C[g] \rightrightarrows X$$

with $h(\Gamma''xK') = \Gamma[g]xK'$. Define $f : C' \rightarrow C''$ by $f(\Gamma'xK') = \Gamma''g_1 xK'$. Then f is well defined, and it is easily seen to be an isomorphism of correspondences. \square

6.11. Narrow tilings. Let $(c_1, c_2) : \bar{C} \rightrightarrows \bar{X}$ be a Hecke correspondence defined by some element $g \in \mathbf{G}(\mathbb{Q})$, so $C = \Gamma' \backslash D$ with $\Gamma' = \Gamma[g] = \Gamma \cap g^{-1} \Gamma g$. Let $b \in \mathcal{B}$ be a sufficiently large Γ -invariant regular parameter. Then it is also Γ' -invariant, it gives rise to tilings $\{C^Q\}$ of \bar{C} and $\{X^Q\}$ of \bar{X} , and the mapping $c_1 : \bar{C} \rightarrow \bar{X}$ takes tiles to tiles (although the same cannot necessarily be said of c_2). Let us say this tiling is *narrow* with respect to the Hecke correspondence if, for every stratum C_Q of \bar{C} , the following holds:

$$c_1(\bar{T}(\bar{C}_Q)) \cap c_2(\bar{T}(\bar{C}_Q)) \neq \emptyset \Leftrightarrow c_1(C_Q) = c_2(C_Q)$$

and if, in this case, $c_1(\bar{T}(\bar{C}_Q)) \cup c_2(\bar{T}(\bar{C}_Q))$ is a Γ -parabolic neighborhood of \bar{X}_Q in \bar{X} .

6.12. Proposition. *Fix a Hecke correspondence $\bar{C} \rightrightarrows \bar{X}$. If the Γ -invariant regular*

parameter $b \in \mathcal{B}$ is chosen sufficiently large then the resulting tiling $\{C^Q\}$ of \bar{C} is narrow for that Hecke correspondence.

6.13. Proof. Let C_Q be a stratum of \bar{C} and suppose $c_1(C_Q) = X_P$ and $c_2(C_Q) = X_{P'}$. Then \mathbf{Q} is Γ -conjugate to \mathbf{P} while $g\mathbf{Q}g^{-1}$ is Γ -conjugate to \mathbf{P}' (cf. Lemma 7.4). In particular, \mathbf{P} and \mathbf{P}' are $\mathbf{G}(\mathbb{Q})$ -conjugate, which implies that either $X_P = X_{P'}$ or $\bar{X}_P \cap \bar{X}_{P'} = \emptyset$ (2.10.4). In the latter case there exist neighborhoods U of \bar{X}_P and U' of $\bar{X}_{P'}$ which do not intersect. Choose the tiling parameter so large that $\bar{T}(\bar{C}_Q) \subset c_1^{-1}(U) \cap c_2^{-1}(U')$. Since there are finitely many strata C_Q in \bar{C} , this amounts to finitely many conditions on the tiling parameter. On the other hand, if $c_1(C_Q) = c_2(C_Q) = X_P$, then we may take $\mathbf{P} = \mathbf{Q}$. Choose any parabolic neighborhood $U \subset \bar{X}$ of \bar{X}_Q and then choose the tiling so small that $\bar{T}(\bar{C}_Q) \subset c_1^{-1}(U) \cap c_2^{-1}(U)$. This guarantees that $c_1(\bar{T}(\bar{C}_Q)) \cup c_2(\bar{T}(\bar{C}_Q))$ is a Γ -parabolic set in \bar{X} . \square

7. Restriction to the boundary

7.1. Parabolic Hecke correspondence. As in §2, \mathbf{G} denotes a connected linear reductive algebraic group defined over \mathbb{Q} , D denotes the associated symmetric space, $K' = A_G K(x_0)$ is the stabilizer in G of a fixed basepoint $x_0 \in D$, $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is an arithmetic group and $X = \Gamma \backslash D$. Fix a rational parabolic subgroup $\mathbf{P} \subset \mathbf{G}$ and let $X_P = \Gamma_P \backslash D_P \subset \bar{X}$ be the corresponding stratum in the reductive Borel-Serre compactification of X . Each $y \in \mathbf{P}(\mathbb{Q})$ determines a correspondence on a $\mathbf{P}(\mathbb{Q})$ -invariant neighborhood of X_P which we now describe. Set $\Gamma'_P = \Gamma_P[y] = \Gamma_P \cap y^{-1}\Gamma_P y$. Define the *parabolic Hecke correspondence*

$$(7.1.1) \quad (c_1, c_2) : \Gamma'_P \backslash D[P] \rightrightarrows \Gamma_P \backslash D[P]$$

determined by $y \in \mathbf{P}(\mathbb{Q})$ to be the unique continuous extension of the correspondence $\Gamma'_P \backslash D \rightrightarrows \Gamma_P \backslash D$ which is given by

$$(7.1.2) \quad \Gamma'_P x K'_P \mapsto (\Gamma_P x K'_P, \Gamma_P y x K'_P)$$

where we identify $D = P/K'_P$. It follows from (6.4.2) (by taking $\mathbf{P} = \mathbf{Q}$) that this correspondence commutes with the geodesic action of A'_P , that is,

$$(7.1.3) \quad c_i(x \cdot a) = c_i(x) \cdot a$$

(for $i = 1, 2$) for any $x \in \Gamma'_P \backslash D[P]$ and for any $a \in A'_P$. Therefore the parabolic Hecke correspondence preserves the corner structure near C_P , that is, if $\mathbf{Q} \supset \mathbf{P}$ is a rational parabolic subgroup then each mapping c_i takes the stratum $\Gamma'_P \backslash D_Q \subset \Gamma'_P \backslash D[P]$ to the stratum $\Gamma_P \backslash D_Q \subset \Gamma_P \backslash D[P]$.

There is also an associated (global) correspondence, $\bar{C} = \Gamma' \backslash \bar{D} \rightrightarrows \bar{X} = \Gamma \backslash \bar{D}$ (where $\Gamma' = \Gamma \cap y^{-1}\Gamma y$). If $V \subset D[P] \subset \bar{D}$ is a Γ -parabolic neighborhood of D_P then it is also a Γ' -parabolic neighborhood of D_P , as is $y^{-1} \cdot V$. Hence $V \cap y^{-1}V$ is also a Γ' -parabolic neighborhood of D_P . It follows that, if $U \subset \Gamma_P \backslash D[P] \subset \Gamma_P \backslash \bar{D}$ is a Γ -parabolic neighborhood of X_P then $U' = c_1^{-1}(U) \cap c_2^{-1}(U) \subset \Gamma'_P \backslash \bar{D}$ is a Γ' -parabolic neighborhood of $C_P = \Gamma'_P \backslash D_P$. We will say that any correspondence isomorphic to such a $U' \rightrightarrows U$ is *modeled* on the parabolic Hecke correspondence (7.1.1).

$$\begin{array}{ccccccc}
 U' & \subset & \Gamma'_P \backslash D[P] & \xrightarrow{\cong} & \Gamma_P \backslash D[P] & \supset & U \\
 \parallel & & \cap & & \cap & & \parallel \\
 U' & \subset & \Gamma'_P \backslash \bar{D} & \xrightarrow{\cong} & \Gamma_P \backslash \bar{D} & \supset & U \\
 \cong \downarrow & & \downarrow \beta' & & \downarrow \beta & & \downarrow \cong \\
 U' & \subset & \bar{C} & \xrightarrow{\cong} & \bar{X} & \supset & U.
 \end{array}$$

7.2. Now suppose $g \in \mathbf{G}(\mathbb{Q})$ gives rise to the Hecke correspondence $C = C[g] \rightrightarrows X$ with its canonical extension $(c_1, c_2) : \bar{C} \rightrightarrows \bar{X}$, where $C = \Gamma[g] \backslash D$ as in §6.5.1. If \mathbf{P} is a rational parabolic subgroup of \mathbf{G} and if X_P denotes the corresponding RBS stratum, then we may consider the part $c_1^{-1}(X_P)$ of \bar{C} which lies over this stratum. It will consist of several RBS boundary strata C_Q of C . Some of these boundary strata may be mapped back to X_P by the mapping c_2 . In this case, we shall say that the Hecke correspondence \bar{C} has a *restriction* to X_P consisting of the union of those boundary strata C_Q such that $(c_1, c_2) | C_Q : C_Q \rightrightarrows X_Q$.

7.3. Proposition. *Let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be a neat arithmetic group. Let*

$$(c_1, c_2) : C = C[g] \rightrightarrows X$$

be the Hecke correspondence which is determined by an element $g \in \mathbf{G}(\mathbb{Q})$. Let \mathbf{P} be a rational parabolic subgroup of \mathbf{G} , with corresponding boundary stratum $X_P \subset \bar{X}$. Decompose the intersection $\Gamma g \Gamma \cap P$ into a union of Γ_P -double cosets,

$$(7.3.1) \quad \Gamma g \Gamma \cap P = \coprod_{j=1}^m \Gamma_P g_j \Gamma_P$$

with $g_j \in \mathbf{P}(\mathbb{Q})$. Then $m < \infty$ and, over a sufficiently small parabolic neighborhood of X_P , the Hecke correspondence $\bar{C} \rightrightarrows \bar{X}$ breaks into a disjoint union of m correspondences which are given by g_j and which are modeled on the parabolic Hecke correspondences

$$\Gamma_P [g_j] \backslash D[P] \rightrightarrows \Gamma_P \backslash D[P]$$

for $j = 1, 2, \dots, m$, where $\Gamma_P [g_j] = \Gamma \cap g_j^{-1} \Gamma g_j \cap P$.

(A similar procedure is described in the adelic setting in [H].) The proof will take the rest of §7. First we establish a one-to-one correspondence between the components of the restriction of the Hecke correspondence to X_P and the double cosets which appear in (7.3.1). Let $\Gamma_P y \Gamma_P \subset \Gamma g \Gamma \cap P$ be a double coset from (7.3.1). Write $y = \gamma_2 g \gamma_1$ for some $\gamma_1, \gamma_2 \in \Gamma$. Set $\Gamma [g] = \Gamma \cap g^{-1} \Gamma g$. Define

$$(7.3.2) \quad \Xi(\Gamma_P y \Gamma_P) = \gamma_1 \mathbf{P} \gamma_1^{-1} = g^{-1} \gamma_2^{-1} \mathbf{P} \gamma_2 g.$$

7.4. Lemma. *The mapping Ξ gives a well defined one-to-one correspondence between*

- (a) *double cosets $\Gamma_P y \Gamma_P \subset \Gamma g \Gamma \cap P$,*

- (b) $\Gamma[g]$ -conjugacy classes of rational parabolic subgroups $\mathbf{Q} = \Xi(\Gamma_P y \Gamma_P) \subset \mathbf{G}$ such that
- (i) \mathbf{Q} is Γ -conjugate to \mathbf{P} and
 - (ii) $g\mathbf{Q}g^{-1}$ is Γ -conjugate to \mathbf{P} ,
- (c) boundary strata $C_Q \subset \bar{C}$ such that $(c_1, c_2) : C_Q \rightrightarrows X_P$.

In particular, this set is finite.

7.5. Proof of Lemma 7.4. First compare the sets (b) and (c). The boundary strata in \bar{C}' are in one-to-one correspondence with $\Gamma[g]$ -conjugacy classes of rational parabolic subgroups of \mathbf{G} , while the boundary strata in \bar{X} are in one-to-one correspondence with Γ conjugacy classes of rational parabolic subgroups. Condition (i) is equivalent to the statement that c_1 maps C_Q to X_P while condition (ii) is equivalent to the statement that c_2 maps C_Q to X_P .

Now verify that the mapping Ξ is well defined, i.e. that the $\Gamma[g]$ -conjugacy class of $\Xi(\Gamma_P g_j \Gamma_P) = \gamma_1 \mathbf{P} \gamma_1^{-1}$ is independent of the choices. Let $y' = \gamma'_2 g \gamma'_1 \in \Gamma_P g_j \Gamma_P$ be another element in the same double coset, and set $\mathbf{Q}' = \gamma'_1 \mathbf{P} (\gamma'_1)^{-1}$. Since $y' \in \Gamma_P y \Gamma_P$, there exists $\gamma_a, \gamma_b \in \Gamma_P$ such that $y' = \gamma_a y \gamma_b$. So $y' = \gamma'_2 g \gamma'_1 = \gamma_a \gamma_2 g \gamma_1 \gamma_b$, which gives

$$h := g^{-1} \gamma'_2{}^{-1} \gamma_a \gamma_2 g = \gamma_1 \gamma_b \gamma'_1{}^{-1} \in g^{-1} \Gamma g \cap \Gamma = \Gamma[g].$$

Then,

$$(7.5.1) \quad h^{-1} \mathbf{Q} h = (\gamma_1 \gamma_b \gamma'_1{}^{-1})^{-1} (\gamma_1 \mathbf{P} \gamma_1^{-1}) (\gamma_1 \gamma_b \gamma'_1{}^{-1}) = \mathbf{Q}'$$

which verifies that \mathbf{Q} and \mathbf{Q}' are $\Gamma[g]$ conjugate.

Next we show that Ξ is surjective. Suppose that \mathbf{Q} and $g\mathbf{Q}g^{-1}$ are both Γ -conjugate to \mathbf{P} . Say, $\mathbf{Q} = \gamma_1 \mathbf{P} \gamma_1^{-1}$ and $g\mathbf{Q}g^{-1} = \gamma_2^{-1} \mathbf{P} \gamma_2$ for some $\gamma_1, \gamma_2 \in \Gamma$. Then

$$(7.5.2) \quad \gamma_2 g \gamma_1 \mathbf{P} \gamma_1^{-1} g^{-1} \gamma_2^{-1} = \mathbf{P}$$

so the element $y = \gamma_2 g \gamma_1 \in \mathbf{P}(\mathbb{Q}) \cap \Gamma g \Gamma$ (since a parabolic subgroup is its own normalizer) and $\Xi(\Gamma_P y \Gamma_P) = \mathbf{Q}$.

Finally we show that Ξ is injective. Suppose $y, y' \in \Gamma g \Gamma \cap P$, say $y = \gamma_2 g \gamma_1$ and $y' = \gamma'_2 g \gamma'_1$. Set

$$\mathbf{Q} = \gamma_1 \mathbf{P} \gamma_1^{-1} = g^{-1} \gamma_2^{-1} \mathbf{P} \gamma_2 g,$$

$$\mathbf{Q}' = \gamma'_1 \mathbf{P} (\gamma'_1)^{-1} = g^{-1} (\gamma'_2)^{-1} \mathbf{P} \gamma'_2 g.$$

Suppose \mathbf{Q} and \mathbf{Q}' are $\Gamma[g]$ -conjugate, say $\mathbf{Q} = \gamma \mathbf{Q}' \gamma^{-1}$ for some $\gamma \in \Gamma \cap g^{-1} \Gamma g$. Comparing these two relations gives

$$\mathbf{Q} = \gamma_1 \mathbf{P} \gamma_1^{-1} = \gamma \gamma_1' \mathbf{P} (\gamma_1')^{-1} \gamma^{-1},$$

$$\mathbf{Q} = g^{-1} \gamma_2^{-1} \mathbf{P} \gamma_2 g = \gamma g^{-1} (\gamma_2')^{-1} \mathbf{P} \gamma_2' g \gamma^{-1},$$

from which it follows that $h_1 = (\gamma_1')^{-1} \gamma^{-1} \gamma_1 \in P$ and $h_2 = \gamma_2 g \gamma g^{-1} (\gamma_2')^{-1} \in P$. Moreover, $h_1, h_2 \in \Gamma$. But $h_2 \gamma' h_1 = \gamma$ hence $\Gamma_P \gamma' \Gamma_P = \Gamma_P \gamma \Gamma_P$ as claimed. \square

7.6. Proof of Proposition 7.3. By Lemma 7.4, the restriction of the Hecke correspondence $\bar{C} = \bar{C}[g] \rightrightarrows \bar{X}$ to the stratum X_P breaks into a union of m correspondences, indexed by the elements g_1, g_2, \dots, g_m . Fix j (with $1 \leq j \leq m$) and set $\Gamma[g_j] = \Gamma \cap g_j^{-1} \Gamma g_j$ and $\Gamma_P[g_j] = \Gamma[g_j] \cap P$. The following commutative diagram of correspondences provides an explicit isomorphism between the parabolic Hecke correspondence given by g_j with the corresponding piece of the Hecke correspondence given by g :

$$\begin{array}{ccc} \Gamma_P[g_j] \backslash D[P] & \rightrightarrows & \Gamma_P \backslash D[P] \\ \beta_j \downarrow & & \downarrow \beta \\ \Gamma[g_j] \backslash \bar{D} & \rightrightarrows & \Gamma \backslash \bar{D} \\ f \downarrow & & \parallel \\ \Gamma[g] \backslash \bar{D} & \rightrightarrows & \Gamma \backslash \bar{D}. \end{array}$$

The first line is the parabolic Hecke correspondence (7.1.1) defined by g_j , i.e., it is the continuous extension of the mapping $\Gamma_P[g_j] x K'_P \mapsto (\Gamma_P x K'_P, \Gamma_P g_j x K'_P)$. The second line is the Hecke correspondence (6.5.1) defined by g_j , i.e. it is the continuous extension of $\Gamma[g_j] x K' \mapsto (\Gamma x K', \Gamma g_j x K')$. The vertical mapping β (resp. β_j) is described in §3.1; it is a homeomorphism over any Γ -parabolic neighborhood of X_P (resp. over any $\Gamma[g_j]$ -parabolic neighborhood of $\Gamma_P[g_j] \backslash D_P$). The top square of this diagram commutes by direct computation. The third line is the given Hecke correspondence (6.5.1). The vertical mapping f is the isomorphism of Hecke correspondences given in Lemma 6.6 and equation (6.7.1). In other words, if $g_j = \gamma_1 g \gamma_2$ then $f(\Gamma[g_j] x K') = \Gamma[g] \gamma_2 x K'$. The bottom square also commutes. This completes the construction of an explicit isomorphism with the parabolic Hecke correspondence, and hence of the proof of Proposition 7.3. \square

8. Counting the fixed points

8.1. As in §2, \mathbf{G} denotes a connected linear reductive algebraic group defined over \mathbb{Q} , D denotes the associated symmetric space, $K' = A_G K(x_0)$ is the stabilizer in G of a fixed basepoint $x_0 \in D$, $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is an arithmetic group and $X = \Gamma \backslash D$. Through this section we assume that Γ is *neat*. Fix a rational parabolic subgroup $\mathbf{P} \subseteq \mathbf{G}$, set $\Gamma_P = \Gamma \cap P$, $\Gamma_L = \nu_P(\Gamma_P) \subset \mathbf{L}_P(\mathbb{Q})$ and denote by $X_P = \Gamma_P \backslash D_P = \Gamma_L \backslash D_P$ the corresponding stratum in the reductive Borel-Serre compactification \bar{X} . An element $y \in \mathbf{P}(\mathbb{Q})$ gives rise to a parabolic Hecke correspondence $(c_1, c_2) : \Gamma'_P \backslash D[P] \rightrightarrows \Gamma_P \backslash D[P]$ where $\Gamma'_P = \Gamma_P[y] = \Gamma_P \cap y^{-1} \Gamma_P y$. Let $\Gamma'_L = \nu_P(\Gamma'_P)$. The restriction $C_P \rightrightarrows X_P$ of this parabolic correspondence to the boundary stratum $C_P = \Gamma'_L \backslash D_P$ is given by

$$(8.1.1) \quad (c_1, c_2)(\Gamma'_L x K_P A_P) = (\Gamma_L x K_P A_P, \Gamma_L \bar{y} x K_P A_P)$$

where $\bar{y} = \nu_P(y)$.

8.2. Characteristic element. Let us suppose that $w \in C_P$ is a *fixed point* of the parabolic Hecke correspondence, that is, $c_1(w) = c_2(w)$. Choose any lift $\tilde{w} \in D_P$ of w and write $\tilde{w} = zK_P A_P \in L_P/K_P A_P$. Since the point $w = \Gamma'_L z K_P A_P$ is fixed, we have

$$(8.2.1) \quad \Gamma_L z K_P A_P = \Gamma_L \bar{y} z K_P A_P.$$

Since Γ is neat, there exists a unique $\gamma \in \Gamma_L$ such that the element $e = \gamma \bar{y}$ fixes the point $\tilde{w} \in D_P$, that is,

$$(8.2.2) \quad e z K_P A_P = \gamma \bar{y} z K_P A_P = z K_P A_P \in D_P.$$

8.3. Definition. The element $e = \gamma \bar{y} \in \mathbf{L}_P(\mathbb{Q})$ is called a *characteristic element* for the fixed point w , or *the characteristic element* corresponding to the lift \tilde{w} of w .

Denote by $F_P(e) \subset C_P$ the set of fixed points in C_P for which e is a characteristic element. We refer to $F_P(e)$ as a *fixed point constituent*; it may consist of several connected components. Let D_P^e denote the fixed points of the mapping $T_e : D_P \rightarrow D_P$ which is given by translation by e . Then $F_P(e)$ is the image of D_P^e under the projection $D_P \rightarrow C_P$. Let $L_e \subset L_P$ denote the centralizer of e in L_P . We say an element of L_P is *elliptic* (or is elliptic modulo A_P) if it is $\mathbf{L}_P(\mathbb{R})$ -conjugate to an element of $K_P A_P$.

8.4. Proposition. *Let $e = a_e m_e \in A_P M_P$ be the characteristic element corresponding to a lift $\tilde{w} \in D_P$ of the fixed point $w \in C_P$. Then the following statements hold.*

(1) *The characteristic element $e \in L_P$ is semisimple and is elliptic (modulo A_P). The group L_e is reductive, algebraic, and defined over \mathbb{Q} . The torus factors $a_y = a_e \in A_P$ are equal (§2.2). The fixed point constituent $F_P(e)$ is a smooth submanifold of C_P .*

(2) *If $y \in P$ is changed by multiplication by an element $u \in \mathcal{U}_P$, or if $\gamma \in \Gamma_L$ is replaced by another element of Γ_L which also satisfies (8.2.2), or if a different representative $z' \in L_P$ of $\tilde{w} \in D_P$ is chosen, then the characteristic element $e \in L_P$ does not change.*

(3) *If a different lift $\tilde{w}' \in D_P$ of w is chosen, or if y is changed within its double coset $\Gamma_P y \Gamma_P$ then e changes at most by Γ_L -conjugacy.*

(4) *The characteristic element e is a rigid invariant of the fixed point set: if $w_t \in C_P$ is a one parameter family of fixed points (with $t \in [0, 1]$) and if $z_t \in D_P$ is a lift to a one parameter family of points in D_P then the resulting characteristic elements e_t do not vary with t .*

(5) *The group L_e acts transitively on D_P^e . Set $\Gamma_e = \Gamma_L \cap L_e$, $\Gamma'_e = \Gamma'_L \cap L_e$, and $K'_e = L_e \cap (z(K_P A_P)z^{-1})$. Then K'_e contains a maximal compact subgroup of L_e . The action of L_e on D_P^e induces diffeomorphisms*

$$(8.4.1) \quad F_P(e) \cong \Gamma'_e \backslash L_e / K'_e \quad \text{and} \quad c_i(F_P(e)) \cong \Gamma_e \backslash L_e / K'_e.$$

The projection $F_P(e) \rightarrow c_i(F_P(e))$ is a covering of degree

$$(8.4.2) \quad d_e = [\Gamma_e : \Gamma'_e] = [\Gamma_L \cap \bar{y}^{-1} \Gamma_L \bar{y} : \nu_P(\Gamma_P \cap y^{-1} \Gamma_P y)].$$

(6) Conversely, let $\mathcal{C} \subset \Gamma_L \bar{y} \Gamma_L$ be any Γ_L -conjugacy class which is elliptic (modulo A_P). Then $\mathcal{C} \cap \Gamma_L \bar{y}$ consists of a single element e' , and there exists a fixed point $w' \in C_P$ for which e' is a characteristic element. In particular, $F_P(e') \neq \emptyset$.

Hence the constituents of the fixed point set in C_P are in one-to-one correspondence with Γ_L -conjugacy classes of elliptic (modulo A_P) elements $e \in \Gamma_L \bar{y} \Gamma_L$.

8.5. Proof. By (8.2.2), $e = \gamma \bar{y} \in zK_P A_P z^{-1}$ which is compact modulo A_P , so e is also semisimple. The elements e and \bar{y} are in the same Γ_L -double coset so they have the same torus component $a_e = a_y \in A_P$. Since Γ is neat, the group Γ_e acts freely on D_P^e . This proves (1). Next, consider (2) and suppose different choices $y' = yu$, $\gamma' \in \Gamma_L$, and $z' \in L_P$ were made, with $u \in \mathcal{U}_P$, with $\hat{w} = zK_P A_P = z'K_P A_P \in D_P$ and, as in (8.2.2), $z'K_P A_P = \gamma' \bar{y}' z' K_P A_P$ (where $\bar{y}' = v_P(y')$). Then $\bar{y}' = \bar{y}$ and

$$\gamma \bar{y} z K_P A_P = z K_P A_P = \gamma' \bar{y}' z' K_P A_P = \gamma' \bar{y} z K_P A_P$$

so $\gamma^{-1} \gamma' \in (\bar{y} z) K_P A_P (\bar{y} z)^{-1}$. Since Γ_L is torsion-free, this implies $\gamma = \gamma'$, hence the characteristic element $e = \gamma \bar{y}$ is unchanged. This proves (2). Since Γ_L is discrete, the characteristic element is constant in a continuous family of fixed points, which proves (4).

Now consider changing y within its double coset $\Gamma_P y \Gamma_P$ and consider changing the lift $\tilde{w} \in D_P$ of the fixed point. Let $\hat{y} = \gamma_1 y \gamma_2$ with $\gamma_1, \gamma_2 \in \Gamma_P$. Set $\bar{y}_1 = v_P(\gamma_1)$, $\bar{y}_2 = v_P(\gamma_2)$ and $\tilde{y} = v_P(\hat{y})$. As in §8.1, the element \tilde{y} determines a Hecke correspondence $(\hat{c}_1, \hat{c}_2) : \hat{C}_P \rightrightarrows X_P$ as follows: Set $\hat{\Gamma}_P = \Gamma_P \cap \hat{y}^{-1} \Gamma_P \hat{y}$, $\hat{\Gamma}_L = v_P(\hat{\Gamma}_P)$ and $\hat{C}_P = \hat{\Gamma}_L \backslash D_P = \hat{\Gamma}_L \backslash L_P / K_P A_P$. Then $(\hat{c}_1, \hat{c}_2)(\hat{\Gamma}_L x K_P A_P) = (\Gamma_L x K_P A_P, \Gamma_L \tilde{y} x K_P A_P)$ for any $x \in L_P$. As in equation (6.7.1), an isomorphism of correspondences $f : \hat{C}_P \rightarrow C_P$ is given by

$$f(\hat{\Gamma}_L x K_P A_P) = \Gamma'_L \bar{y}_2 x K_P A_P.$$

Choose any lift $\hat{z} K_P A_P \in D_P$ of the fixed point $\hat{w} = f^{-1}(w)$ (with $\hat{z} \in L_P$). We obtain a new characteristic element $\hat{e} = \hat{\gamma} \hat{y}$ (for some $\hat{\gamma} \in \Gamma_L$) such that

$$(8.5.1) \quad \hat{\gamma} \hat{y} \hat{z} K_P A_P = \hat{z} K_P A_P.$$

We need to show that $\hat{e} = \hat{\gamma} \hat{y}$ is Γ_L -conjugate to $e = \gamma \bar{y}$. Since $f(\hat{w}) = w$ we have

$$\Gamma'_L \bar{y}_2 \hat{z} K_P A_P = \Gamma'_L z K_P A_P$$

so there exists a unique $h \in \Gamma'_L$ such that $h \bar{y}_2 \hat{z} K_P A_P = z K_P A_P$ or

$$(8.5.2) \quad \hat{z} K_P A_P = \bar{y}_2^{-1} h^{-1} z K_P A_P.$$

Substituting (8.5.2) into both sides of (8.5.1) and using (8.2.2) gives

$$h \bar{y}_2 \hat{\gamma} \bar{y}_2^{-1} h^{-1} z K_P A_P = z K_P A_P = \gamma \bar{y} z K_P A_P$$

or

$$h \bar{y}_2 \hat{\gamma} \bar{y}_1 \bar{y} h^{-1} z K_P A_P = \gamma \bar{y} z K_P A_P.$$

Since $h, h^{-1} \in \nu_P(y^{-1}\Gamma_P y) = \bar{y}^{-1}\Gamma_L\bar{y}$ there exists $h' \in \Gamma_L$ such that $\bar{y}h^{-1} = h'\bar{y}$, which gives

$$\gamma^{-1}h\bar{\gamma}_2\hat{\gamma}\bar{\gamma}_1h'(\bar{y}zK_P A_P) = \bar{y}zK_P A_P.$$

This implies that $\gamma^{-1}h\bar{\gamma}_2\hat{\gamma}\bar{\gamma}_1h' = 1$ since it is both in the group $(\bar{y}z)K_P A_P(\bar{y}z)^{-1}$ and in Γ_L . Therefore $1.\bar{y} = \gamma^{-1}h\bar{\gamma}_2\hat{\gamma}\bar{\gamma}_1\bar{y}h^{-1}$ or

$$\gamma\bar{y} = h\bar{\gamma}_2\hat{\gamma}\bar{\gamma}_1\bar{y}\bar{\gamma}_2\bar{\gamma}_2^{-1}h^{-1} = (h\bar{\gamma}_2)\hat{\gamma}\bar{\gamma}_1(h\bar{\gamma}_2)^{-1}.$$

Thus, the characteristic elements $\gamma\bar{y}$ and $\hat{\gamma}\bar{y}$ are Γ_L -conjugate, which proves (3).

Now let us prove (5). It is easy to see that L_e acts on D_P^e . To see that this action is transitive, let $v_1, v_2 \in D_P^e$, say $v_1 = z_1K_P A_P$ and $v_2 = z_2K_P A_P$ with $z_i \in L_P$ (for $i = 1, 2$). Then there exists $k_1, k_2 \in K_P A_P$ so that $ez_1 = z_1k_1$ and $ez_2 = z_2k_2$, hence k_1 and k_2 are L -conjugate (by $z_2z_1^{-1}$). It follows from [Bo3], §24.7 that k_1 and k_2 are also $K_P A_P$ conjugate. Say, $k_2 = mk_1m^{-1}$ for some $m \in K_P A_P$. Define $x = z_2mz_1^{-1}$. Then $v_2 = xv_1$ and moreover, $x \in L_e$ since

$$xex^{-1} = z_2mz_1^{-1}ez_1m^{-1}z_2^{-1} = z_2mk_1m^{-1}z_2^{-1} = e.$$

This completes the verification that L_e acts transitively on D_P^e .

Using the chosen lift $\tilde{w} = zK_P A_P \in D_P^e$ as a basepoint, we obtain a diffeomorphism $L_e/K_e' \cong D_P^e$ where $K_e' = L_e \cap (zK_P A_P z^{-1})$ is the stabilizer (in L_e) of \tilde{w} . This induces a surjection $(L_e \cap \Gamma_L') \backslash L_e/K_e' \rightarrow F_P(e)$ which we will now show to be injective.

Suppose $x_1, x_2 \in L_e$ and that $x_1\tilde{w}, x_2\tilde{w} \in D_P^e$ map to the same point in C_P , that is, $\Gamma_L'x_1zK_P A_P = \Gamma_L'x_2zK_P A_P$. Then there exists $\gamma \in \Gamma_L'$ so that

$$(8.5.3) \quad \gamma x_1 z K_P A_P = x_2 z K_P A_P.$$

We need to show that $\gamma \in L_e$. Acting by e on the left hand side of (8.5.3) and using (8.2.2) gives the quantity

$$e\gamma e^{-1}e x_1 z K_P A_P = e\gamma e^{-1}x_1 e z K_P A_P = e\gamma e^{-1}x_1 z K_P A_P$$

while acting by e on the right hand side of (8.5.3) gives

$$e x_2 z K_P A_P = x_2 z K_P A_P = \gamma x_1 z K_P A_P.$$

So $\gamma^{-1}e\gamma e^{-1} \in (x_1 z)K_P A_P(x_1 z)^{-1}$. But $\gamma^{-1}e\gamma e^{-1} \in \Gamma_L$ so this element is trivial, that is, $\gamma e = e\gamma$, hence $\gamma \in L_e$. Therefore $F_P(e) = \Gamma_e' \backslash D_P^e$. The equality $c_1(F_P(e)) = c_2(F_P(e)) = \Gamma_e \backslash L_e/K_e'$ is similar. Equation (8.4.2) will be proven in §8.6.

Now let us verify part (6). Suppose $e'' = \gamma_2\bar{y}\gamma_1 \in \mathbf{L}_P(\mathbb{Q})$ is elliptic (modulo A_P). Then e'' is Γ_L -conjugate to the element $e' = \gamma_1\gamma_2\bar{y}$ which is also elliptic modulo A_P . There exists $z \in L_P$ so that $e' \in zK_P A_P z^{-1}$ hence $e'zK_P A_P = zK_P A_P$. In other words, e' is a characteristic element for the point $w' = \Gamma_L'zK_P A_P \in C_P$ (which is easily seen to be fixed under the Hecke correspondence). \square

8.6. Proof of (8.4.2). (We refer to the notation of §8.4 and §8.5.) Unfortunately the arithmetic group $\Gamma_L[\bar{y}] = \Gamma_L \cap \bar{y}^{-1}\Gamma_L\bar{y}$ may be larger than $\Gamma'_L = \nu_P(\Gamma_P \cap y^{-1}\Gamma_P y)$, so the correspondence (8.1.1) is not necessarily a Hecke correspondence, but rather it is a covering of the following Hecke correspondence:

$$(\tilde{c}_1, \tilde{c}_2) : \tilde{C}_P = \Gamma_L[\bar{y}] \backslash D_P \rightrightarrows X_P,$$

$$\Gamma_L[\bar{y}]xK_P A_P \mapsto (\Gamma_L x K_P A_P, \Gamma_L \bar{y} x K_P A_P).$$

This covering $\phi : \Gamma'_L \backslash D_P \rightarrow \Gamma_L[\bar{y}] \backslash D_P$ has degree $d = [\Gamma_L[\bar{y}] : \Gamma'_L]$. A point $w \in C_P$ is fixed iff the point $\phi(w) \in \tilde{C}_P$ is fixed.

Let $\tilde{F}_P(e) \subset \tilde{C}_P$ be the set of fixed points within this (smaller) Hecke correspondence with characteristic element e . We claim that the restriction of \tilde{c}_i to $\tilde{F}_P(e) \subset \tilde{C}_P$ is a diffeomorphism: $\tilde{F}_P(e) \cong c_i(F_P(e))$ (for $i = 1, 2$). As in (8.4.1) it is clear that $\tilde{F}_P(e) \cong (\Gamma_L[\bar{y}] \cap L_e) \backslash L_e / K'_e$ so it suffices to verify that the inclusion $\Gamma_L[\bar{y}] \cap L_e \subset \Gamma_e$ is an isomorphism. If $\gamma_1 \in \Gamma_e$ then $\gamma_1 = e^{-1}\gamma_1 e$. But the left side of this equation is in Γ_L and the right side is in $\bar{y}^{-1}\Gamma_L\bar{y}$, hence $\gamma_1 \in \Gamma_L[\bar{y}]$, which proves the claim.

In summary, we have a diagram

$$\begin{array}{ccccc} C_P & \xrightarrow{\phi} & \tilde{C}_P & \rightrightarrows & X_P \\ \cup & & \cup & & \cup \\ F_P(e) & \longrightarrow & \tilde{F}_P(e) & \xrightarrow{\cong} & c_i(F_P(e)) \end{array}$$

and in particular the degree d of the covering ϕ coincides with the degree d_e of the mapping $F_P(e) \rightarrow c_i(F_P(e))$ which gives (8.4.2). \square

8.7. Remark. The codimension of $F_P(e)$ in D_P is odd if and only if the action of e reverses orientations in the normal bundle of $F_P(e)$ in D_P . This is because e preserves an appropriately chosen normal slice through any point in $F_P(e)$, the boundary of which is a sphere on which e then acts as a diffeomorphism without fixed points, so its Lefschetz number is 0. In the odd codimension case this sphere is even dimensional, so by the Lefschetz fixed point theorem, its action on the top degree cohomology is given by multiplication by -1 , that is, it reverses the orientation. In the even codimension case, it preserves orientation.

9. Hyperbolic properties of Hecke correspondences

9.1. Expanding and contracting roots. As in §2, \mathbf{G} denotes a connected linear reductive algebraic group defined over \mathbb{Q} , D denotes the associated symmetric space, $K' = A_G K(x_0)$ is the stabilizer in G of a fixed basepoint $x_0 \in D$, $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is an arithmetic group and $X = \Gamma \backslash D$. Throughout this section we fix a Hecke correspondence $(c_1, c_2) : \bar{C} \rightrightarrows \bar{X}$ defined by some element $g \in \mathbf{G}(\mathbb{Q})$. So $C = \Gamma' \backslash D$ with $\Gamma' = \Gamma \cap g^{-1}\Gamma g$.

Let $\mathbf{P} \subset \mathbf{G}$ be a rational parabolic subgroup and suppose that $c_1(C_P) = c_2(C_P) = X_P$. By Proposition 7.3, near C_P the correspondence is modeled on a parabolic Hecke corre-

spondence $\Gamma'_P \backslash D[P] \rightrightarrows \Gamma_P \backslash D[P]$ (7.1.1) which is determined by some $y \in \mathbf{P}(\mathbb{Q})$ (where $\Gamma'_P = \Gamma_P \cap y^{-1}\Gamma_P y$). Suppose $y = u_y a_y m_y$ is the Langlands decomposition (2.2.2) of $y \in \mathbf{P}(\mathbb{Q})$. If y is allowed to vary within the double coset $\Gamma_P y \Gamma_P$ then the element $a_y \in A_P$ will remain fixed, so we may write $a_P = a_y$. We refer to a_P as the *torus factor* associated to the Hecke correspondence near C_P . The torus factor may be used to define a partition of the simple roots Δ_P into three subsets

$$\Delta_P^+ = \{\alpha \in \Delta_P \mid \alpha(a_P) < 1\},$$

$$\Delta_P^- = \{\alpha \in \Delta_P \mid \alpha(a_P) > 1\},$$

$$\Delta_P^0 = \{\alpha \in \Delta_P \mid \alpha(a_P) = 1\},$$

consisting of those simple roots which are *expanding*, *contracting*, or *neutral*, respectively, near the stratum C_P . (See also §11.7.) The terminology is motivated by the following fact, whose proof follows immediately from (3.2.2) and the definition (7.1.2) of the correspondence. For all $\alpha \in \Delta_P$ and for all $z \in \Gamma'_P \backslash D[P]$ the root function f_α^P satisfies:

$$(9.1.1) \quad f_\alpha^P(c_2(z)) = \alpha(a_P) f_\alpha^P(c_1(z)).$$

Now suppose $\mathbf{P} \subset \mathbf{Q}$ are rational parabolic subgroups of \mathbf{G} , with $\Delta_P = i(\Delta_Q) \amalg J$ as in (2.4.1). Suppose that $c_1(C_P) = c_2(C_P)$, giving rise to a torus factor $a_P \in A_P$ and a decomposition of Δ_P into expanding, contracting and neutral roots as above. Then $c_1(C_Q) = c_2(C_Q)$ (by §7.1) so we obtain a torus factor $a_Q \in A_Q$ and a decomposition of Δ_Q into expanding, contracting and neutral roots also.

9.2. Proposition. *Suppose that $J \subset \Delta_P^0$. Then:*

(1) *The torus factors $a_P = a_Q$ are equal; in particular a_P lies in the sub-torus $A_Q \subset A_P$.*

(2) *The expanding, contracting, and neutral simple roots for \mathbf{P} and for \mathbf{Q} are related as follows:*

$$(9.2.1) \quad \Delta_P^+ = i(\Delta_Q^+), \quad \Delta_P^- = i(\Delta_Q^-), \quad \Delta_P^0 = i(\Delta_Q^0) \amalg J.$$

(3) *For all $z \in \Gamma'_P \backslash D[P]$ and for all $\beta \in \Delta_Q$ we have,*

$$\frac{f_{i(\beta)}^P(c_2(z))}{f_{i(\beta)}^P(c_1(z))} = i(\beta)(a_P) = \beta(a_P) = \frac{f_\beta^Q(c_2(z))}{f_\beta^Q(c_1(z))}$$

provided the denominators do not vanish.

In this case we say that \mathbf{Q} is a *neutral* parabolic subgroup containing \mathbf{P} and we write $\mathbf{P} \prec \mathbf{Q}$. Intuitively, the Hecke correspondence is neutral in those directions normal to C_P which point into C_Q ; cf. §3.6.

9.3. Proof. Locally near C_Q the Hecke correspondence is isomorphic to a parabolic Hecke correspondence given by some $y' = u_{y'} a_{y'} m_{y'} \in Q$ with torus factor $a_Q = a_{y'} \in A_Q$. In a neighborhood of C_P the correspondence is isomorphic to the parabolic Hecke corre-

spondence given by some $y = u_y a_y m_y \in P$ (with $a_P = a_y$). Moreover y may be chosen to lie in the double coset $\Gamma_Q y' \Gamma_Q$ since the correspondence $\bar{C}_P \rightrightarrows \bar{X}_P$ is the restriction to \bar{C}_P of the correspondence $\bar{C}_Q \rightrightarrows \bar{X}_Q$; cf. Proposition 7.3. By assumption, $\alpha(a_P) = 1$ for all $\alpha \in J$ which implies that $a_P \in A_Q$. It follows that $a_P = a_Q$ because the homomorphism $Q \rightarrow A_Q$ (which associates to any $z \in Q$ its torus factor a_z) kills Γ_Q . Therefore, for any $\beta \in \Delta_Q$ we have: $\beta(a_Q) = i(\beta)(a_P)$. This proves (1) and (2). The first equality in part (3) is just (9.1.1). The last equality in part (3) follows from part (1) and from (9.1.1) (with f_α^P replaced by f_β^Q). \square

9.4. Maximal neutrality. Suppose $\mathbf{P} \subset \mathbf{Q} \subset \mathbf{R}$ are rational parabolic subgroups and that $c_1(C_P) = c_2(C_P)$. Write $\Delta_P = i(\Delta_Q) \amalg I$ and $\Delta_Q = j(\Delta_R) \amalg J$ for the disjoint union of (2.4.1). Suppose moreover that $\mathbf{P} \prec \mathbf{Q}$ and $\mathbf{Q} \prec \mathbf{R}$, that is, that $I \subset \Delta_P^0$ and $J \subset \Delta_Q^0$. Then it follows from Proposition 9.2 that $\mathbf{P} \prec \mathbf{R}$. Hence there is a greatest neutral parabolic subgroup \mathbf{P}^\dagger containing \mathbf{P} ; in fact it is $\mathbf{P}^\dagger = \mathbf{P}(\Delta_P^0)$ in the notation of §2.4 and §2.3. It is easy to see that

$$(9.4.1) \quad \Delta_{P^\dagger}^0 = \phi \quad \text{and} \quad \Delta_{P^\dagger}^\pm = i'(\Delta_{P^\dagger}^\pm)$$

(where $i' : \Delta_{P^\dagger} \hookrightarrow \Delta_P$ is the natural inclusion). Moreover,

$$(9.4.2) \quad \mathbf{P} \prec \mathbf{Q} \Rightarrow \mathbf{P}^\dagger = \mathbf{Q}^\dagger.$$

10. Structure of the fixed point set

10.1. As in §2, \mathbf{G} denotes a connected reductive linear algebraic group defined over \mathbb{Q} , $D = G/K'$ is its associated symmetric space with basepoint $x_0 \in D$ and stabilizer $K' = A_G K(x_0)$, $\Gamma \subset \mathbf{G}(\mathbb{Q})$ denotes an arithmetic subgroup, and $X = \Gamma \backslash D$. Throughout this section we fix a Hecke correspondence $(c_1, c_2) : \bar{C} \rightrightarrows \bar{X}$ defined by some element $g \in \mathbf{G}(\mathbb{Q})$. So $C = \Gamma' \backslash D$ with $\Gamma' = \Gamma \cap g^{-1} \Gamma g$. We also fix a Γ -equivariant tiling of D which is narrow with respect to the Hecke correspondence (cf. §6.11, §4.5), and denote by $\{C^P\}$ and $\{X^P\}$ the resulting tilings of \bar{C} and \bar{X} respectively. Let $F \subset \bar{C}$ be a connected component of the set of fixed points. The following lemma says that if F spans two strata $C_P \subset \bar{C}_Q$ then the Hecke correspondence is neutral in those directions which point from C_P into C_Q ; cf. §3.6.

10.2. Lemma. *Let $\mathbf{P} \subset \mathbf{Q} \subseteq \mathbf{G}$ be rational parabolic subgroups and write $\Delta_P = i(\Delta_Q) \amalg J$ as in (2.4.1). Suppose $F \cap C_Q \cap T(\bar{C}_P) \neq \emptyset$ (that is, C_Q contains fixed points which lie in the Γ -parabolic neighborhood $T(\bar{C}_P)$ of \bar{C}_P). Then:*

- (1) $J \subset \Delta_P^0$ (hence the conclusions of Proposition 9.2 hold).
- (2) $F \cap T(\bar{C}_P)$ is invariant under the geodesic action of $A'_P (\geq 1)$.
- (3) $\pi_P(F \cap T(\bar{C}_P)) \subset F$, that is, each fixed point in this Γ' -parabolic neighborhood projects to a fixed point in C_P .

10.3. Proof. Part (1) follows from (3.5.2) and (9.1.1) by taking $z \in F \cap C_Q \cap T(\bar{C}_P)$ to be a fixed point. Part (2) follows from (7.1.3). Part (3) follows by continuity. \square

10.4. Proposition. *Let $F \subset \bar{C}$ be a connected component of the fixed point set. Let \mathbf{Q} be a rational parabolic subgroup and suppose $F \cap C_{\mathbf{Q}} \neq \emptyset$. Let $a_{\mathbf{Q}} \in A_{\mathbf{Q}}$ be the torus factor for this stratum. Let $\mathbf{Q}^{\dagger} = \mathbf{Q}(\Delta_{\mathbf{Q}}^0)$ be the maximal neutral parabolic subgroup containing \mathbf{Q} . Then:*

(1) *The whole connected component F of the fixed point set is contained in the closure*

$$F \subset \bar{C}_{\mathbf{Q}^{\dagger}}$$

of the single stratum $C_{\mathbf{Q}^{\dagger}}$.

(2) *The Hecke correspondence $\bar{C} \rightrightarrows \bar{X}$ restricts to a correspondence $\bar{C}_{\mathbf{Q}^{\dagger}} \rightrightarrows \bar{X}_{\mathbf{Q}^{\dagger}}$ on this stratum-closure. Within this restricted correspondence, near each point $c \in F$, every simple root is neutral: If $\mathbf{P} \subset \mathbf{Q}^{\dagger}$, if $F \cap C_{\mathbf{P}} \neq \emptyset$, if $i' : \Delta_{\mathbf{Q}^{\dagger}} \hookrightarrow \Delta_{\mathbf{P}}$ is the inclusion, and if $\Delta_{\mathbf{P}} = i'(\Delta_{\mathbf{Q}^{\dagger}}) \amalg J$ as in (2.4.1) then $J = \Delta_{\mathbf{P}}^0$.*

(3) *There exists a neighborhood $U(F) \subset \bar{C}$ of the fixed point set such that for all $\alpha \in \Delta_{\mathbf{Q}}$ and for all $x \in U(F)$, if $x \in T(\bar{C}_{\mathbf{Q}})$ and if $c_2(x) \in T(\bar{X}_{\mathbf{Q}})$ then*

$$(10.4.1) \quad r_x^{\mathbf{Q}}(c_2(x)) = \alpha(a_{\mathbf{Q}})^{-1} r_x^{\mathbf{Q}}(c_1(x)).$$

10.5. Remarks. Part (1) does *not* imply that $F \cap C_{\mathbf{Q}^{\dagger}} \neq \emptyset$. In fact, the fixed point component F may be “reducible”: it does not necessarily coincide with the closure of its intersection $F \cap C_{\mathbf{P}}$ with any single stratum $C_{\mathbf{P}}$. (See §16.1.)

10.6. Proof. Suppose that F has a nontrivial intersection with some other stratum, say $F \cap C_{\mathbf{R}} \neq \emptyset$. Suppose for the moment that $\mathbf{R} \supset \mathbf{Q}$ and that $F \cap C_{\mathbf{Q}}$ contains limit points from $F \cap C_{\mathbf{R}}$, that is,

$$(10.6.1) \quad (F \cap C_{\mathbf{Q}}) \cap \overline{F \cap C_{\mathbf{R}}} \neq \emptyset.$$

Then Lemma 10.2 part (1) implies that \mathbf{R} is a neutral parabolic subgroup containing \mathbf{Q} so (9.4.2) implies that $\mathbf{R}^{\dagger} = \mathbf{Q}^{\dagger}$, hence $F \cap C_{\mathbf{R}} \subset F \cap \bar{C}_{\mathbf{R}^{\dagger}} = F \cap \bar{C}_{\mathbf{Q}^{\dagger}}$. Now we drop the assumption (10.6.1). Since F is connected, the stratum $C_{\mathbf{R}}$ is related to the stratum $C_{\mathbf{Q}}$ through a chain of strata $C_{\mathbf{R}_i}$ (say, $1 \leq i \leq m$), each having nontrivial intersection with F , with each step in the chain related to the next by

$$(F \cap C_{\mathbf{R}_i}) \cap \overline{F \cap C_{\mathbf{R}_{i+1}}} \neq \emptyset \quad \text{or} \quad (F \cap C_{\mathbf{R}_{i+1}}) \cap \overline{F \cap C_{\mathbf{R}_i}} \neq \emptyset.$$

Repeated application of (9.4.2) implies that

$$(10.6.2) \quad \mathbf{R}^{\dagger} = \mathbf{R}_1^{\dagger} = \dots = \mathbf{R}_m^{\dagger} = \mathbf{Q}^{\dagger}.$$

So once again, $F \cap C_{\mathbf{R}} \subset F \cap \bar{C}_{\mathbf{Q}^{\dagger}}$. This verifies part (1).

Consider part (2). Since the stratum $C_{\mathbf{Q}}$ is preserved by the Hecke correspondence, the same holds for each larger stratum, especially $C_{\mathbf{Q}^{\dagger}}$. Suppose $F \cap C_{\mathbf{P}} \neq \emptyset$. By (10.6.2), $\mathbf{P}^{\dagger} = \mathbf{Q}^{\dagger}$ so by (9.4.1), $\Delta_{\mathbf{P}} = i'(\Delta_{\mathbf{Q}^{\dagger}}) \amalg \Delta_{\mathbf{P}}^0$.

Next we verify part (3). Suppose $\mathbf{P} \subset \mathbf{Q}$ and suppose $F \cap C_P \neq \emptyset$. By Proposition 9.2 part (3), for all $\alpha \in \Delta_Q$ and for all $w \in T(\bar{C}_Q) \cap c_2^{-1}(T(\bar{X}_Q))$, the root function $f_{i(\alpha)}^P$ satisfies

$$(10.6.3) \quad f_{i(\alpha)}^P(c_2(w)) = \alpha(a_Q) f_{i(\alpha)}^P(c_1(w))$$

where, as in (2.4.1), we have written $\Delta_P = i(\Delta_Q) \amalg J$. However this does not yet prove (10.4.1). The problem is that the partial distance function $r_\alpha^Q(w)$ is patched together (4.2.2) from these root functions $f_{i(\alpha)}^P$ in a way that depends on which tile contains the point $c_i(w)$. So we need to show that the Hecke correspondence preserves the tile boundaries in some neighborhood $U(F)$ of the fixed point set. This in turn will follow from the neutrality properties of Lemma 10.2 and Proposition 9.2.

10.7. Lemma. *Suppose $F \subset \bar{C}$ is a connected component of the fixed point set of the Hecke correspondence $\bar{C} \rightrightarrows \bar{X}$. Then there exists a neighborhood $U(F) \subset \bar{C}$ of F such that for all $w \in U(F)$ and for any rational parabolic subgroup $\mathbf{P} \subseteq \mathbf{G}$,*

$$(10.7.1) \quad c_1(w) \in X^P \Leftrightarrow c_2(w) \in X^P.$$

10.8. Proof. Assume not. Then there is a sequence of points $x_i \in \bar{C}$ converging to F so that for each i , $c_1(x_i)$ and $c_2(x_i)$ are in different tiles. By taking subsequences if necessary we may assume the sequence x_i converges to some point $x_0 \in F$, that x_i are all contained in a single tile C^P (so $c_1(x_i) \in X^P$) and that $c_2(x_i)$ all lie in a single tile X^Q . Since $c_1(x_0) \in \bar{C}^P \cap \bar{C}^Q$ is a fixed point, the Hecke correspondence must preserve the strata C_P and C_Q (meaning that $c_1(C_P) = c_2(C_P) = X^P$ and $c_1(C_Q) = c_2(C_Q) = X^Q$) and we may assume that either $\mathbf{P} \subseteq \mathbf{Q}$ or $\mathbf{Q} \subseteq \mathbf{P}$. Since the tiling is narrow this implies that $F \cap C_P \neq \emptyset$, that $F \cap C_Q \neq \emptyset$, and that either $F \cap C_P$ contains limit points from $F \cap C_Q$ (if $\mathbf{P} \subseteq \mathbf{Q}$) or else $F \cap C_Q$ contains limit points from $F \cap C_P$ (if $\mathbf{Q} \subseteq \mathbf{P}$).

Let us first consider the case that $\mathbf{P} \subseteq \mathbf{Q} = \mathbf{G}$, meaning that $X^Q = X^0$. Let $a_P \in A_P$ denote the torus factor for the Hecke correspondence near C_P as in §9.1. Then it follows from Lemma 10.2 that $\mathbf{Q} = \mathbf{G}$ is a neutral parabolic subgroup containing \mathbf{P} , that is, $\alpha(a_P) = 1$ for all $\alpha \in \Delta_P$. Within any Γ -parabolic neighborhood W of \bar{X}_P , the tile X^P is given by (4.1.3):

$$X^P = \{x \in W \mid \pi_P(x) \in X_P^0 \text{ and } f_\alpha^P(x) > \alpha(b_P) \text{ for all } \alpha \in \Delta_P\}.$$

Since $c_2(x_i) \in X^0$ it follows that for at least one $\alpha \in \Delta_P$ we have:

$$\alpha(b_P) \geq f_\alpha^P(c_2(x_i)) = \alpha(a_P) f_\alpha^P(c_1(x_i)) = f_\alpha^P(c_1(x_i)) > \alpha(b_P)$$

(using equation (9.1.1)), which is a contradiction.

Next consider the case $\mathbf{P} \subset \mathbf{Q} \neq \mathbf{G}$. For sufficiently large i the points x_i will lie in some Γ' -parabolic neighborhood of \bar{C}_Q , and the same argument applied to the sequence $z_i = \pi_Q(x_i) \rightarrow z_0 = \pi_Q(x_0) \in F \cap C_Q$ also leads to a contradiction.

The case $\mathbf{Q} \subseteq \mathbf{P}$ may be handled by reversing the roles of \mathbf{P} and \mathbf{Q} in these arguments. This completes the proof of Lemma 10.7 and also the proof of Proposition 10.4. \square

11. Modified Hecke correspondence

11.1. As in §2, \mathbf{G} denotes a connected linear reductive algebraic group defined over \mathbb{Q} , D denotes the associated symmetric space, $K' = A_G K(x_0)$ is the stabilizer in G of a chosen basepoint $x_0 \in D$, $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is an arithmetic group and $X = \Gamma \backslash D$. Throughout this section we fix a Hecke correspondence $(c_1, c_2) : \bar{C} \rightrightarrows \bar{X}$ defined by some element $g \in \mathbf{G}(\mathbb{Q})$, with $C = \Gamma' \backslash D$ and $\Gamma' = \Gamma \cap g^{-1} \Gamma g$. Let $F = \{w \in \bar{C} \mid c_1(w) = c_2(w)\}$ denote the fixed point set. Fix a sufficiently large regular Γ -equivariant parameter $b \in \mathcal{B}$ which is so large that the resulting tilings $\{D^P\}$ of \bar{D} , $\{X^P\}$ of \bar{X} and $\{C^P\}$ of \bar{C} are narrow (§6.11) with respect to the Hecke correspondence. Fix $\mathbf{t} \in A_{P_0}$ (>1) dominant and regular, with resulting shrink homeomorphism $Sh(\mathbf{t}) : \bar{X} \rightarrow \bar{X}$ as in §5. Define the (shrink-) modified correspondence

$$(11.1.1) \quad (c'_1, c'_2) : \bar{C} \rightrightarrows \bar{X}$$

by $c'_1 = c_1$ and $c'_2 = Sh(\mathbf{t}) \circ c_2$. Let $\hat{F} = \{w \in \bar{C} \mid c'_1(w) = c'_2(w)\}$ denote the fixed point set of the modified correspondence.

11.2. Proposition. *If $\mathbf{t} \in A_{P_0}$ (>1) is chosen regular and sufficiently close to $\mathbf{1}$, then*

$$(11.2.1) \quad \hat{F} \cap C_Q = F \cap C_Q^0$$

for each stratum $C_Q \subset \bar{C}$, and

$$(11.2.2) \quad c_1(\hat{F}) \cap X_Q = c_1(F) \cap X_Q^0$$

for each stratum $X_Q \subset \bar{X}$, where C_Q^0 (resp. X_Q^0) denotes the central tile in C_Q (resp. X_Q).

11.3. Proof. The correspondence \bar{C} has finitely many boundary strata C_P with the property that $c_1(C_P) = c_2(C_P)$. For each such stratum C_P , according to Proposition 7.3, the Hecke correspondence is locally isomorphic near C_P to a parabolic Hecke correspondence $\Gamma'_P \backslash D[P] \rightrightarrows \Gamma_P \backslash D[P]$ which is given by some $y \in P(\mathbb{Q})$ and to which we may uniquely associate a torus factor $a_y = a_P \in A_P$ as in §7.1. Conjugating all these torus factors back to \mathbf{S}_{P_0} gives a collection $\{a_1, a_2, \dots, a_N\} \subset A_{P_0}$ of finitely many *standard torus factors* (some of which may coincide and some of which may equal $\mathbf{1}$) associated to the Hecke correspondence g . If $\mathbf{t} \in A_{P_0}$ (>1) is chosen to be regular and sufficiently close to $\mathbf{1}$ then we can guarantee that the following condition holds: For all $\alpha \in \Delta$ and for all $i = 1, 2, \dots, N$, if $\alpha(a_i) < 1$ then $\alpha(a_i \mathbf{t}) < 1$ while if $\alpha(a_i) \geq 1$ then $\alpha(a_i \mathbf{t}) > 1$. Therefore, for any ρ with $0 < \rho \leq 1$, for all $\alpha \in \Delta$ and for all $i = 1, 2, \dots, N$, the following holds:

$$(11.3.1) \quad \begin{cases} \alpha(a_i) < 1 \Rightarrow \alpha(a_i) \alpha(\mathbf{t})^\rho < 1, \\ \alpha(a_i) \geq 1 \Rightarrow \alpha(a_i) \alpha(\mathbf{t})^\rho > 1. \end{cases}$$

Having made these choices, let us now prove Proposition 11.2. Certainly $F \cap C_Q^0 = \hat{F} \cap C_Q^0$ because the shrink acts as the identity on C_Q^0 . So we only need to show that $\hat{F} \cap C_Q \subset C_Q^0$, that is, we must show that the fixed points of the modified Hecke correspondence which appear in the stratum C_Q are all contained in the central tile of that stratum. Suppose otherwise and let $w \in C_Q$ be a fixed point of the modified correspondence which lies in

some tile C^P for $\mathbf{P} \subset \mathbf{Q}$, and $\mathbf{P} \neq \mathbf{Q}$. Since the shrink preserves tiles, it follows that $c_1(C^P) \cap c_2(C^P) \neq \emptyset$. The tiling is narrow so this implies that $c_1(C_P) = c_2(C_P)$. Set $\Delta_P = i(\Delta_Q) \amalg J$ as in (2.4.1); then $J \neq \emptyset$. By Proposition 7.3, locally near C_P we may replace the Hecke correspondence by a parabolic correspondence: in other words, we may assume that $g \in \mathbf{P}$. Let $a_P \in A_P$ be the torus factor for the correspondence near C_P , that is, if $g = u_g a_g m_g \in \mathcal{U}_P A_P M_P$ is the Langlands decomposition then $a_P = a_g$. The point $c_1(w) = c_1'(w) = c_2'(w)$ lies in $X_Q \cap X^P \subset \bar{X}$. Since the shrink preserves tiles, $c_2(w) \in X_Q \cap X^P$ also.

For any $\alpha \in \Delta_P$,

$$\begin{aligned} f_\alpha^P(c_2'(w)) &= f_\alpha^P(c_2(w)) \alpha_0(\mathbf{t})^{\rho(r_\alpha^P(c_2(w)))} && \text{by (5.5.2)} \\ &= \alpha(a_P) \alpha_0(\mathbf{t})^{\rho(r_\alpha^P(c_2(w)))} f_\alpha^P(c_1(w)) && \text{by (9.1.1)} \end{aligned}$$

where $\alpha_0 \in \Delta$ is the unique simple root which, after conjugation and restriction to \mathbf{S}_P , agrees with α .

This gives a contradiction: First note that $\rho(r_\alpha^P(c_2(w))) \neq 0$, for otherwise we would have $r_\alpha^P(c_2(w)) = 1$ or $c_2(w) \notin X^P$. As the shrink preserves tiles, this would imply that $c_2'(w) \notin X^P$ which is absurd. So by (11.3.1) the factor $\alpha(a_P) \alpha_0(\mathbf{t})^{\rho(r_\alpha^P(c_2(w)))} \neq 1$. If we choose $\alpha \in J$ then the assumption $c_1(w) \in X_Q$ implies that $f_\alpha^P(c_1(w)) \neq 0$. Therefore the point w cannot be fixed by the modified correspondence, which proves (11.2.1).

There are finitely many strata C_R such that $c_1(C_R) = X_Q$. To prove (11.2.2) it suffices to show, for each of these strata, that

$$c_1(\hat{F} \cap C_R) \cap X_Q = c_1(F \cap C_R) \cap X_Q^0.$$

Write $F \cap C_R = (F \cap C_R^0) \cup \tilde{F}$ as a disjoint union. Then \tilde{F} is contained in a union of tiles $C^{R'}$ with $\mathbf{R}' \subset \mathbf{R}$ a proper inclusion. Since c_1 takes tiles to tiles, it follows that $c_1(\tilde{F}) \cap X_Q^0 = \emptyset$ hence

$$\begin{aligned} c_1(F \cap C_R) \cap X_Q^0 &= c_1(F \cap C_R^0) \cap X_Q^0 = c_1(F \cap C_R^0) \\ &= c_1(\hat{F} \cap C_R) = c_1(\hat{F} \cap C_R) \cap X_Q \end{aligned}$$

by (11.2.1). \square

11.4. Tangential distance. Choose a regular Γ -invariant parameter so that the associated tiling is narrow (§6.11) with respect to the Hecke correspondence. Choose $\mathbf{t} \in A_{P_0}$ (>1) to be regular and sufficiently close to $\mathbf{1}$ as in Proposition 11.2. Suppose C_Q is a stratum of \bar{C} for which $F \cap C_Q \neq \emptyset$. Then the Hecke correspondence restricts to a correspondence $C_Q \rightrightarrows X_Q$. Fix $e \in L_Q$ and let $F_Q(e) \subset C_Q$ denote the set of fixed points in C_Q for which e is a characteristic element as in §8.2. The corresponding set of fixed points for the modified Hecke correspondence is the “truncation”

$$F_Q^0(e) = F_Q(e) \cap C_Q^0.$$

Let $E = c_1(F_Q^0(e)) = c_1(F_Q(e)) \cap X_Q^0$ denote its image in X_Q . In this section we construct a good function which measures the distance from E .

Let $R_{X_Q} : X_Q \rightarrow X_Q^0$ and $R_{C_Q} : C_Q \rightarrow C_Q^0$ be the retraction(s) and let $W_Q : \overline{X_Q} \rightarrow [0, 1]$ be the exhaustion function of §4.6. A choice of G -invariant Riemannian metric on D induces Riemannian metrics on C , X , C_Q , and X_Q . Define the tangential distance $d_E : X_Q \rightarrow [0, \infty]$ by

$$(11.4.1) \quad d_E(x) = W_Q(x) + \text{dist}_{X_Q}(R_{X_Q}(x), E)$$

where dist_{X_Q} denotes the distance in X_Q with respect to the Riemannian metric. Then $d_E^{-1}(0) = E$. Although the restriction of the Hecke correspondence to $\overline{C_Q}$ is locally an isometry, composing with $Sh(\mathbf{t})$ has the following effect: points near the boundary of X_Q are moved even closer to the boundary of X_Q and hence they are moved away from E . This is the intuition behind the following lemma.

11.5. Lemma. *There exists a neighborhood $V \subset C_Q$ of $F_Q^0(e) = F_Q(e) \cap C_Q^0$ such that*

$$(11.5.1) \quad d_E(c'_2(w)) \geq d_E(c'_1(w))$$

for all $w \in V$.

11.6. Proof. The stratum closure $\overline{C_Q}$ is tiled by the collection of intersections $C_Q^P = \overline{C_Q} \cap C^P$ with $\mathbf{P} \subseteq \mathbf{Q}$. Let $\overline{C_Q^P}$ denote the closure of such a tile. Let $U_1 \subset \overline{C_Q}$ be a neighborhood of the closure $\overline{F_Q(e)}$ so that for any rational parabolic subgroup $\mathbf{P} \subset \mathbf{Q}$,

$$\overline{F_Q(e)} \cap \overline{C_Q^P} = \emptyset \Leftrightarrow U_1 \cap \overline{C_Q^P} = \emptyset.$$

By Lemma 10.7 we may also assume that the Hecke correspondence preserves tile boundaries in U_1 . The mapping c_1 preserves tiles, and the points in $F_Q(e)$ are fixed, hence

$$E = c_1(F_Q^0(e)) = c_2(F_Q^0(e)).$$

By Proposition 8.4 (6), and for $i = 1, 2$, the mapping c_i is one-to-one on $F_Q(e)$. Moreover, it is locally an isometry. It follows that,

$$(11.6.1) \quad \text{dist}_{X_Q}(c_1(w), E) = \text{dist}_{C_Q}(w, F_Q^0(e)) = \text{dist}_{X_Q}(c_2(w), E)$$

for $w \in C_Q$ in some neighborhood $U_2 \subset C_Q$ of $F_Q^0(e)$. The desired neighborhood is

$$V = U_1 \cap U_2 \subset C_Q.$$

By Proposition 10.4 the restricted correspondence $\overline{C_Q} \rightrightarrows \overline{X_Q}$ is neutral near $F_Q(e)$. So if $\mathbf{P} \subset \mathbf{Q}$ and if $F_Q(e) \cap C^P \neq \emptyset$ and if $\Delta_P = i(\Delta_Q) \amalg J$ as in (2.4.1) then by (10.4.1),

$$(11.6.2) \quad r_\alpha^P(c_2(w)) = r_\alpha^P(c_1(w))$$

for all $w \in C_Q^P$ and for all $\alpha \in J$. Moreover, by Lemma 10.7 the correspondence preserves tiles near $F_Q(e)$, that is, for all $w \in U_1$ and for all $\mathbf{P} \subseteq \mathbf{Q}$ we have

$$w \in \bar{C}_Q^P \Leftrightarrow c_1(w) \in \bar{X}_Q^P \Leftrightarrow c_2(w) \in \bar{X}_Q^P.$$

By (7.1.3) the correspondence commutes with the geodesic action of A_P . Therefore

$$R_{X_Q}(c_1(w)) = c_1(R_{C_Q}(w)) \quad \text{and} \quad R_{X_Q}(c_2(w)) = c_2(R_{C_Q}(w)).$$

Now suppose $w \in V$ and $w \in C_Q^P$ for some $\mathbf{P} \subseteq \mathbf{Q}$. If $\mathbf{P} = \mathbf{Q}$ (that is, if $w \in C_Q^0$ lies in the central tile) then $c_i(w) \in C_Q^0$ as well, in which case $R_{X_Q}(c_i(w)) = c_i(w)$, $W_Q(c_i(w)) = 0$, and $c_i(w) = c'_i(w)$. Then (11.5.1) follows from (11.6.1) and in fact, equality holds.

Now suppose $w \in V \cap C_Q^P$ for some $\mathbf{P} \neq \mathbf{Q}$. Then $\mathbf{P} \subset \mathbf{Q}$, $c_1(C_P) = c_2(C_P) = X_P$ by Proposition 7.3, and locally near X_P this correspondence is isomorphic to a parabolic Hecke correspondence, that is, we may assume that $g \in \mathbf{P}(\mathbb{Q})$. In the tile C^P the retraction R commutes with the geodesic action of A_P , cf. (4.6.1), and so does the Hecke correspondence, (7.1.3), hence

$$R_{X_Q}(c'_2(w)) = R_{X_Q}(Sh(\mathbf{t})c_2(w)) = R_{X_Q}(c_2(w)) = c_2(R_{C_Q}(w)).$$

So the second terms in (11.4.1) are equal:

$$\begin{aligned} \text{dist}_{X_Q}(R_{X_Q}c'_2(w), E) &= \text{dist}_{X_Q}(c_2(R_{C_Q}(w)), c_2(F_Q^0(e))) = \text{dist}_{C_Q}(R_{C_Q}(w), F_Q^0(e)) \\ &= \text{dist}_{X_Q}(c_1(R_{C_Q}(w)), E) = \text{dist}_{X_Q}(R_{X_Q}c_1(w), E) \end{aligned}$$

because both morphisms c_1 and c_2 are local isometries. Now consider the first terms in (11.4.1). Fix $w \in C_Q^P$. For $\alpha \in \Delta_P$ set $\rho(\alpha) = \rho(r_\alpha^P(c_2(w)))$. Using (4.6.3), (5.5.1), and (11.6.2) we find,

$$\begin{aligned} W_Q(c'_2(w)) &= 1 - \inf_{\alpha \in J} \{r_\alpha^P Sh(\mathbf{t})c_2(w)\} \\ &= 1 - \inf_{\alpha \in J} \{r_\alpha^P(c_2(w))\alpha_0(\mathbf{t})^{-\rho(\alpha)}\} \\ &= 1 - \inf_{\alpha \in J} \{r_\alpha^P(c_1(w))\alpha_0(\mathbf{t})^{-\rho(\alpha)}\} \\ &\geq 1 - \inf_{\alpha \in J} \{r_\alpha^P(c_1(w))\} \\ &= W_Q(c'_1(w)) \end{aligned}$$

which completes the proof of (11.5.1). \square

11.7. Hyperbolic correspondences. Recall that the correspondence $(c_1, c_2) : \bar{C} \rightrightarrows \bar{X}$ is *weakly hyperbolic* ([GM2], [GM5]) near a connected component $F \subset \bar{C}$ of the fixed point set, if there is a neighborhood $N(F') \subset \bar{X}$ of the image $F' = c_1(F) = c_2(F)$ and an *indicator mapping* $t = (t_1, t_2) : N(F') \rightarrow \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ such that

- (1) the mapping t is proper and subanalytic;
- (2) the pre-image of the origin $t^{-1}(0) = F'$ consists precisely of F' ;

(3) there is a neighborhood $N(F) \subset \bar{C}$ so that $c_i(N(F)) \subset N(F')$ (for $i = 1, 2$) and

$$N(F) \cap c_1^{-1}(F') \cap c_2^{-1}(F') = F;$$

(4) for any $x \in N(F)$,

$$t_1(c_1(x)) \leq t_1(c_2(x)),$$

$$t_2(c_1(x)) \geq t_2(c_2(x)).$$

(Due to an error in [GM2], condition (3) above was omitted from the original definition of weakly hyperbolic, cf. [GM5].)

11.8. The modified correspondence is hyperbolic. Choose a tiling parameter $\mathbf{b} \in \mathcal{B}$ so that the associated tiling is narrow with respect to the Hecke correspondence $(c_1, c_2) : \bar{C} \rightrightarrows \bar{X}$. Choose $\mathbf{t} \in A_{p_0}$ (>1) to be dominant, regular, and sufficiently close to $\mathbf{1}$ as in Proposition 11.2 and equation (11.3.1). Let $(c'_1, c'_2) : \bar{C} \rightrightarrows \bar{X}$ be the modified correspondence.

Suppose C_Q is a stratum of \bar{C} for which $F \cap C_Q \neq \emptyset$. Then $c_1(C_Q) = c_2(C_Q) = X_Q$. By Proposition 7.3 we may, locally near C_Q , replace the Hecke correspondence with a parabolic Hecke correspondence determined by some $g = u_g a_g m_g \in \mathbf{Q}(\mathbb{Q})$. For any $e \in L_Q$ let $F_Q(e) \subset C_Q$ denote the corresponding fixed point constituent: the set of fixed points in C_Q for which e is a characteristic element as in §8.2. By Proposition 11.2 the fixed point set in C_Q of the modified Hecke correspondence is a union of “truncated” constituents $F_Q^0(e) = F_Q(e) \cap C_Q^0$ (as e varies over elliptic elements in $\Gamma_L \bar{g} \Gamma_L$, cf. Proposition 8.4). (Although $F_Q^0(e)$ may have finitely many connected components we will treat them all simultaneously.)

Fix such an element e and let $E = c_1(F_Q(e)) \cap X_Q^0 = c_1(F_Q^0(e))$ be the image in X_Q of the truncated fixed point constituent as in §11.4. Write $\Delta_Q = \Delta_Q^+ \cup \Delta_Q^- \cup \Delta_Q^0$ according to whether the simple root is expanding, contracting, or neutral near X_Q as in §7.1. Define $t = (t_1, t_2) : T(X_Q) \rightarrow \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ by

$$t_1(x) = \sum_{\alpha \in \Delta_Q^+} r_\alpha^Q(x) + d_E(\pi_Q(x)),$$

$$t_2(x) = \sum_{\alpha \in \Delta_Q^-} r_\alpha^Q(x) + \sum_{\alpha \in \Delta_Q^0} r_\alpha^Q(x).$$

Here,

$$T(X_Q) = T(\bar{X}_Q) \cap X[Q]$$

denotes the open neighborhood of X_Q on which the above mappings are defined: it consists of the part of $T(\bar{X}_Q)$ which is contained in those strata X_P such that $\mathbf{Q} \subseteq \mathbf{P}$.

11.9. Theorem. *The mapping (t_1, t_2) is an indicator mapping, with respect to which the modified Hecke correspondence is hyperbolic near $F_Q^0(e)$.*

11.10. Proof. The idea is that the composition with the $Sh(\mathbf{t})$ converts neutral directions (normal to a given stratum) into contracting directions but it does not change the nature of the expanding or contracting (normal) directions. It converts distances within the stratum (which are preserved by the Hecke correspondence and hence neutral) into expanding directions.

We must display a neighborhood $N(F_Q^0(e)) \subset \bar{C}$ which satisfies conditions (3) and (4) of §11.7 (but with F replaced by $F_Q^0(e)$). First we find a neighborhood N_1 so that condition (3) holds. Since $c_1 : \bar{C} \rightarrow \bar{X}$ is stratum preserving,

$$c_1^{-1}(E) \cap T(C_Q) = c_1^{-1}(E) \cap C_Q$$

where $T(C_Q) = T(\bar{C}_Q) \cap C[Q]$. The mapping $c_1 : C_Q \rightarrow X_Q$ is a finite (unramified) covering. Therefore, if $W \subset C_Q$ is a sufficiently small neighborhood of $F_Q^0(e)$ in C_Q then

$$c_1^{-1}(E) \cap W = F_Q^0(e).$$

Take $N_1 = \pi_Q^{-1}(W) \cap T(C_Q)$. This neighborhood of $F_Q^0(e)$ satisfies condition (3) because

$$c_1^{-1}(E) \cap c_2^{-1}(E) \cap N_1 \subset c_1^{-1}(E) \cap N_1 = c_1^{-1}(E) \cap W = F_Q^0(e)$$

and the reverse inclusion is obvious.

Now consider the conditions (4). Let $a_Q = a_g \in A_Q$ be the torus factor for the correspondence near C_Q . It is easy to check that $t_1^{-1}(0) \cap t_2^{-1}(0) = E$. For any $w \in T(\bar{C}_Q)$ and for all $\alpha \in \Delta_Q$ we have

$$\begin{aligned} r_x^Q(c_2'(w)) &= r_x^Q(Sh(\mathbf{t})c_2(w)) = \alpha_0(\mathbf{t})^{-\rho(r_x^Q c_2(w))} r_x^Q c_2(w) \quad \text{by (5.5.1)} \\ &= \alpha_0(\mathbf{t})^{-\rho(r_x^Q c_2(w))} \alpha(a_Q)^{-1} r_x^Q(c_1(w)) \quad \text{by (10.4.1)}. \end{aligned}$$

If $\alpha \in \Delta_Q^- \cup \Delta_Q^0$ then $\alpha_0(\mathbf{t})^{-\rho(r_x^Q c_2(w))} \alpha(a_Q)^{-1} \leq 1$ since both factors are ≤ 1 . This proves that

$$t_2 c_2'(w) \leq t_2 c_1'(w).$$

If $\alpha \in \Delta_Q^+$ then $\alpha(a_Q) < 1$ so $\alpha_0(\mathbf{t})^{-\rho(r_x^Q c_2(w))} \alpha(a_Q)^{-1} > 1$ by (11.3.1). Let $V \subset C_Q$ be the neighborhood of $F_Q^0(e)$ described in Lemma 11.5. Then for all $w \in \pi_Q^{-1}(V)$ we have

$$\begin{aligned} d_E \pi_Q c_2'(w) &= d_E \pi_Q Sh(\mathbf{t})c_2(w) \\ &= d_E Sh(\mathbf{t})\pi_Q c_2(w) \quad \text{by §5.5(4)} \\ &= d_E Sh(\mathbf{t})c_2 \pi_Q(w) \quad \text{by (6.4.3)} \\ &\geq d_E c_1' \pi_Q(w) \quad \text{by (11.5.1)} \\ &= d_E \pi_Q c_1'(w) \end{aligned}$$

which proves that

$$t_1 c'_2(w) \geq t_1 c'_1(w)$$

for all $w \in \pi_Q^{-1}(V)$. This completes the verification of condition (4) of §11.7. In summary, the neighborhood

$$N(F_Q^0(e)) = N_1 \cap \pi_Q^{-1}(V) \subset \bar{C}$$

satisfies both conditions (3) and (4). \square

12. Local weighted cohomology with supports

12.1. Quadrants (See [GKM], §7.14, p. 534). As in previous sections we suppose \mathbf{G} is a connected reductive linear algebraic group defined over \mathbb{Q} and we denote the greatest \mathbb{Q} -split torus in its center by $\mathbf{S}_{\mathbf{G}}$. Let \mathbf{P} be a rational parabolic subgroup with $\mathbf{S}_{\mathbf{P}}$ the greatest \mathbb{Q} -split torus in the center of its Levi quotient $\mathbf{L}_{\mathbf{P}}$. Let Δ_P denote the simple positive roots of $\mathbf{S}_{\mathbf{P}}$ occurring in $\mathfrak{N}_P = \text{Lie}(\mathcal{U}_P)$. The elements $\alpha \in \Delta_P$ are trivial on $\mathbf{S}_{\mathbf{G}}$ and form a basis of $\chi_{\mathbb{Q}}^*(\mathbf{S}'_{\mathbf{P}}) \subset \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}})$ where $\mathbf{S}'_{\mathbf{P}} = \mathbf{S}_{\mathbf{P}}/\mathbf{S}_{\mathbf{G}}$. For any subset $J \subset \Delta_P$ as in §2.4 let $\mathbf{Q} = \mathbf{P}(J)$ be the parabolic subgroup containing \mathbf{P} for which the corresponding torus

$$\mathbf{S}_{\mathbf{J}} = \mathbf{S}_{\mathbf{P}(J)}$$

is the identity component of the intersection $\bigcap_{\alpha \in J} \ker(\alpha)$. Let $\{t_{\alpha}\}$ be the basis of the cocharacter group $\chi_*(\mathbf{S}'_{\mathbf{P}}) \otimes \mathbb{Q}$ which is dual to the basis Δ_P so that $\langle \alpha, t_{\beta} \rangle = \delta_{\alpha, \beta}$ (with respect to the canonical pairing $\langle \cdot, \cdot \rangle$). The cocharacter group $\chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}}/\mathbf{S}_{\mathbf{J}})$ is spanned by $\{t_{\alpha} \mid \alpha \in J\}$ while $\chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{J}}/\mathbf{S}_{\mathbf{G}})$ is spanned by $\{t_{\alpha} \mid \alpha \in \bar{J}\}$, where $\bar{J} = \Delta_P - J$ denotes the complement.

Fix $v_P \in \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}})$ and $J \subset \Delta_P$. Let $\gamma \in \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}})$ and suppose that $\gamma|_{\mathbf{S}_{\mathbf{G}}} = v_P|_{\mathbf{S}_{\mathbf{G}}}$. Then $\gamma - v_P$ may be regarded as an element of $\chi_{\mathbb{Q}}^*(\mathbf{S}'_{\mathbf{P}})$ so we may define

$$(12.1.1) \quad I_{v_P}(\gamma) = \{\alpha \in \Delta_P \mid \langle \gamma - v_P, t_{\alpha} \rangle < 0\},$$

$$(12.1.2) \quad \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}})_{[v_P, J]} = \{\gamma \in \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}}) \mid I_{v_P}(\gamma) = J \text{ and } \gamma|_{\mathbf{S}_{\mathbf{G}}} = v_P|_{\mathbf{S}_{\mathbf{G}}}\}.$$

This last set is called the *quadrant of type J*. The disjoint union of the $2^{|\Delta_P|}$ quadrants,

$$\coprod_{J \subseteq \Delta_P} \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}})_{[v_P, J]} = \{\gamma \in \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}}) \mid \gamma|_{\mathbf{S}_{\mathbf{G}}} = v_P|_{\mathbf{S}_{\mathbf{G}}}\}$$

is the subset of all characters whose restriction to $\mathbf{S}_{\mathbf{G}}$ agrees with that of v_P . Taking $J = \emptyset$ gives

$$\chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}})_{[v_P, \emptyset]} = \{\gamma \in \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}}) \mid \gamma|_{\mathbf{S}_{\mathbf{G}}} = v_P|_{\mathbf{S}_{\mathbf{G}}} \text{ and } \langle \gamma - v_P, t_{\alpha} \rangle \geq 0 \text{ for all } \alpha \in \Delta_P\}$$

which was denoted $\chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}})_+$ in [GHM] and was denoted $\chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}})_{\geq v_P}$ in [GKM]. It is the translate by v_P of the positive cone $\left\{ \sum_{\alpha \in \Delta_P} m_{\alpha} \alpha \right\}$ with $m_{\alpha} \geq 0$. More generally, for $J \subset \Delta_P$ define $\chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}})_{\geq v_P(J)}$ to be the translate by v_P of the cone $\left\{ \sum_{\alpha \in \bar{J}} m_{\alpha} \alpha \mid m_{\alpha} \geq 0 \right\}$. That is,

$$\begin{aligned}\chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}})_{\geq v_P(J)} &= \{\gamma \in \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}}) \mid \gamma|_{\mathbf{S}_{\mathbf{G}}} = v_P|_{\mathbf{S}_{\mathbf{G}}} \text{ and } \langle \gamma - v_P, t_{\alpha} \rangle \geq 0 \text{ for all } \alpha \in \bar{J}\} \\ &= \bigcup_{K \subseteq J} \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}})_{[v_P, K]}.\end{aligned}$$

Then

$$(12.1.3) \quad \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}})_{[v_P, J]} = \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}})_{\geq v_P(J)} - \bigcup_{K \subseteq J} \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}})_{\geq v_P(K)}.$$

Equation (12.1.3) remains valid if we replace the union on the right hand side by the union over those $K \subset J$ such that $|K| = |J| - 1$.

(This apparently backward notation was chosen so as to simplify the computation in §12.6. It can be reconciled with the notation of [GHM] as follows. There are $|\Delta_P|$ proper maximal parabolic subgroups containing \mathbf{P} . Each $J \subset \Delta_P$ corresponds to a collection \hat{J} of these maximal parabolic subgroups, with $\mathbf{Q} \in \hat{J}$ iff $\mathbf{S}_{\mathbf{Q}} \subset \bigcap_{\alpha \in J} \ker(\alpha)$. Then the subset $\chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}})_{\geq v_P(J)}$ in this paper coincides with the subset $\chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}})_{+(\hat{J})}$ in [GHM].)

If H is an $\mathbf{S}_{\mathbf{P}}$ module such that $\mathbf{S}_{\mathbf{G}}$ acts on H through the character $v_P|_{\mathbf{S}_{\mathbf{G}}}$ then one may define $H_{[v_P, J]}$ (resp. $H_{\geq v_P}$, resp. $H_{\geq v_P(J)}$) to be the sum of those weight spaces H_{γ} for which $\gamma \in \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}})_{[v_P, J]}$ (resp. $\gamma \in \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}})_{\geq v_P}$, resp. $\gamma \in \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}})_{\geq v_P(J)}$).

12.2. Weighted cohomology. As in §2, let D denote the symmetric space associated to G , $K' = A_G K(x_0)$ denote the stabilizer in G of a fixed basepoint $x_0 \in D$, $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be a neat arithmetic group and $X = \Gamma \backslash D$. Let $G \rightarrow GL(E)$ be a finite dimensional irreducible representation of \mathbf{G} on some complex vector space E . It gives rise to a local system $\mathbf{E} = (G/K') \times_{\Gamma} E$ on $X = \Gamma \backslash G/K$. Let \mathbf{P}_0 be the standard minimal rational parabolic subgroup with $\mathbf{S}_0 = \mathbf{S}_{\mathbf{P}_0}$. Fix $v \in \chi_{\mathbb{Q}}^*(\mathbf{S}_0)$ so that $v|_{\mathbf{S}_{\mathbf{G}}}$ coincides with the character by which $\mathbf{S}_{\mathbf{G}}$ acts on E . Then v defines a *weight profile* in the sense of [GHM]: if $\mathbf{Q} \supseteq \mathbf{P}_0$ is a standard rational parabolic subgroup then set $v_{\mathbf{Q}} = v|_{\mathbf{S}_{\mathbf{Q}}}$ and

$$\chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{Q}})_{+} = \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{Q}})_{\geq v_{\mathbf{Q}}} = \chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{Q}})_{[v_{\mathbf{Q}}, \phi]}.$$

These definitions may be extended to arbitrary rational parabolic subgroups by conjugation. We obtain from [GHM] a complex of fine sheaves, $\mathbf{W}^v \mathbf{C}^{\bullet}(\mathbf{E})$ on the reductive Borel-Serre compactification \bar{X} of X , whose (hyper)-cohomology groups $W^v H^*(\bar{X}, \mathbf{E})$ are the weighted cohomology groups. Let $i : X \rightarrow \bar{X}$ denote the inclusion. Recall from [GHM], §13 that a choice of basepoint induces an isomorphism

$$\mathbf{H}_x^j(Ri_*(\mathbf{E})) \cong H^j(\mathfrak{R}_Q, E)$$

between the stalk cohomology at a point $x \in X_Q$ of the complex of sheaves $Ri_*(\mathbf{E})$ and the Lie algebra cohomology of \mathfrak{R}_Q . The weighted cohomology complex is obtained by applying a weight truncation to the complex $Ri_*(\mathbf{E})$ with the result that its stalk cohomology becomes

$$(12.2.1) \quad \mathbf{H}_x^j(\mathbf{W}^v \mathbf{C}^{\bullet}(\mathbf{E})) = H^j(\mathfrak{R}_Q, E)_{\geq v_{\mathbf{Q}}}.$$

12.3. Remarks on sheaf theory. In the next few sections we will need to use the formalism of the derived category of sheaves, and some relations between the standard functors, for which we refer to [GM4], [GM2], [Bo5], [I], [KS], [GeM]. Specifically, if X is a subanalytic set we denote by $D^b(X)$ the bounded (cohomologically-) constructible derived category of sheaves of complex vector spaces on X . An element $\mathbf{S}^\bullet \in D^b(X)$ is a complex of sheaves, bounded from below, whose cohomology sheaves $\mathbf{H}^i(\mathbf{S}^\bullet)$ are finite dimensional and are locally constant on each stratum of some subanalytic stratification of X . The hypercohomology of \mathbf{S}^\bullet will be denoted $H^*(\mathbf{S}^\bullet)$ and the stalk cohomology at a point $x \in X$ will be denoted $H_x^*(\mathbf{S}^\bullet)$. Denote by $\mathbf{S}[n]^\bullet$ the shifted sheaf, $\mathbf{S}[n]^p = \mathbf{S}^{n+p}$. The derived category $D^b(X)$ supports the standard operations of $R\mathbf{Hom}$, \otimes , Rf_* , $Rf^!$, f^* , and $f^!$. There are many relations between these functors, of which we mention a few that we will use:

If $f : Y \rightarrow X$ is a normally nonsingular embedding ([GM4], §5.4) then there is a canonical isomorphism

$$(12.3.1) \quad f^!(\mathbf{S}^\bullet) \cong f^*(\mathbf{S}^\bullet) \otimes \mathcal{O}_{X/Y}[-d]$$

where $\mathcal{O}_{X/Y}$ denotes the orientation bundle (or top exterior power) of the normal bundle of Y in X , and where d denotes the codimension of Y in X . If $f : X \rightarrow \text{pt}$ is the map to a point then $\mathbb{D}_X = f^!(\mathbb{C})$ is the dualizing complex. If X is an n -dimensional manifold (or even a rational homology manifold) then $\mathbb{D}_X \cong \mathcal{O}_X[n]$ where \mathcal{O}_X denotes the orientation bundle.

12.4. Cohomology with supports. Let \bar{X} be a compact subanalytic set and let \mathbf{S}^\bullet be a (cohomologically) constructible complex of sheaves on \bar{X} . Suppose $Y \subset W \subset \bar{X}$ are locally closed subsets with inclusions

$$Y \xrightarrow{h_Y} W \xrightarrow{j_W} \bar{X}.$$

Define the *restriction of \mathbf{S}^\bullet to Y with compact supports in W* to be the complex of sheaves

$$(12.4.1) \quad \mathbf{B}^\bullet = h_Y^! j_W^* \mathbf{S}^\bullet.$$

If $Y = \{y\}$ is a single point, then the cohomology of this complex is the relative cohomology group

$$(12.4.2) \quad H^m(\mathbf{B}^\bullet) = H^m(B_e \cap \bar{X}, \partial B_e \cap W; \mathbf{S}^\bullet),$$

where B_e is a sufficiently small ball around y (with respect to some subanalytic embedding in Euclidean space) and ∂B_e is its boundary.

Now suppose \bar{X} is the reductive Borel-Serre compactification of a locally symmetric space $X = \Gamma \backslash G/K$ as in §12.2, and that $\mathbf{S}^\bullet = \mathbf{W}^\nu \mathbf{C}^\bullet(\mathbf{E})$ is the weighted cohomology sheaf constructed with respect to some weight profile ν and local system \mathbf{E} as in §12.2. Let $Y = X_P$ be some stratum and let $W = \bar{X}_Q$ be the closure of a larger stratum, corresponding to some rational parabolic subgroup $\mathbf{Q} \supset \mathbf{P}$. Form $\mathbf{B}^\bullet = h_Y^! j_W^* \mathbf{W}^\nu \mathbf{C}^\bullet(\mathbf{E})$ as above. Write $\Delta_P = i(\Delta_Q) \amalg I$ as in (2.4.1).

12.5. Theorem. *The cohomology sheaf $\mathbf{H}^m(\mathbf{B}^\bullet)$ is isomorphic to the local system on X_P which is associated to the following L_P -submodule of the \mathfrak{R}_P -cohomology,*

$$(12.5.1) \quad H^{m-|I|}(\mathfrak{R}_P, E)_{[v_P, I]}.$$

12.6. Proof of Theorem 12.5. The proof follows closely the computation [GHM], §18 of the weighted cohomology of the link \mathcal{L}_y . First let us recall some generalities. Each stratum X_P of the reductive Borel-Serre compactification \bar{X} is a rational homology manifold. If Γ is neat, then each stratum is a smooth manifold. Suppose \mathbf{S}^\bullet is a complex of sheaves whose cohomology sheaves are locally constant on each stratum of \bar{X} . Let $Y = X_P \subset W = \bar{X}_Q$ as above. The choice of basepoint $x_0 \in D$ determines a basepoint $y \in Y$. Let $N_y \subset \bar{X}$ be a *normal slice* (cf. [GM3], §5.4) to the stratum Y at the point y . Let $k : N_y \cap W \rightarrow \bar{X}$ denote the inclusion, and let i_y and a_y denote the inclusions of y into Y and $N_y \cap W$ respectively.

Then the stalk cohomology of $\mathbf{B}^\bullet = h_Y^! j_W^* \mathbf{S}^\bullet$ is given by

$$(12.6.1) \quad H_y^m(\mathbf{B}^\bullet) = H^m(i_y^* h_Y^! j_W^* \mathbf{S}^\bullet) \cong H^m(a_y^! k^* \mathbf{S}^\bullet)$$

which in turn may be identified with the relative cohomology group

$$(12.6.2) \quad H^m(B_\epsilon \cap N_y, \partial B_\epsilon \cap N_y \cap \bar{X}_Q; \mathbf{S}^\bullet)$$

(where B_ϵ is a sufficiently small ball around y , chosen with respect to some locally defined subanalytic embedding of \bar{X} into some Euclidean space).

These isomorphisms are deduced from the following fiber squares

$$(12.6.3) \quad \begin{array}{ccccc} y & \xrightarrow{\quad} & N_y \cap W & \xrightarrow{\quad} & N_y \\ & & \downarrow & & \downarrow k_N \\ i_y \downarrow & & & & \\ Y & \xrightarrow{\quad} & W & \xrightarrow{\quad} & \bar{X} \\ & & \downarrow h_Y & & \downarrow j_W \end{array}$$

where $k = k_N k_W$. In the case that $\mathbf{S}^\bullet = \mathbf{W}^v \mathbf{C}^\bullet$ we will compute (12.6.2) using the long exact cohomology sequence for the pair.

Step 1. Construct an isomorphism of L_P -modules,

$$(12.6.4) \quad H^c(\partial B_\epsilon \cap N_y \cap \bar{X}_Q; \mathbf{W}^v \mathbf{C}^\bullet(\mathbf{E})) \cong H^{c-|I|+1}(\mathfrak{R}_P, E)_{[v_P, I]} \oplus H^c(\mathfrak{R}_P, E)_{\geq v_P}.$$

In order to simplify notation, let us choose a labeling $\{\alpha_1, \alpha_2, \dots, \alpha_s\} = \Delta_P$ of the simple roots. As in [GHM], §8.8, the link $\mathcal{L}_y = \partial B_\epsilon \cap N_y$ comes with a natural mapping $\delta : \mathcal{L}_y \rightarrow \triangleright^{s-1}$ to the $s - 1$ dimensional simplex,

$$\triangleright^{s-1} = \left\{ (x_1, x_2, \dots, x_s) \in \mathbb{R}^s \mid 0 \leq x_i \leq 1 \text{ and } \sum_{i=1}^s x_i = 1 \right\}.$$

For any subset $J \subset \{1, 2, \dots, s\}$ let \bar{J} denote its complement. Associated to J there is a (closed) face of dimension $|J| - 1$,

$$\triangleright_J = \{x \in \triangleright^{s-1} \mid x_j = 0 \text{ for all } j \in \bar{J}\}$$

whose interior we denote by \triangleright_J^o . Each $\triangleright_{\{j\}}$ is a vertex of \triangleright^{s-1} ; the face \triangleright_J is spanned by the vertices $\triangleright_{\{j\}}$ such that $j \in J$. Let $U_{\{j\}} = St(\triangleright_{\{j\}})$ be the open star of the vertex $\triangleright_{\{j\}}$. These form a covering of \triangleright^{s-1} whose multi-intersections we denote by

$$U_J = \bigcap_{j \in J} U_{\{j\}}.$$

Then

$$U_J = St(\triangleright_J^o) = \bigcup \{F^o \mid F \text{ is a face of } \triangleright^{s-1} \text{ and } F \supseteq \triangleright_J\}$$

is the open star of the interior of the face \triangleright_J .

If we think of stratifying the simplex \triangleright^{s-1} by the interiors of its faces, then the mapping $\delta : \mathcal{L}_y \rightarrow \triangleright^{s-1}$ is a stratified mapping: for any $J \subset \{1, 2, \dots, s\}$ it maps $\mathcal{L}_y \cap X_{P(J)}$ to the interior \triangleright_J^o of the face \triangleright_J , and in particular

$$\mathcal{L}_y \cap \bar{X}_Q = \delta^{-1}(\triangleright_I).$$

The fiber over any interior point $s \in \triangleright_J^o$ is the nilmanifold $(\Gamma \cap \mathcal{U}_Q) \backslash \mathcal{U}_Q$. As in [GHM], §18.5, the (weighted) cohomology of $\mathcal{L}_y \cap \bar{X}_Q$ can be computed using the Mayer-Vietoris spectral sequence for the covering by open stars (for $i \in I$),

$$V_{\{i\}} = \delta^{-1}(U_{\{i\}} \cap \triangleright_I)$$

of the vertices of \triangleright_I . Set $V_J = \delta^{-1}(U_J \cap \triangleright_I)$. The groups $E_1^{a,b}$ are cohomology groups of multi-intersections of open sets in this covering, and were computed in [GHM], Lemma 18.5,

$$\begin{aligned} E_1^{a,b} &= \bigoplus_{\substack{|J|=a+1 \\ J \subset I}} W^v H^b \left(\bigcap_{j \in J} V_{\{j\}}; \mathbf{E} \right) = \bigoplus_{\substack{|J|=a+1 \\ J \subset I}} W^v H^b(V_J; \mathbf{E}) \\ &= \bigoplus_{\substack{|J|=a+1 \\ J \subset I}} H^b(\mathfrak{R}_P, E)_{\geq v_{P(J)}}. \end{aligned}$$

The E_1 differential is given (up to sign) by inclusion, so the argument of [GHM], §18.7 applies here as well: the spectral sequence collapses at E_2 , which has only two possibly nonzero columns: $E_2^{0,b} = H^b(\mathfrak{R}_P, E)_{\geq v_P}$ and, using (12.1.3),

$$E_2^{|I|-1,b} = \frac{H^b(\mathfrak{R}_P, E)_{\geq v_{P(I)}}}{\sum_{\substack{|K|=|I|-1 \\ K \subset I}} H^b(\mathfrak{R}_P, E)_{\geq v_{P(K)}}} = H^b(\mathfrak{R}_P, E)_{[v_P, I]}$$

which contributes to $W^\nu H^*(\delta^{-1}St(\triangleright_I), \mathbf{E})$ in degree $|I| - 1 + b$. So we obtain a split short exact sequence (with $c = |I| - 1 + b$),

$$0 \rightarrow H^{c-|I|+1}(\mathfrak{R}_P, E)_{\lceil v_P, I \rceil} \rightarrow W^\nu H^c(\mathcal{L}_y \cap \bar{X}_Q; \mathbf{E}) \rightarrow H^c(\mathfrak{R}_P, E)_{\geq v_P} \rightarrow 0$$

which completes the proof of (12.6.4).

Step 2. As in [GHM], §18.11, the long exact sequence for the pair (12.6.2) splits into split short exact sequences,

$$0 \rightarrow H^c(B_\varepsilon \cap N_y) \rightarrow H^c(\partial B_\varepsilon \cap N_y \cap \bar{X}_Q) \rightarrow H^{c+1}(B_\varepsilon \cap N_y, \partial B_\varepsilon \cap N_y \cap \bar{X}_Q) \rightarrow 0.$$

But $H^*(B_\varepsilon \cap N_y) = H^*(B_\varepsilon) = H^*(\mathfrak{R}_P, E)_{\geq v_P}$ is the stalk cohomology at y of the weighted cohomology sheaf. This kills the second summand in (12.6.4), leaving

$$(12.6.5) \quad \mathbf{H}^m(\mathbf{B}^\bullet) \cong H^m(B_\varepsilon \cap N_y, \partial B_\varepsilon \cap N_y \cap \bar{X}_Q; \mathbf{W}^\nu \mathbf{C}^\bullet(\mathbf{E})) \cong H^{m-|I|}(\mathfrak{R}_P, E)_{\lceil v_P, I \rceil}.$$

Step 3. We briefly indicate why the isomorphism (12.6.5) extends to an isomorphism of flat vector bundles on X_P ,

$$\mathbf{H}^m(\mathbf{B}^\bullet) \cong H^{m-|I|}(\mathfrak{R}_P, E)_{\lceil v_P, I \rceil} \times_{\Gamma_{L(P)}} L_P / K_P A_P$$

(where $\Gamma_{L(P)} = v_P(\Gamma \cap P)$ is the projection of $\Gamma \cap P$ to the Levi quotient L_P and where it acts on $H^*(\mathfrak{R}_P, E)$ by conjugation). Let $i : X \rightarrow \bar{X}$ denote the inclusion. In [GHM], §17, special differential forms are used in order to identify the restriction $\mathbf{H}^m(Ri_* \mathbf{E})|_{X_P}$ with the flat vector bundle

$$H^m(\mathfrak{R}_P, E)_{\lceil v_P, \phi \rceil} \times_{\Gamma_{L(P)}} L_P / K_P A_P.$$

But each of the cohomology groups appearing in Step 2 (above) is an L_P -submodule of $H^*(\mathfrak{R}_P, E)$ and the corresponding bundle on X_P is a sub-bundle of $\mathbf{H}^* Ri_* (\mathbf{E})|_{X_P}$ (while the shift by $|I|$ corresponds to tensoring with a trivial vector bundle on X_P). So it suffices to verify that the stalk cohomology modules agree at the basepoint, which we have done. \square

12.7. Kostant's theorem. In this section we will use Kostant's theorem [Ko] to explicitly evaluate the cohomology group (12.5.1). Let $\mathbf{B} \subset \mathbf{G}$ be a Borel subgroup (over \mathbb{C}), chosen so that $\mathbf{B}(\mathbb{C}) \subset \mathbf{P}_0(\mathbb{C}) \subset \mathbf{P}(\mathbb{C})$. Choose a maximal torus \mathbf{T} (over \mathbb{C}) of \mathbf{G} so that

$$(12.7.1) \quad \mathbf{S}_P(\mathbb{C}) \subset \mathbf{S}_0(\mathbb{C}) \subset \mathbf{T}(\mathbb{C}) \subset \mathbf{B}_L(\mathbb{C})$$

where $\mathbf{B}_L = \mathbf{B} \cap \mathbf{L}_P$ is the corresponding Borel subgroup of \mathbf{L}_P . This gives rise to root systems $\Phi_G = \Phi(\mathbf{G}(\mathbb{C}), \mathbf{T}(\mathbb{C}))$ and $\Phi_L = \Phi(\mathbf{L}_P(\mathbb{C}), \mathbf{T}(\mathbb{C}))$ with positive roots $\Phi_G^+ = \Phi(\mathcal{U}_B(\mathbb{C}), \mathbf{T}(\mathbb{C}))$ and $\Phi_L^+ = \Phi_L \cap \Phi_G^+$ (determined by the Borel subgroups $\mathbf{B} \subset \mathbf{G}$ and $\mathbf{B}_L \subset \mathbf{L}_P$ respectively). Let $\rho_B = \frac{1}{2} \sum_{\alpha \in \Phi_G^+} \alpha$.

Let $W_G = W(\mathbf{G}(\mathbb{C}), \mathbf{T}(\mathbb{C}))$ denote the Weyl group of $\mathbf{G}(\mathbb{C})$ and let $W_P = W(\mathbf{L}_P(\mathbb{C}), \mathbf{T}(\mathbb{C}))$ denote the Weyl group of $\mathbf{L}_P(\mathbb{C})$. The choice of \mathbf{B} determines a length function ℓ on W_G . Let $W_P^1 \subset W_G$ denote the set of Kostant representatives: the

unique elements of minimal length from each of the cosets $W_P x \in W_P \backslash W_G$. As in [Sp], §10.2 or [Vo], §3.2.1, it may also be described as the set

$$W_P^1 = \{w \in W_G \mid w^{-1}(\Phi_L^+) \subset \Phi_G^+\}$$

(and depends on the choice of $\mathbf{B}_L \subset \mathbf{L}_P$).

If $\beta \in \chi^*(\mathbf{T}(\mathbb{C}))$ is \mathbf{B}_L -dominant, let us write V_β^L for the irreducible \mathbf{L}_P -representation with highest weight β . Let $\lambda_B \in \chi^*(\mathbf{T}(\mathbb{C}))$ be the highest weight of the irreducible representation E of \mathbf{G} . Kostant's theorem states that for all $w \in W_P^1$, the weight $w(\lambda_B + \rho_B) - \rho_B$ is \mathbf{B}_L -dominant, and that as an \mathbf{L}_P -module, the cohomology group $H^i(\mathfrak{N}_P, E)$ is isomorphic to

$$\bigoplus \{V_{w(\lambda_B + \rho_B) - \rho_B}^L \mid w \in W_P^1 \text{ and } \ell(w) = i\}.$$

If $w \in W_G$ then the character $w(\lambda_B + \rho_B) - \rho_B - \nu$ of \mathbf{S}_0 is trivial on \mathbf{S}_G so we may define

$$(12.7.2) \quad I_\nu(w) = \{\alpha \in \Delta_P \mid \langle (w(\lambda_B + \rho_B) - \rho_B - \nu) \mid \mathbf{S}'_\alpha, t_\alpha \rangle < 0\}$$

where $\{t_\alpha\}$ form the basis of the cocharacter group $\chi_*^{\mathbb{Q}}(\mathbf{S}'_\alpha)$ which is dual to the basis Δ_P of simple roots, cf. (12.1.1). So in the notation of (12.1.1),

$$I_\nu(w) = I_\nu(\gamma) \quad \text{where } \gamma = (w(\lambda_B + \rho_B) - \rho_B) \mid \mathbf{S}_P.$$

To summarize we have:

12.8. Proposition. *Let \mathbf{P} be a standard rational parabolic subgroup of \mathbf{G} . Let $\nu_P = \nu \mid \mathbf{S}_P \in \chi_{\mathbb{Q}}^*(\mathbf{S}_P)$ be the character which is determined by the weight profile $\nu \in \chi_{\mathbb{Q}}^*(\mathbf{S}_0)$. Let λ_B denote the highest weight of the irreducible representation E of \mathbf{G} . Let $I \subset \Delta_P$ be a subset corresponding to a choice of standard rational parabolic subgroup $\mathbf{Q} \supset \mathbf{P}$. Then Kostant's theorem determines an isomorphism of graded \mathbf{L}_P -modules,*

$$(12.8.1) \quad H^*(\mathfrak{N}_P, E)_{[\nu_P, I]} \cong \bigoplus_{\substack{w \in W_P^1 \\ I_\nu(w) = I}} V_{w(\lambda_B + \rho_B) - \rho_B}^L[-\ell(w)]$$

where the sum is taken over all $w \in W_P^1$ such that $I_\nu(w) = I$, and where $V_\beta^L[-m]$ means that the irreducible \mathbf{L}_P -module V_β^L appears in degree m .

13. Lefschetz numbers

13.1. In this section we recall the Lefschetz fixed point theorem for hyperbolic correspondences from [GM2], §10.3.

Suppose \bar{C} , \bar{X} and \bar{Y} are compact subanalytic spaces and that $c = (c_1, c_2) : \bar{C} \rightarrow \bar{X} \times \bar{Y}$ is a subanalytic mapping. (The bars are used so that the notation here will agree with that in the rest of the paper.) Let $\mathbf{S}^\bullet \in D^b(\bar{X})$ be a (bounded from below) complex of (cohomologically) constructible sheaves on \bar{X} and let $\mathbf{T}^\bullet \in D^b(\bar{Y})$ be a (bounded from below) complex of (cohomologically) constructible sheaves on \bar{Y} . Since

c is proper we have $c_* = c_!$. A lift of the correspondence \bar{C} to the sheaf level ([Ve], [GI], [Bo5]) is a morphism

$$(13.1.1) \quad \Phi : c_2^* \mathbf{T}^\bullet \rightarrow c_1^! \mathbf{S}^\bullet.$$

Such a morphism induces a homomorphism $H^*(\bar{Y}; \mathbf{T}^\bullet) \rightarrow H^*(\bar{X}; \mathbf{S}^\bullet)$ as follows. First apply $(c_1)_!$ and adjunction to obtain a morphism

$$(13.1.2) \quad (c_1)_! c_2^* \mathbf{T}^\bullet \rightarrow (c_1)_! c_1^! \mathbf{S}^\bullet \rightarrow \mathbf{S}^\bullet.$$

Let $p : \bar{X} \rightarrow \text{pt}$ and $q : \bar{Y} \rightarrow \text{pt}$ be the map to a point. Then the diagram

$$\begin{array}{ccc} \bar{X} \times \bar{Y} & \xrightarrow{\quad} & \bar{Y} \\ \pi_1 \downarrow & & \downarrow q \\ \bar{X} & \xrightarrow[p]{} & \text{pt} \end{array}$$

is a fiber square so there is an adjunction natural transformation [GM2], (2.6b), $q_!(\pi_2)_* \rightarrow p_*(\pi_1)_!$. Apply $q_!$ to the adjunction morphism $\mathbf{T}^\bullet \rightarrow (c_2)_* c_2^* \mathbf{T}^\bullet$ and use (13.1.2) to obtain

$$\begin{aligned} q_! \mathbf{T}^\bullet &\rightarrow q_!(c_2)_* c_2^* \mathbf{T}^\bullet = q_!(\pi_2)_* c_* c_2^* \mathbf{T}^\bullet \\ &\rightarrow p_*(\pi_1)_! c_* c_2^* \mathbf{T}^\bullet = p_*(c_1)_! c_2^* \mathbf{T}^\bullet \rightarrow p_* \mathbf{S}^\bullet. \end{aligned}$$

This morphism induces the desired mapping on cohomology. (It may also be constructed by applying $p_!(c_2)_*$ to (13.1.1) rather than $q_*(c_1)_!$.)

In what follows, we suppose $\bar{X} = \bar{Y}$ and $\mathbf{S}^\bullet = \mathbf{T}^\bullet$, so $c = (c_1, c_2) : \bar{C} \rightarrow \bar{X} \times \bar{X}$ is a correspondence on \bar{X} and $\Phi : c_2^* \mathbf{S}^\bullet \rightarrow c_1^! \mathbf{S}^\bullet$ is a lift to the sheaf level. The Lefschetz fixed point theorem states that the resulting Lefschetz number

$$L(\mathbf{S}^\bullet, \bar{C}) = \sum_{i \geq 0} \text{Tr}(\Phi^* : H^i(\bar{X}; \mathbf{S}^\bullet) \rightarrow H^i(\bar{X}; \mathbf{S}^\bullet)) = \sum_F L(\mathbf{S}^\bullet, \bar{C}, F)$$

is a sum of locally defined contributions $L(\mathbf{S}^\bullet, \bar{C}, F)$, one for each connected component $F \subset \bar{C}$ of the fixed point set of the correspondence \bar{C} .

Let $F \subset \bar{C}$ be a connected component of the fixed point set and suppose that the correspondence \bar{C} is weakly hyperbolic (§11.7) near $F' = c_1(F) = c_2(F)$ with indicator mapping $t : W \rightarrow \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$. (This means that $W \subset \bar{X}$ is a neighborhood of F' , that t is a proper subanalytic mapping such that $t^{-1}(0, 0) = F'$, and that for all $x \in c_1^{-1}(W) \cap c_2^{-1}(W)$ we have $t_1 c_1(x) \leq t_1 c_2(x)$ and $t_2 c_1(x) \geq t_2 c_2(x)$.) Denote by h and j the inclusions

$$F' \xrightarrow{h} t^{-1}(\mathbb{R}_{\geq 0} \times \{0\}) \xrightarrow{j} \bar{X}$$

of F' into the ‘‘expanding set’’ or ‘‘unstable manifold’’ $F^- = t^{-1}(\mathbb{R}_{\geq 0} \times \{0\})$, and of F^- into \bar{X} .

Let $\mathbf{A}^\bullet = h^!j^*(\mathbf{S}^\bullet)$ as in §12.4. Then the lift Φ determines a lift $\Psi' : c_2^*\mathbf{A}^\bullet \rightarrow c_1^*\mathbf{A}^\bullet$ which, by adjunction, induces an endomorphism $\Psi : \mathbf{A}^\bullet \rightarrow \mathbf{A}^\bullet$ (which covers the identity mapping on F'). In [GM2] we prove:

13.2. Theorem. *The contribution $L(\mathbf{S}^\bullet, \bar{C}, F)$ of F to the global Lefschetz number $L(\mathbf{S}^\bullet, \bar{C})$ is given by*

$$L(\mathbf{S}^\bullet, \bar{C}, F) = \sum_{i \geq 0} (-1)^i \text{Tr}(\Psi^* : H^i(F'; \mathbf{A}^\bullet) \rightarrow H^i(F'; \mathbf{A}^\bullet)).$$

Moreover, if $F' = \coprod_{\alpha=1}^m F'_\alpha$ is stratified so that the pointwise Lefschetz number $n(x) = \sum_{i \geq 0} (-1)^i \text{Tr}(\Psi_x^* : H_x^i(\mathbf{A}^\bullet) \rightarrow H_x^i(\mathbf{A}^\bullet))$ is constant on each stratum, then the local contribution is the sum over strata,

$$(13.2.1) \quad L(\mathbf{S}^\bullet, \bar{C}, F) = \sum_{\alpha=1}^m \chi_c(F'_\alpha) n(x_\alpha)$$

where $x_\alpha \in F'_\alpha$ and where χ_c denotes the Euler characteristic with compact supports. (See [KS], Prop. 9.6.12 for a related result.) The right hand side of (13.2.1) is the Euler characteristic $\chi(F'; n)$ of the constructible function $n(x)$, as discussed in [Mac].

13.3. Morphisms and weighted cohomology. In this section we show how to lift any morphism to the weighted cohomology sheaf. As in §2, \mathbf{G} denotes a connected linear reductive algebraic group defined over \mathbb{Q} , D denotes the associated symmetric space, $K' = A_G K(x_0)$ is the stabilizer in G of a fixed basepoint $x_0 \in D$, $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is a neat arithmetic group and $X = \Gamma \backslash D$. As in §12.2 let $\tau : G \rightarrow GL(E)$ be a finite dimensional irreducible representation on some complex vector space. It gives rise to the local coefficient system (flat homogeneous vector bundle) $\mathbf{E} = (G/K') \times_\Gamma E$ which is the quotient of $(G/K') \times E$ under the equivalence relation $(xK', v) \sim (\gamma xK', \tau(\gamma)v)$ for all $\gamma \in \Gamma$. Denote by $[xK', v] \in \mathbf{E}$ the resulting equivalence class. Let \mathbf{P}_0 be the standard minimal rational parabolic subgroup with $\mathbf{S}_0 = \mathbf{S}_{\mathbf{P}_0}$. Fix $\nu \in \chi_{\mathbb{Q}}^*(\mathbf{S}_0)$ so that $\nu|_{\mathbf{S}_G}$ coincides with the character by which \mathbf{S}_G acts on E and let $\mathbf{W}^\nu \mathbf{C}^\bullet(\bar{X}; \mathbf{E})$ denote the resulting weighted cohomology complex of sheaves on \bar{X} .

Suppose $\Gamma' \subset \Gamma$ is a subgroup of finite index, set $C = \Gamma' \backslash G/K$, and let $f : C \rightarrow X$ be a morphism, i.e., there exists $h \in \mathbf{G}(\mathbb{Q})$ such that $h\Gamma'h^{-1} \subset \Gamma$ and $f(\Gamma'xK) = \Gamma hxK$. Let $\mathbf{E}' \rightarrow C$ be the local coefficient system on C which is determined by the representation $\tau : G \rightarrow GL(E)$. The morphism f is covered by a mapping $\mathbf{E}' \rightarrow \mathbf{E}$ of local systems given by $[xK, v] \mapsto [hxK, \tau(h)v]$. This mapping is easily seen to be well defined, and it induces an isomorphism of local systems $\mathbf{E}' \cong f^*(\mathbf{E})$ on C . Since $f : C \rightarrow X$ is an unramified finite covering, it further induces a canonical quasi-isomorphism of the sheaves of smooth differential forms with coefficients in this local system, $f^*\Omega^\bullet(X; \mathbf{E}) \rightarrow \Omega^\bullet(C; \mathbf{E}')$.

The morphism $f : C \rightarrow X$ admits a unique continuous extension $\bar{f} : \bar{C} \rightarrow \bar{X}$ to the reductive Borel-Serre compactifications (Lemma 6.3). If $i_C : C \rightarrow \bar{C}$ and $i_X : X \rightarrow \bar{X}$ denote the inclusions then the adjunction mapping [GM2], equation (2.5a),

$$\bar{f}^*(i_X)_*\Omega^\bullet(X; \mathbf{E}) \xrightarrow{\cong} (i_C)_*f^*\Omega^\bullet(X; \mathbf{E}) \xrightarrow{\cong} (i_C)_*\Omega^\bullet(C; \mathbf{E})$$

is a quasi-isomorphism. It is easy to see that this induces a quasi-isomorphism

$$(13.3.1) \quad \bar{f}^*\mathbf{W}^v\mathbf{C}^\bullet(\bar{X}; \mathbf{E}) \rightarrow \mathbf{W}^v\mathbf{C}^\bullet(\bar{C}; \mathbf{E}')$$

of weighted cohomology sheaves. (In fact the whole construction of the weighted cohomology sheaf on \bar{X} pulls back to the construction of weighted cohomology on \bar{C} .)

13.4. Hecke correspondences and weighted cohomology. Let $g \in \mathbf{G}(\mathbb{Q})$. Then g gives rise to a Hecke correspondence $(c_1, c_2) : \bar{C} \rightarrow \bar{X}$. Here, \bar{C} is the reductive Borel-Serre compactification of $C = \Gamma' \backslash G/K$ with $\Gamma' = \Gamma \cap g^{-1}\Gamma g$. Both mappings c_1 and c_2 are finite so there are natural isomorphisms of functors $c_i^* \cong c_i^!$ and $(c_i)_* \cong (c_i)_!$ (for $i = 1, 2$). From the preceding paragraph we obtain a canonical lift

$$(13.4.1) \quad \Phi : c_2^*\mathbf{W}^v\mathbf{C}^\bullet(\bar{X}; \mathbf{E}) \rightarrow c_1^!\mathbf{W}^v\mathbf{C}^\bullet(\bar{X}; \mathbf{E})$$

to the weighted cohomology sheaves, which is given by the composition

$$c_2^*\mathbf{W}^v\mathbf{C}^\bullet(\bar{X}; \mathbf{E}) \xrightarrow{\cong} \mathbf{W}^v\mathbf{C}^\bullet(\bar{C}; \mathbf{E}) \xleftarrow{\cong} c_1^*\mathbf{W}^v\mathbf{C}^\bullet(\bar{X}; \mathbf{E}) \cong c_1^!\mathbf{W}^v\mathbf{C}^\bullet(\bar{X}; \mathbf{E}).$$

13.5. Computation of the local contribution. For the remainder of §13, fix a Hecke correspondence $\bar{C} \rightrightarrows \bar{X}$ which is determined by some element $g \in \mathbf{G}(\mathbb{Q})$. Fix a regular Γ -equivariant parameter $b \in \mathcal{B}$ which is so large that the resulting tilings $\{D^P\}$ of \bar{D} , $\{X^P\}$ of \bar{X} and $\{C^P\}$ of \bar{C} are narrow (§6.11) with respect to the Hecke correspondence. Choose $\mathbf{t} \in A_{P_0}$ (>1) to be regular, dominant, and sufficiently close to $\mathbf{1}$ as in Proposition 11.2, with resulting shrink homeomorphism $Sh(\mathbf{t})$, and let $(c'_1, c'_2) : \bar{C} \rightrightarrows \bar{X}$ be the resulting modified correspondence. It is easy to see that $Sh(\mathbf{t})^*(\mathbf{W}^v\mathbf{C}^\bullet) \cong \mathbf{W}^v\mathbf{C}^\bullet$ so we may consider (13.4.1) to be a lift of the modified correspondence as well.

Suppose the Hecke correspondence preserves some stratum C_P . According to Proposition 7.3, locally near C_P the correspondence is isomorphic to the parabolic Hecke correspondence $\Gamma'_P \backslash D[P] \rightrightarrows \Gamma_P \backslash D[P]$ which is given by some $y \in \mathbf{P}(\mathbb{Q}) \cap \Gamma g \Gamma$ and to which we may associate a decomposition $\Delta_P = \Delta_P^+ \cup \Delta_P^- \cup \Delta_P^0$ of the simple roots. Suppose that C_P contains fixed points and denote by $F_P(e) \subset C_P$ the set of fixed points with characteristic element $e \in \Gamma_L \bar{y} \Gamma_L \subset \mathbf{L}_P(\mathbb{Q})$.

By Proposition 8.4 the torus factor $a_e \in A_P$ of e coincides with the torus factor a_y , so the set Δ_P^+ (resp. Δ_P^- , resp. Δ_P^0) consists of those simple roots $\alpha \in \Delta_P$ for which $\alpha(a_e) < 0$ (resp. >0 , resp. $=0$). Hence we may write $\Delta_P^+ = \Delta_P^+(e)$ (resp. $\Delta_P^- = \Delta_P^-(e)$, resp. $\Delta_P^0 = \Delta_P^0(e)$).

As in §12.7, choose a Borel pair $\mathbf{T}(\mathbb{C}) \subset \mathbf{B}(\mathbb{C})$ so that (12.7.1) holds. Assume the local system \mathbf{E} arises from an irreducible representation of \mathbf{G} with highest weight $\lambda_B \in \chi^*(\mathbf{T}(\mathbb{C}))$. Let $\rho_B = \frac{1}{2} \sum_{\alpha \in \Phi_G^+} \alpha \in \chi^*(\mathbf{T}(\mathbb{C}))$ denote the half-sum of the positive roots. Let $r = [\Gamma \cap \mathcal{U}_P : \Gamma' \cap \mathcal{U}_P]$.

13.6. Theorem. *The contribution to the Lefschetz number from the fixed point constituent $F_P(e)$ is:*

$$(13.6.1) \quad r\chi_c(F_P(e))(-1)^{|\Delta_P^+|} \sum_{\substack{w \in W_P^1 \\ I_v(w) = \Delta_P^+(e)}} (-1)^{\ell(w)} \text{Tr}(e^{-1}; V_{w(\lambda_B + \rho_B) - \rho_B}^L)$$

where $I_v(w)$ is defined in (12.7.2).

In [GKM], §7.14 the Lefschetz formula in the adelic setting is described but not proven. The missing ingredient is the proof of the formula for the local contribution $L_Q(\gamma)$ which appears on page 534. (This formula differs slightly from (13.6.1) because the factor $r\chi_c(F_P(e))$ is absorbed by the orbital integral in [GKM].) Theorem 13.6 thus provides the proof of this formula, so it completes the proof of Theorem B (7.14) of [GKM]. The proof of Theorem 13.6 will occupy the rest of this section.

13.7. The nilmanifold correspondence. The Hecke correspondence $C \rightrightarrows X$ extends to a correspondence on the Borel-Serre compactification

$$(13.7.1) \quad \tilde{C} \rightrightarrows \tilde{X}$$

which is compatible with the projection $\mu : \tilde{X} \rightarrow \bar{X}$ to the reductive Borel-Serre compactification. Let $w' \in F_P(e)$ and set $w = c_1(w') = c_2(w')$. The restriction of the correspondence to the relevant Borel-Serre stratum is given by

$$(13.7.2) \quad Y'_P = \Gamma'_P \backslash P / K_P A_P \rightrightarrows Y_P = \Gamma_P \backslash P / K_P A_P,$$

$$(13.7.3) \quad \Gamma'_{P,x} K_P A_P \mapsto (\Gamma_{P,x} K_P A_P, \Gamma_{P,y,x} K_P A_P).$$

(Here, $\Gamma'_P = \Gamma_P \cap y^{-1} \Gamma_P y$.) The fibers $N_P = \mu^{-1}(w) \subset Y_P$ and $N'_P = (\mu')^{-1}(w') \subset Y'_P$ are nilmanifolds isomorphic to $\Gamma_{\mathcal{U}} \backslash \mathcal{U}_P$ and $\Gamma'_{\mathcal{U}} \backslash \mathcal{U}$ respectively, where $\Gamma_{\mathcal{U}} = \Gamma_P \cap \mathcal{U}_P$ and $\Gamma'_{\mathcal{U}} = \Gamma'_P \cap \mathcal{U}_P = \Gamma_P \cap y^{-1} \Gamma_{\mathcal{U}} y$. So the correspondence 13.7.1 restricts to a correspondence $N'_P \rightrightarrows N_P$ which will be described below. The following diagram may help in sorting out these spaces.

$$\begin{array}{ccccc} N'_P & \rightrightarrows & N_P & & Y'_P & \rightrightarrows & Y_P & & \tilde{C} & \rightrightarrows & \tilde{X} \\ \downarrow & & \downarrow & \text{in} & \downarrow & & \downarrow & \text{in} & \downarrow & & \downarrow \\ w' & \rightrightarrows & w & & C_P & \rightrightarrows & X_P & & \bar{C} & \rightrightarrows & \bar{X} \end{array}$$

13.8. Lemma. Let $\phi : L_P \rightarrow GL(H^*(\mathfrak{N}_P, E))$ denote the adjoint representation of the Levi quotient L_P on the Lie algebra cohomology of \mathfrak{N}_P . Let $w' \in F_P(e)$ be a fixed point in C_P with characteristic element $e \in L_P$. Then the nilmanifold correspondence $(c_1, c_2) : N'_P \rightrightarrows N_P$ induces a mapping $(c_1)_* c_2^* : H^*(N_P, \mathbf{E}) \rightarrow H^*(N_P, \mathbf{E})$ on cohomology which, under the Nomizu-van Est isomorphism $H^*(N_P, \mathbf{E}) \cong H^*(\mathfrak{N}_P, E)$ may be identified with the homomorphism

$$r\phi(e^{-1})$$

where $r = [\Gamma_{\mathcal{U}} : \Gamma'_{\mathcal{U}}]$.

13.9. Proof. First we find equations for the nilmanifold correspondence. Choose a

lift $xK_P A_P \in D = P/K_P A_P$ of the fixed point $w' = \Gamma'_P x K_P A_P \mathcal{U}_P \in C_P$. This determines a parametrization of the nilmanifold N_P by

$$(13.9.1) \quad \Gamma_{\mathcal{U}} \backslash \mathcal{U} \rightarrow N_P \subset Y_P = \Gamma_P \backslash P/K_P A_P,$$

$$(13.9.2) \quad \Gamma_{\mathcal{U}} z \mapsto \Gamma_P z x K_P A_P,$$

and similarly $\Gamma'_{\mathcal{U}} \backslash \mathcal{U} \rightarrow N'_P$ by $\Gamma'_{\mathcal{U}} z \mapsto \Gamma'_P z x K_P A_P$.

Since w' is fixed, we have $\Gamma_P x \mathcal{U}_P K_P A_P = \Gamma_P y x \mathcal{U}_P K_P A_P$ hence there exists $\gamma \in \Gamma_P$ and $u \in \mathcal{U}_P$ so that $\gamma y u x K_P A_P = x K_P A_P$, in other words, so that $\gamma y u$ fixes the point $x K_P A_P$ in the Borel-Serre boundary component $P/K_P A_P$. Then $e = v_P(\gamma y) = v_P(\gamma y u)$ is the characteristic element of the fixed point w' . Define $\Gamma'_{\mathcal{U}} \backslash \mathcal{U} \rightrightarrows \Gamma_{\mathcal{U}} \backslash \mathcal{U}$ by

$$(13.9.3) \quad \Gamma'_{\mathcal{U}} z \mapsto (\Gamma_{\mathcal{U}} z, \Gamma_{\mathcal{U}}(\gamma y) z u^{-1} (\gamma y)^{-1}).$$

A simple calculation shows that the following diagram commutes:

$$\begin{array}{ccc} \Gamma'_{\mathcal{U}} \backslash \mathcal{U} & \xrightarrow{(13.9.3)} & \Gamma_{\mathcal{U}} \backslash \mathcal{U} \\ (13.9.2) \downarrow \cong & & \cong \downarrow (13.9.2) \\ N'_P & \xrightarrow{\quad} & N_P \\ \downarrow & & \downarrow \\ Y'_P & \xrightarrow{(13.7.3)} & Y_P. \end{array}$$

Next we will apply the theorem of Nomizu [No1] and van Est [E] to this correspondence. The local system $\mathbf{E} \rightarrow X$ which is defined by the representation $\tau : G \rightarrow GL(E)$ extends canonically to a local system on the Borel-Serre compactification \tilde{X} . Its restriction to the nilmanifold N_P is given by the quotient $\mathbf{E}|_{N_P} = \mathcal{U} \times_{\Gamma_{\mathcal{U}}} E$ under the relation $(z, v) \sim (\gamma z, \tau(\gamma)v)$ (for $\gamma \in \Gamma_{\mathcal{U}}$, $z \in \mathcal{U}_P$, and $v \in E$). The complex $\Omega^{\bullet}(N_P, \mathbf{E})$ of smooth \mathbf{E} -valued differential forms on N_P consists of sections of the (flat) vector bundle

$$C^{\bullet}(N_P, \mathbf{E}) = \mathcal{U}_P \times_{\Gamma_P} C^{\bullet}(\mathfrak{N}_P, E)$$

where $C^{\bullet}(\mathfrak{N}_P, E) = \text{Hom}_{\mathbb{C}}(\wedge^{\bullet} \mathfrak{N}_P, E)$ is the complex of Lie algebra cochains. Let ϕ be the representation of P on this complex: if $\wedge^{\bullet} \text{Ad}(p) : \wedge^{\bullet} \mathfrak{N}_P \rightarrow \wedge^{\bullet} \mathfrak{N}_P$ denotes the adjoint action of $p \in P$ on the exterior algebra of \mathfrak{N}_P , then

$$\phi(p)(s) = \tau(p) \circ s \circ \wedge^{\bullet} \text{Ad}(p).$$

Denote by

$$(13.9.4) \quad \Omega^{\bullet}_{\text{inv}}(\mathcal{U}_P, E) = \{ \omega : \mathcal{U}_P \rightarrow C^{\bullet}(\mathfrak{N}_P, E) \mid \omega(ux) = \phi(u)\omega(x) \text{ for all } u, x \in \mathcal{U}_P \}$$

the complex of (left) \mathcal{U}_P -invariant E -valued differential forms on \mathcal{U}_P . Such a differential form is determined by its value $s = \omega(1) \in C^{\bullet}(\mathfrak{N}_P, E)$, and it passes to a differential form

on N_P . Denote by $\Omega_{\text{inv}}^\bullet(N_P, \mathbf{E})$ the collection of all such “left”-invariant differential forms. The Nomizu-van Est theorem ([No1], [E]) states that the inclusion $\Omega_{\text{inv}}^\bullet(N_P, \mathbf{E}) \hookrightarrow \Omega^\bullet(N_P, \mathbf{E})$ induces an isomorphism on cohomology. In summary we have a diagram

$$C^\bullet(\mathfrak{N}_P, E) \xleftarrow{\cong} \Omega_{\text{inv}}^\bullet(\mathcal{U}_P, E) \xleftarrow{\cong} \Omega_{\text{inv}}^\bullet(N_P, \mathbf{E}) \hookrightarrow \Omega^\bullet(N_P, \mathbf{E})$$

of isomorphisms and quasi-isomorphisms. Although the group P does not act on the vector bundle $C^\bullet(N_P, \mathbf{E})$, it does act on the complex $\Omega_{\text{inv}}^\bullet(N_P, \mathbf{E}) \cong \Omega_{\text{inv}}^\bullet(\mathcal{U}_P, E)$ of invariant sections by

$$(p \cdot \omega)(x) = \phi(p)^{-1} \omega(pxp^{-1})$$

and the group \mathcal{U}_P acts on this complex by

$$(u \cdot \omega)(x) = \omega(xu^{-1}).$$

If $\omega \in \Omega_{\text{inv}}^\bullet(N_P, \mathbf{E})$ is given by (13.9.4) then by (13.9.3) its pullback by c_2 is given by

$$c_2^*(\omega)(z) = \phi(\gamma y)^{-1} \omega((\gamma y)zu^{-1}(\gamma y)^{-1}).$$

Evaluating at $z = 1$ and using the fact that ω is left invariant,

$$c_2^*(\omega)(1) = \phi(u)^{-1} \phi(\gamma y)^{-1} \omega(1).$$

Let $s = \omega(1) \in C^\bullet(\mathfrak{N}_P, E)$, suppose $ds = 0$ and let $[s] \in H^*(\mathfrak{N}_P, E)$ be the resulting cohomology class. Since \mathcal{U}_P acts trivially on this cohomology,

$$c_2^*([s]) = \phi(e)^{-1} [s]$$

where $e = v_P(\gamma y)$ is the characteristic element of the fixed point w . Finally, observe that the pushforward mapping $(c_1)_* : H^*(N'_P, \mathbf{E}) \rightarrow H^*(N_P, \mathbf{E})$ is given by multiplication by $r = [\Gamma_{\mathcal{U}} : \Gamma'_{\mathcal{U}}]$. This completes the proof of Lemma 13.8. \square

13.10. Proof of Theorem 13.6. We will apply the Lefschetz fixed point formula to the modified Hecke correspondence. By Proposition 11.2, after modifying the correspondence by composing with $Sh(\mathbf{t})$, the fixed point constituent $F_P(e)$ becomes “truncated”, that is, it becomes replaced by the intersection $F_P^0(e) = F_P(e) \cap C_P^0$ of $F_P(e)$ with the central tile in C_P . Denote by $\partial F^0 = F_P(e) \cap \partial C_P^0$ its intersection with the boundary of the central tile. Set $F' = c_1(F_P(e)) = c_2(F_P(e))$. Set $E = F' \cap X_P^0 = c_i(F_P^0(e))$ and $\partial E = F' \cap \partial X_P^0 = c_i(\partial F^0)$. (Having used up all the letters some time ago, we temporarily re-use the notation E here, hoping the reader will not confuse it with the local system.) Note that $E - \partial E$ is diffeomorphic to F' .

By Theorem 11.9 the (modified) Hecke correspondence is weakly hyperbolic near $F_P(e)$ and an indicator mapping (defined in a neighborhood $U \subset \bar{X}$ of F') is given by

$$(13.10.1) \quad t(x) = \left(\sum_{\alpha \in \Delta_p^+} r_\alpha^P(x) + d_E \pi_P(x), \sum_{\alpha \in \Delta_p^-} r_\alpha^P(x) + \sum_{\alpha \in \Delta_p^0} r_\alpha^P(x) \right).$$

Let $\mathbf{Q} \supset \mathbf{P}$ be the rational parabolic subgroup corresponding to the subset $I = \Delta_P^+ \subset \Delta_P$ consisting of the simple roots for which the Hecke correspondence is (strictly) expanding. Then, in the notation of (2.4.1), $\Delta_P = i(\Delta_Q) \amalg \Delta_P^+$. The partial distance function r_α^P vanishes on the stratum X_Q whenever $\alpha \in \Delta_P^- \cup \Delta_P^0$, cf. (3.5.2). Hence $\bar{X}_Q \cap U = \tau^{-1}(\mathbb{R}_{\geq 0} \times \{0\})$ is the “expanding set” of the correspondence.

According to Theorem 13.2 we need to compute the stalk cohomology (at points $w \in E$) of the sheaf

$$\mathbf{A}^\bullet = h^1 j^* \mathbf{W}^v \mathbf{C}^\bullet(\mathbf{E})$$

where

$$E \xrightarrow{h} \bar{X}_Q \xrightarrow{j} \bar{X}.$$

This is best accomplished by decomposing h ,

$$E \xrightarrow{h_1} F' \xrightarrow{h_2} X_P \xrightarrow{h_3} \bar{X}_Q \xrightarrow{j} \bar{X}.$$

Then $\mathbf{B}^\bullet = h_3^1 j^* \mathbf{W}^v \mathbf{C}^\bullet(\mathbf{E})$ is the sheaf studied in Theorem 12.5, where we have taken $I = \Delta_P^+$. Its stalk cohomology is locally constant on X_P and was shown to be

$$H_w^i(\mathbf{B}^\bullet) \cong H^{j-|\Delta_P^+|}(\mathfrak{R}_P, \mathbf{E})_{\lceil v_P, \Delta_P^+ \rceil}.$$

Since h_2 is a smooth closed embedding we have a canonical quasi-isomorphism (12.3.1)

$$\mathbf{C}^\bullet := h_2^1 \mathbf{B}^\bullet \cong h_2^*(\mathbf{B}^\bullet) \otimes \mathcal{O}[-c]$$

where $c = \dim(X_P) - \dim(F' \cap X_P)$ and where \mathcal{O} is the orientation bundle (i.e. the top exterior power) of the normal bundle of $F' \cap X_P$ in X_P . The complex \mathbf{C}^\bullet is constructible with respect to the stratification of \bar{X} , meaning that its cohomology sheaves are locally constant on X_P , hence also on E . But E is a manifold with boundary, so

$$(13.10.2) \quad h_1^1 \mathbf{C}^\bullet \cong i_! \mathbf{C}^\bullet \mid (E - \partial E)$$

is obtained by first restricting to the interior $E - \partial E$ and then extending by 0. (Here, $i : E - \partial E \rightarrow E$ denotes the inclusion.) Thus the cohomology of $h_1^1 \mathbf{C}^\bullet$ is the compactly supported cohomology $H_c^i(E - \partial E; \mathbf{C}^\bullet) \cong H_c^i(F'; \mathbf{C}^\bullet)$.

Next we must compute the pointwise Lefschetz number $n(w)$ for $w \in E$, that is, the alternating sum of the traces on the stalk cohomology of $\mathbf{A}^\bullet = h_1^1 \mathbf{C}^\bullet$. By (13.10.2) it is 0 when $w \in \partial E$, so let $w \in E - \partial E$. Then

$$(13.10.3) \quad H_w^i(\mathbf{C}^\bullet) = H_w^{i-c}(h_2^* \mathbf{B}^\bullet \otimes \mathcal{O})$$

$$(13.10.4) \quad = H^{i-c-|\Delta_P^+|}(\mathfrak{R}_P, \mathbf{E})_{\lceil v_P, \Delta_P^+ \rceil} \otimes \mathcal{O}_w.$$

By §8.6, the mapping $c_1 : F_P(e) \rightarrow F'$ is a covering of degree

$$d_e = [\Gamma_L \cap \bar{y}^{-1} \Gamma_L \bar{y} : \nu_P(\Gamma_P \cap y^{-1} \Gamma_P y)].$$

Near each fixed point $w' \in c_1^{-1}(w)$ the Hecke correspondence acts on the \mathfrak{R}_P -cohomology through the homomorphism $r\phi(e^{-1})$ (using Lemma 13.8), and by §8.7 it acts on \mathcal{O}_w by $(-1)^c$. Summing these contributions over the d_e different points in $c_1^{-1}(w)$ gives

$$(13.10.5) \quad n(w) = d_e r (-1)^{-c-|\Delta_P^+|} (-1)^c \sum_{i \geq 0} (-1)^i \operatorname{Tr}(\phi(e^{-1}); H^i(\mathfrak{R}_P, E)_{[\nu_P, \Delta_P^+]})$$

$$(13.10.6) \quad = d_e r (-1)^{|\Delta_P^+|} \sum_{\substack{v \in W_P^1 \\ I_v(v) = \Delta_P^+}} (-1)^{\ell(v)} \operatorname{Tr}(e^{-1}; V_{v(\lambda_B + \rho_B) - \rho_B}^L)$$

by Proposition 12.8. The contribution arising from $F_P(e)$ is this quantity times $\chi_c(E - \partial E) = \chi_c(F')$. However (by §8.6), $\chi_c(F_P(e)) = d_e \chi_c(F')$ which absorbs the factor of d_e in (13.10.6) and therefore completes the proof of Theorem 13.6. \square

14. Proof of Theorem 1.5

14.1. As in §2, \mathbf{G} denotes a connected reductive linear algebraic group defined over \mathbb{Q} , $D = G/K'$ is its associated symmetric space with basepoint $x_0 \in D$ and stabilizer $K' = A_G K(x_0)$. Let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ denote an arithmetic subgroup which we assume to be neat, and $X = \Gamma \backslash D$. Throughout this section we fix a Hecke correspondence $(c_1, c_2) : \bar{C} \rightrightarrows \bar{X}$ defined by some element $g \in \mathbf{G}(\mathbb{Q})$. So $C = \Gamma' \backslash D$ with $\Gamma' = \Gamma \cap g^{-1} \Gamma g$. We also fix a Γ -equivariant tiling of D which is narrow with respect to the Hecke correspondence. Choose $t \in A_{P_0}$ (>1) in accordance with Proposition 11.2.

Let $F \subset \bar{C}$ denote the (full) fixed point set of the Hecke correspondence $\bar{C} \rightrightarrows \bar{X}$ and let E denote the (full) fixed point set of the modified Hecke correspondence (11.1.1). Then

$$F = \coprod_{\{\mathbf{P}\}} F \cap C_P \quad \text{and} \quad E = \coprod_{\{\mathbf{P}\}} F \cap C_P^0$$

where the union is over the strata of \bar{C} , that is, over Γ' -conjugacy classes of rational parabolic subgroups $\mathbf{P} \subseteq \mathbf{G}$. Each $F \cap C_P^0$ is a union of connected components of E by Proposition 11.2. The Lefschetz fixed point theorem (Theorem 13.2) may be used to write the Lefschetz number as a sum over these individual strata.

14.2. Contribution from a single stratum. Let $\mathbf{P} \subseteq \mathbf{G}$ be a rational parabolic subgroup and suppose that $c_1(C_P) = c_2(C_P) = X_P$. By Proposition 7.3, in a neighborhood of C_P the correspondence is isomorphic to the parabolic Hecke correspondence determined by some $y \in \Gamma g \Gamma \cap P$ and moreover (in this neighborhood) the fixed points of the modified correspondence coincide with those of $E \cap C_P = F \cap C_P^0$.

If $F_P(e)$ denotes the set of fixed points in C_P with characteristic element $e \in \Gamma_L \bar{y} \Gamma_L$, then by Proposition 8.4,

$$(14.2.1) \quad F \cap C_P = \coprod_{\{e\}} F_P(e) \quad \text{and} \quad E \cap C_P = \coprod_{\{e\}} F_P(e) \cap C_P^0$$

where the union is over Γ_L -conjugacy classes of elements $\{e\} \subset \Gamma_L \bar{y} \Gamma_L$ which are elliptic modulo A_P . (Here, $\Gamma_L = v_P(\Gamma \cap P) \subset L_P$ and $\bar{y} = v_P(y)$.) For each such conjugacy class $\{e\}$, the set $F_P(e)$ consists of finitely many connected components, say, F_1, F_1, \dots, F_m . The contribution to the Lefschetz number from the component F_j is given by Theorem 13.6. By (8.4.1) (see also §8.6, §15.8),

$$\sum_{j=1}^m \chi_c(F_j) = \chi_c(F_P(e)) = \chi_c(\Gamma'_e \backslash L_e / K'_e).$$

So the contribution to the Lefschetz number from the stratum $C_P \rightrightarrows X_P$ is

$$(14.2.2) \quad L(P, y) := \sum_{\{e\}} \chi_c(\Gamma'_e \backslash L_e / K'_e) r(-1)^{|\Delta_P^+|} \sum_{\substack{w \in W_P^1 \\ L_v(w) = \Delta_P^+(e)}} (-1)^{\ell(w)} \text{Tr}(e^{-1}; V_{w(\lambda_B + \rho_B) - \rho_B}^L)$$

where the index set for the first sum is the same as that for the union in (14.2.1). This quantity $L(P, y)$ depends only on the local system \mathbf{E} , the choice of parabolic subgroup \mathbf{P} and the element $y \in P$.

14.3. Sum over strata. Let $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_t$ denote a collection of representatives, one from each Γ -conjugacy class of rational parabolic subgroups $\mathbf{P} \subseteq \mathbf{G}$. These index the strata of \bar{X} . For each such i the intersection $\Gamma g \Gamma \cap P_i$ decomposes:

$$\Gamma g \Gamma \cap P_i = \coprod_j \Gamma_{P_i} y_{ij} \Gamma_{P_i}.$$

Lemma 7.4 gives a one-to-one correspondence between this collection $\{y_{ij}\}$ and strata C_{ij} of \bar{C} such that $c_1(C_{ij}) = c_2(C_{ij})$. Moreover the restriction of the Hecke correspondence to a neighborhood of C_{ij} is locally isomorphic to the parabolic Hecke correspondence defined by y_{ij} so the local contribution to the Lefschetz number from C_{ij} equals the number $L(P_i, y_{ij})$ given in (14.2.2). In summary, the total Lefschetz number is

$$(14.3.1) \quad L(g) = \sum_{i=1}^t \sum_j L(P_i, y_{ij})$$

as claimed in Theorem 1.5. \square

14.4. Another formula. If a little expansion $Sh(\mathbf{t})^{-1}$ is used instead of the shrink, this will convert neutral directions normal to each stratum into expanding directions, and it will convert the tangential distance into a contracting direction. An indicator mapping replacing (13.10.1) is

$$t(x) = \left(\sum_{\alpha \in \Delta_P^+ \cup \Delta_P^0} r_\alpha^P(x), \sum_{\alpha \in \Delta_P^-} r_\alpha^P(x) + d_E \pi_P(x) \right).$$

This changes the nature of the sheaf \mathbf{A}^\bullet with the result that the Euler characteristic (rather than the Euler characteristic with compact supports) appears in the formula. So, in equa-

tion (14.3.1), the contribution $L(P, y)$ (14.2.2) from the stratum $C_P \rightrightarrows X_P$ will be replaced by the quantity

$$L'(P, y) = \sum_{\{e\}} r(-1)^{|\Delta_P^+ \cup \Delta_P^0|} \chi(\Gamma_e' \backslash L_e / K_e') \sum_{\substack{w \in W_P^1 \\ I_V(w) = \Delta_P^+ \cup \Delta_P^0}} (-1)^{\ell(w)} \mathrm{Tr}(e^{-1}; V_{w(\lambda_B + \rho_B) - \rho_B}^L)$$

where the summations are over the same index sets as in (14.2.2), and where $\Delta_P^+ = \Delta_P^+(e)$ and $\Delta_P^0 = \Delta_P^0(e)$.

15. Remarks on the Euler characteristic

As in §2, \mathbf{G} denotes a connected linear reductive algebraic group defined over \mathbb{Q} ; D denotes the associated symmetric space; $\mathbf{S}_{\mathbf{G}}$ denotes the greatest \mathbb{Q} -split torus in the center of G ; $A_G = \mathbf{S}_{\mathbf{G}}(\mathbb{R})^0$ denotes the identity component of its real points; $K' = A_G K$ is the stabilizer in G of a fixed basepoint $x_0 \in D$; $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is an arithmetic group, and $X = \Gamma \backslash D$.

15.1. Proposition. *Suppose $(\mathbf{G}/\mathbf{S}_{\mathbf{G}})(\mathbb{R})$ does not contain a compact maximal torus. Then $\chi(X) = \chi_c(X) = 0$, that is, both the Euler characteristic and the Euler characteristic with compact supports vanish.*

The proof will appear in §15.6.

15.2. Lemma. *Let $X = \Gamma \backslash G / KA_G$. Then the Euler characteristic and the Euler characteristic with compact supports coincide: $\chi(X) = \chi_c(X)$.*

15.3. Proof. Let \tilde{X} denote the Borel-Serre compactification of X . Topologically, it is a manifold with boundary $\partial\tilde{X} = \tilde{X} - X$. Since $H_c^i(X) = H^i(\tilde{X}, \partial\tilde{X})$, it suffices to show that $\chi(\partial\tilde{X}) = 0$. The boundary $\partial\tilde{X}$ is a union of finitely many boundary strata Y_P , each of which fibers over the corresponding stratum X_P (of the reductive Borel-Serre compactification) with fiber a nilmanifold N_P (cf. §2.5, 13.7). So $\chi(Y_P) = \chi(N_P)\chi(X_P) = 0$. It follows from Mayer-Vietoris that $\chi(\partial\tilde{X}) = 0$. \square

For completeness we also include a proof of the following often-cited fact.

15.4. Lemma. *Suppose the real Lie group G/A_G does not contain a compact maximal torus. Then the Euler form vanishes identically on X .*

15.5. Proof. By replacing \mathbf{G} by the algebraic group ${}^0\mathbf{G}$ (and noting that $X = \Gamma \backslash {}^0G/K$), we may assume that $\mathbf{S}_{\mathbf{G}}$ is trivial. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathrm{Lie}(G)$ corresponding to the choice K of maximal compact subgroup. Choose a K -invariant inner product on \mathfrak{p} . This determines a G -invariant Riemannian metric on $D = G/K$ which passes to a Riemannian metric on X . Let Ω be the curvature form of the torsion-free Levi-Civita connection which is associated to this metric. The resulting Euler form Eu is defined to be 0 if $\dim(D)$ is odd. If $\dim(D) = 2k$ then Eu is the G -invariant differential form on D whose value at the basepoint x_0 is

$$Eu(\dot{x}_1, \dot{y}_1, \dot{x}_2, \dot{y}_2, \dots, \dot{x}_k, \dot{y}_k) = P(\Omega(\dot{x}_1, \dot{y}_1), \dots, \Omega(\dot{x}_k, \dot{y}_k))$$

(for any $\dot{x}_1, \dots, \dot{y}_k \in \mathfrak{p} = T_{x_0}D$), where P is the polarization of the Pfaffian $\text{Pf} : \text{End}(\mathfrak{p})^- \rightarrow \mathbb{R}$. (Here, $\text{End}(\mathfrak{p})^-$ denotes the skew-adjoint endomorphisms of \mathfrak{p} .) The form Eu on D passes to a differential form on $X = \Gamma \backslash D$, which is the Euler form for X .

Let $\text{Ad} : K \rightarrow GL(\mathfrak{p})$ be the adjoint representation and let $\text{ad} : \mathfrak{k} \rightarrow \text{End}(\mathfrak{p})^-$ be its derivative. We claim that $\det(\text{ad}(\dot{k})) = 0$ for any $\dot{k} \in \mathfrak{k}$. Modify \dot{k} by conjugacy if necessary, so as to guarantee that \dot{k} lies in a maximal torus $\mathfrak{t} \subset \mathfrak{g}$ which is stable under the Cartan involution ([Wa], §1.2, 1.3). Then $\mathfrak{t} = \mathfrak{t}_+ \oplus \mathfrak{t}_-$ with $\mathfrak{t}_+ \subset \mathfrak{k}$ and $\mathfrak{t}_- \subset \mathfrak{p}$. By assumption, \mathfrak{t}_- contains a nonzero vector \dot{i} , and $\text{ad}(\dot{k})(\dot{i}) = [\dot{k}, \dot{i}] = 0$, which proves the claim.

The principal K -bundle $G \rightarrow D = G/K$ admits a canonical G -invariant connection ([KN], Chapt. II, Thm. 11.5). Its curvature form $\omega \in \mathcal{A}^2(D, \mathfrak{k})$ is the G -invariant differential form whose value at the basepoint x_0 is given by $\omega_0(\dot{p}_1, \dot{p}_2) = -[\dot{p}_1, \dot{p}_2] \in \mathfrak{k}$ for any $\dot{p}_1, \dot{p}_2 \in \mathfrak{p}$. By a theorem of Nomizu [No2], for any real representation $\lambda : K \rightarrow GL(E)$, the resulting connection in the associated G -homogeneous vector bundle $\mathbf{E} = G \times_K E$ coincides with the torsion-free metric (Levi-Civita) connection of any G -invariant metric on E . Its curvature is the G -invariant $\text{End}(\mathbf{E})$ -valued differential form whose value at the basepoint is $\Omega_0 = \lambda' \circ \omega_0$ where $\lambda' : \mathfrak{k} \rightarrow \text{End}(E)$ is the differential of λ . Taking $\lambda = \text{Ad} : K \rightarrow GL(\mathfrak{p})$ as above gives $\Omega_0(\dot{p}_1, \dot{p}_2) = -\text{ad}([\dot{p}_1, \dot{p}_2])$. By the above claim, this has determinant 0 hence its Pfaffian vanishes also. Therefore the Euler form is zero on D , so it is also zero on X . \square

15.6. The proof of Proposition 15.1 is then a consequence of the following classical result of Harder [H] (a more streamlined proof of which may be found in [Le2]).

15.7. Theorem. *The Euler characteristic $\chi(X)$ is given by the integral over X of the Euler form with respect to any invariant Riemannian metric on X . \square*

15.8. Euler characteristic of a fixed point component. Now suppose that $X_P \subset \bar{X}$ is a boundary stratum corresponding to a rational parabolic subgroup $P = \mathcal{U}_P L_P$. Let $F_P(e) \subset X_P$ be the set of fixed points with some fixed (elliptic) characteristic element $e \in \mathbf{L}_P(\mathbb{Q})$. Let L_e be the centralizer of e in L_P . By (8.4.1), $F_P(e) \cong \Gamma_e' \backslash L_e / K_e'$ where $\Gamma_e' = \Gamma_L' \cap L_e$ and where $K_e' = L_e \cap (z(K_P A_P)z^{-1})$ (for appropriate z). By (8.4.2),

$$(15.8.1) \quad \chi_c(\Gamma_e' \backslash L_e / K_e') = d_e \chi_c(\Gamma_e \backslash L_e / K_e')$$

where $\Gamma_e = \Gamma_L \cap L_e$ and $d_e = [\Gamma_e : \Gamma_e']$. This expression has the following merit. The contribution (14.2.2) to the Lefschetz number from the stratum C_P depends on the subgroup $\Gamma_p' \subset \Gamma_P$. However once this expression (15.8.1) has been substituted into (14.2.2), the dependency on this subgroup Γ_p' occurs only in the two integers r and d_e .

15.9. Descent. Let \mathbf{S}_e be the greatest \mathbb{Q} -split torus in the center of L_e and let A_e be the identity component of its group of real points. As explained in [GKM], §7.11, the group K_e' does not necessarily contain $\mathbf{S}_e(\mathbb{R})$, so although $F_P(e)$ is not necessarily a “locally sym-

metric space" in the sense of §2, it fibers over the locally symmetric space $\Gamma_e' \backslash L_e / K_e' A_e$ with fiber A_e / A_P which is diffeomorphic to a Euclidean space. Therefore

$$\chi_c(F_P(e)) = (-1)^{\dim(A_e/A_P)} \chi_c(\Gamma_e' \backslash L_e / K_e A_e) = (-1)^{\dim(A_e/A_P)} d\chi_c(\Gamma_e \backslash L_e / K_e A_e)$$

(where $K_e = L_e \cap (zK_P z^{-1})$).

Now suppose that L_P/A_P does not contain a compact maximal torus. According to the preceding remarks, $\chi_c(C_P) = \chi_c(X_P) = 0$. However the contribution (14.2.2) to the Lefschetz number from the stratum C_P does not necessarily vanish. Assuming L_P/A_P does not contain a compact maximal torus, the same will be true of L_e/A_P . If, moreover, $A_P = A_e$ then $\chi_c(F_P(e)) = 0$. However if A_P differs from A_e , it is possible that $\chi(F_P(e)) \neq 0$ (in which case $F_P(e)$ is necessarily non-compact since it is fibered by A_e/A_P as described above). See Example 16.4 in which $L_e = A_e$ and $F_P(e) \cong A_e/A_P$ is the orbit of a split torus. In such cases it is possible to re-attribute the contribution (14.2.2) from the stratum $C_P \rightrightarrows X_P$ to smaller strata in the correspondence. This procedure is carried out in [GKM], p. 531, resulting in a Lefschetz formula in which the only nonzero contributions come from strata $C_P \rightrightarrows X_P$ such that L_P has a compact maximal torus.

15.10. In the adelic setting, the Euler characteristic with compact support $\chi_c(F_P(e))$ can be expressed in terms of orbital integrals (cf. [GKM], §7.11 and §7.14).

16. Examples and special cases

16.1. Reducible fixed point components. For $G = SL(3, \mathbb{R})$, $D = G/K$, and $\Gamma \subset SL(3, \mathbb{Z})$ a neat principal congruence subgroup, the reductive Borel-Serre compactification \bar{X} contains a singular 0-dimensional stratum X_B corresponding to the standard Borel subgroup \mathbf{B} . This stratum is contained in the closures of the strata X_{P_1} and X_{P_2} corresponding to the standard maximal parabolic subgroups $\mathbf{P}_1, \mathbf{P}_2$ containing \mathbf{B} . Let g be a generic element of $\mathbf{G}(\mathbb{Q}) \cap \mathcal{U}(P_1) \cap \mathcal{U}(P_2)$ which is not in Γ , for example,

$$g = \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $\bar{C} \rightrightarrows \bar{X}$ be the resulting Hecke correspondence. Then these three strata X_{P_1} , X_{P_2} , and X_B are fixed by the correspondence. However points in X which are sufficiently close to these strata are not fixed.

16.2. Middle weight for \mathbf{Sp}_4 . Let $\mathbf{G} = \mathbf{Sp}_4$, fix a neat arithmetic subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$, and choose a Hecke correspondence $\bar{C} \rightrightarrows \bar{X}$ which is determined by some $g \in \mathbf{G}(\mathbb{Q})$. If \mathbf{P} is a minimal parabolic subgroup of \mathbf{G} then its Levi quotient $\mathbf{L}_{\mathbf{P}} = \mathbf{S}_{\mathbf{P}}$ is a maximal split torus and the boundary stratum C_P consists of a single point. Suppose this point is an isolated fixed point of the Hecke correspondence. Let e be its characteristic element. The vector

space $\chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}})$ has a basis consisting of the simple roots $\Delta_P = \{\alpha, \beta\}$. Let $\{t_{\alpha}, t_{\beta}\}$ be the dual basis of $\chi_{\mathbb{Q}}^*(\mathbf{S}_{\mathbf{P}})$, so that $\langle \beta, t_{\beta} \rangle = 1$, $\langle \beta, t_{\alpha} \rangle = 0$ and the same with α and β interchanged. See Figure 6.

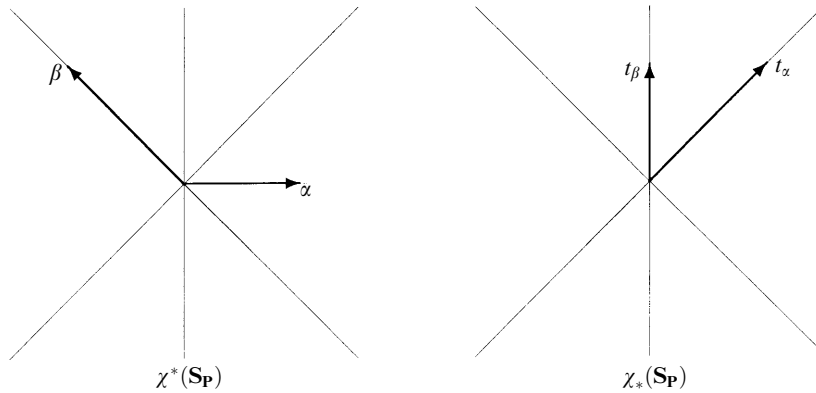


Figure 6. Simple roots and dual basis.

Let us take \mathbf{E} to be the trivial local system, and the weight profile $\nu = -\rho_B$ to be the middle weight (where ρ_B is the half-sum of the positive roots). The cohomology $H^*(\mathfrak{R}_P, \mathbb{C})$ decomposes into a sum of 1-dimensional weight spaces,

$$V_{w\rho_B - \rho_B} \subset H^{\ell(w)}(\mathfrak{R}_P, \mathbb{C})$$

as $w \in W$ varies over the elements of the full Weyl group. These weights are the dots in the left hand part of Figure 7, in which the origin is at $-\rho_B$. For each weight space indexed by a given $w \in W$ we have indicated the corresponding set

$$I_{\nu}(w) = \{\theta \in \Delta_P \mid \langle w\rho_B, t_{\theta} \rangle < 0\}$$

of simple roots. The cohomology $H^*(\mathfrak{R}_P)$ is divided into four ‘‘quadrants’’ according to the value of $I_{\nu}(w)$.

If necessary, project the characteristic element e to the identity component A_P of the torus $\mathbf{S}_{\mathbf{P}}(\mathbb{R})$ and let $t \in \mathfrak{A}_P = \text{Lie}(A_P)$ denote its log. The right hand half of Figure 7 may be identified with the Lie algebra \mathfrak{A}_P . The chamber containing t determines the expanding-contracting nature of the Hecke correspondence near this fixed point. In each chamber we have indicated the set of expanding roots,

$$\Delta_P^+(e) = \{\theta \in \Delta_P \mid \theta(t) < 0\}$$

(where now $\theta \in \Delta_P$ has been identified with a homomorphism $\mathfrak{A}_P \rightarrow \mathbb{R}$). The Lie algebra \mathfrak{A}_P is divided into four ‘‘quadrants’’ according to the value of $\Delta_P^+(e)$ (although we have not indicated which quadrant contains a given ‘‘wall’’).

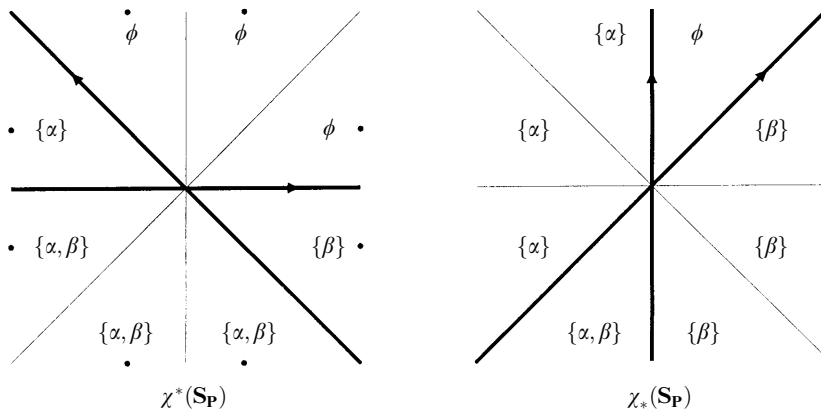


Figure 7. Diagram of $I_v(w)$ and of $\Delta_p^+(e)$.

Theorem 13.6 states that the portion of $H^*(\mathfrak{N}_p)$ which contributes to the Lefschetz number at this fixed point depends on the quadrant in which $t = \log(e)$ lies: if $\Delta_p^+(e) = J \subset \Delta_p$ then only the portion of $H^*(\mathfrak{N}_p)$ which lies in the quadrant indexed by J contributes to the Lefschetz number. A further degree shift by $|J|$ occurs when this portion $H^*(\mathfrak{N}_p)_{[v_p, J]}$ is identified (in Theorem 12.5) with the local weighted cohomology with supports.

It is a remarkable fact that, globally in the Hecke correspondence, the fixed points occur in Weyl group orbits. Assuming t is regular (does not lie on a wall) then, after summing over all the fixed points, each chamber will appear the same number of times. It is the sum of these local contributions over a W -orbit of fixed points ([GKM], p. 529, last paragraph) which gives rise to the combinatorial formula for the averaged discrete series characters as described in [GKM].

16.3. Very positive and very negative weights. Let $i : X \rightarrow \bar{X}$ denote the inclusion. Suppose the weight profile $v = -\infty$ (or is very negative). Then the weight truncation does nothing, and the weighted cohomology sheaf $\mathbf{W}^v \mathbf{C}^\bullet(\mathbf{E}) \cong Ri_*(\mathbf{E})$ becomes the “full” direct image of \mathbf{E} . For any stratum X_Q , $I_v(w) = \emptyset$ for any $w \in W_Q^1$. Theorem 13.6 then says that a fixed point stratum $F \cap C_Q$ (with characteristic element e) makes a contribution to the Lefschetz number only if $\Delta_Q^+(e) = \emptyset$, which is to say, only if the Hecke correspondence is either contracting or neutral in every direction normal to the stratum X_Q .

In this case the local contribution to the Lefschetz number may be expressed in terms of the character of the finite dimensional representation $G \rightarrow GL(E)$. We briefly recall the argument in [GKM], §7.18. The quantity $\sum_i (-1)^i \text{Tr}(e^{-1}; H^i(\mathfrak{N}_p, E))$ is equal to $\text{Tr}(e^{-1}; E)$ times the following quantity:

$$\begin{aligned} \sum_i (-1)^i \text{Tr}(e^{-1}; \wedge^i(\mathfrak{N}_p^*)) &= \det(1 - \text{Ad}(e); \mathfrak{N}_p(\mathbb{C})) \\ &= \prod_{\alpha \in \Phi_L^+} (1 - \alpha^{-1}(e)) \prod_{\alpha \in \Phi_L^+} \alpha(e) (-1)^{\dim \mathfrak{N}_p} \\ &= \Delta_p(e) \det(e; \mathfrak{N}_p) (-1)^{\dim \mathfrak{N}_p} \end{aligned}$$

where $\Delta_P(e) = \prod_{\alpha \in \Phi_L^+} (1 - \alpha^{-1}(e))$ denotes the (partial) Weyl denominator. (These quantities may be further expressed in terms of $|D_L^G(e)|$, $\delta_P(e)$, and $\chi_G(e)$ using [GKM], (7.16.11), (7.18.3) and [GKM], p. 497.)

Similarly, suppose the weight profile is $\nu = +\infty$ or is a very large positive weight. The stalk cohomology (at a point $x \in X_Q$ in some boundary stratum X_Q) of the weighted cohomology sheaf $\mathbf{W}^\nu \mathbf{C}^\bullet(\mathbf{E})$ vanishes because the weight truncation (12.2.1) kills everything. In this case, the weighted cohomology sheaf is quasi-isomorphic to the sheaf $Ri_!(\mathbf{E})$ which is obtained as the extension by 0 of the local system \mathbf{E} . Its cohomology is the compact support cohomology $H_c^*(X; \mathbf{E})$ of the locally symmetric space. For any stratum X_Q , according to (12.7.2), $I_\nu(w) = \Delta_Q$ for any $w \in W_Q^1$. Theorem 13.6 then says that a fixed point stratum $F \cap C_Q$ (with characteristic element $e \in L_Q$) makes a contribution to the Lefschetz number only if $\Delta_Q^+(e) = \Delta_Q$, that is, only if the Hecke correspondence is strictly expanding in all directions normal to the stratum X_Q . Then the same quantity $\sum_i (-1)^i \text{Tr}(e^{-1}; H^i(\mathfrak{R}_P, E))$ occurs in the formula, but with a (possibly) different sign.

In these cases (of $\nu = \pm\infty$) the Lefschetz formula of Franke [F] can be recovered, cf. [GKM], §7.17, 7.18.

16.4. Hyperbolic 3-space. For $\mathbf{G}(\mathbb{R}) = SL_2(\mathbb{C})$ the symmetric space $D = G/K$ may be identified with hyperbolic 3-space. If Γ is a torsion-free arithmetic group, then $X = \Gamma \backslash D$ is a hyperbolic 3-manifold. The group G does not contain a compact maximal torus. Consequently, $\chi(X) = 0$ (cf. §15.1). However, when X is not compact, there exists a Hecke correspondence on X whose fixed point set consists of a smooth curve which passes from one cusp to another cusp. The Euler characteristic of this fixed point set is not zero, although the Euler form vanishes identically. The fixed point set is not a “locally symmetric space” in the sense of §2.1 because it contains (and in fact consists of) a Euclidean factor, cf. §15.9. It is possible to find particular weight profiles such that the (global) Lefschetz number of this correspondence on the weighted cohomology is nonzero. However, the formula [GKM] (thm. 7.14.B) would attribute the contribution from this fixed curve to the cusps, rather than to the interior stratum. This re-attribution is a result of equation (7.14.2) of [GKM].

16.5. Nielsen fixed point theory. Suppose X is a compact manifold with fundamental group $\Gamma = \pi_1(X, x_0)$. Let $f : X \rightarrow X$ be a self-map. A choice of path from the basepoint x_0 to its image $f(x_0)$ determines a homomorphism $\phi : \Gamma \rightarrow \Gamma$. Two elements $\gamma_1, \gamma_2 \in \Gamma$ are said to be ϕ -conjugate if there exists $\gamma \in \Gamma$ so that $\gamma_2 = \gamma \gamma_1 \phi(\gamma)^{-1}$. Let $(\Gamma)_\phi$ denote the set of ϕ -conjugacy classes in Γ and let $\mathbb{R}(\Gamma)_\phi$ be the vector space of finite formal linear combinations of such classes. For each connected component F of the fixed point set of f , let $L(F) \in \mathbb{R}$ denote the contribution of F to the Lefschetz number L , that is,

$$L = \sum_i (-1)^i \text{Tr}(f^* : H^i(X) \rightarrow H^i(X)) = \sum_F L(F).$$

The Nielsen theory (see [GN]) assigns

- a ϕ -conjugacy class $\{F\} \in (\Gamma)_\phi$ to each connected component F of the fixed point set, and

• a (cohomologically defined) Nielsen number $N(\{\gamma\}, f)$ to each ϕ -conjugacy class $\{\gamma\}$ such that

$$(16.5.1) \quad \sum_{\{\gamma\} \in (\Gamma)_\phi} N(\{\gamma\}, f) \{\gamma\} = \sum_F L(F) \{F\} \in \mathbb{R}(\Gamma)_\phi$$

thereby “refining” the Lefschetz fixed point formula. (The sum on the left is over ϕ -conjugacy classes and the sum on the right is over connected components of the fixed point set.)

Now suppose that $X = \Gamma \backslash D$ is a compact locally symmetric space. Fix $g \in \mathbf{G}(\mathbb{Q})$ and let $C \rightrightarrows X$ be the resulting Hecke correspondence. Let $(\Gamma g \Gamma)_1$ be the set of Γ -conjugacy classes of elements $e \in \Gamma g \Gamma$. Let \mathbf{E} be the local system corresponding to a representation $\tau : G \rightarrow GL(E)$. Theorem 1.5 then says that the Lefschetz number of this correspondence is:

$$(16.5.2) \quad L = \sum_{\{e\}} \chi(F(e)) \operatorname{tr}(\tau(e)^{-1}; E).$$

Here, the sum is taken over all conjugacy classes $\{e\} \in (\Gamma g \Gamma)_1$, and $F(e)$ denotes the set of fixed points which have characteristic element equal to e . This set is empty unless e is elliptic (modulo A_G). If $F(e)$ is not empty, then it is compact.

It turns out that if the local system \mathbf{E} is trivial, and if the correspondence $C \rightrightarrows X$ is actually a self-map $f : X \rightarrow X$ then the terms in (16.5.2) are exactly the terms in the Nielsen formula (16.5.1). The group Γ may be identified with the fundamental group $\pi_1(X, x_0)$. The Hecke correspondence is actually a self-map iff the element g normalizes Γ . In this case, the automorphism $\phi : \Gamma \rightarrow \Gamma$ is given by conjugation: $\phi(\gamma) = g\gamma g^{-1}$. Finally, the association $a \mapsto ag$ (for $a \in \Gamma$) determines a one-to-one correspondence

$$(\Gamma)_\phi \rightarrow (\Gamma g \Gamma)_1.$$

There is a slightly more general Nielsen formula for correspondences, also with coefficients in a local system. The terms in this formula again coincide with the terms in the sum (16.5.2).

References

- [Ar1] *J. Arthur*, The L^2 Lefschetz numbers of Hecke operators, *Inv. Math.* **97** (1989), 257–290.
- [Ar2] *J. Arthur*, A trace formula for reductive groups I, *Duke Math. J.* **45** (1978), 911–952.
- [Ar3] *J. Arthur*, Characters, harmonic analysis, and an L^2 -Lefschetz formula, *The Mathematical Heritage of Hermann Weyl*, Proc. Symp. Pure Math. **48**, Amer. Math. Soc., Providence, RI, 1998.
- [AMRT] *A. Ash, D. Mumford, M. Rapoport, and Y.-S. Tai*, Smooth Compactification of Locally Symmetric Varieties, *Math. Sci. Press*, Brookline, Mass., 1975.
- [BaM] *D. Barbasch and H. Moscovici*, L^2 -index and the Selberg trace formula, *J. Funct. Anal.* **53** (1983), 151–201.
- [Be] *J. Bewersdorff*, Eine Lefschetzsche Fixpunktformel für Hecke-Operatoren, *Bonner Math. Schr.* **164**, Univ. Bonn, 1985.
- [Bo3] *A. Borel*, *Linear Algebraic Groups* (second edition), *Grad. Texts Math.* **126**, Springer Verlag, New York 1991.
- [Bo4] *A. Borel*, Stable real cohomology of arithmetic groups, *Ann. Sci. Éc. Norm. Sup.* **7** (1974), 235–272.

- [Bo5] *A. Borel*, Sheaf theoretic intersection cohomology, Seminar on Intersection Cohomology, Progr. Math. **50**, Birkhäuser, Boston 1984.
- [BS] *A. Borel* and *J.-P. Serre*, Corners and Arithmetic Groups, Comm. Math. Helv. **48** (1973), 436–491.
- [E] *W. T. van Est*, A generalization of the Cartan-Leray sequence, i, ii, Indag. Math. **XX** (1958), 399–413.
- [F] *J. Franke*, Harmonic analysis in weighted L_2 -spaces, Ann. Sci. Éc. Norm. Sup. **31** (1998), 181–279.
- [GeM] *S. Gelfand* and *Y. Manin*, Homological Algebra, in: Algebra V, Encycl. Math. Sci. **38**, Springer-Verlag, Berlin 1994.
- [Gi] *C. Gibson*, *K. Wirthmüller*, *A. Du Plessis*, *E. Looijenga*, Topological Stability of Smooth Mappings, Lect. Notes Math. **552**, Springer Verlag, New York 1976.
- [GN] *R. Geoghegan* and *A. Nicas*, Parametrized Lefschetz-Nielsen fixed point theory and Hochschild homology traces, Amer. J. Math **116** (1994), 397–446.
- [GHM] *M. Goresky*, *G. Harder*, and *R. MacPherson*, Weighted cohomology, Inv. Math. **116** (1994), 139–213.
- [GKM] *M. Goresky*, *R. Kottwitz*, and *R. MacPherson*, Discrete series characters and the Lefschetz formula for Hecke operators, Duke Math. J. **89** (1997), 477–554.
- [GM1] *M. Goresky* and *R. MacPherson*, Lefschetz numbers of Hecke correspondences, in: The Zeta Functions of Picard Modular Surfaces, R. Langlands and D. Ramakrishnan, eds., Publ. C.R.M., Univ. de Montréal (1992), 465–478.
- [GM2] *M. Goresky* and *R. MacPherson*, Local contribution to the Lefschetz fixed point formula, Inv. Math. **111** (1993), 1–33.
- [GM3] *M. Goresky* and *R. MacPherson*, Stratified Morse Theory, Ergeb. Math. **14**, Springer Verlag, Heidelberg 1989.
- [GM4] *M. Goresky* and *R. MacPherson*, Intersection homology II, Inv. Math. **71** (1983), 77–129.
- [GM5] *M. Goresky* and *R. MacPherson*, Errata to: Local contribution to the Lefschetz fixed point formula, 2003, <http://arxiv.org/math.AT.0301289>.
- [GI] *A. Grothendieck*, *L. Illusie*, Formule de Lefschetz, S.G.A. 5, Cohomologie l -adique et Fonctions L, Lect. Notes Math. **589**, Springer Verlag, Heidelberg (1977), 73–137.
- [H] *G. Harder*, Anhang (Letter to Goresky and MacPherson) in Eisensteinkohomologie und die Konstruktion gemischter Motive, Lect. Notes Math. **1562**, Springer Verlag, Heidelberg 1993.
- [I] *B. Iverson*, Cohomology of Sheaves, Universitext, Springer-Verlag, Heidelberg 1986.
- [KS] *M. Kashiwara* and *P. Schapira*, Sheaves on Manifolds, Grundle. Math. Wiss. **292**, Springer Verlag, Heidelberg 1990.
- [KN] *S. Kobayashi* and *K. Nomizu*, Foundations of Differential Geometry I, II, John Wiley and Sons, New York 1966.
- [Ko] *B. Kostant*, Lie algebra cohomology and the generalized Borel-Weil theorem, Ann. Math. **74** (1961), 329–387.
- [Ku] *M. Kuga* and *J. H. Sampson*, A coincidence formula for locally symmetric spaces, Amer. J. Math **94** (1972), 486–500.
- [L1] *R. Langlands*, Modular forms and ℓ -adic representations, in: Modular Functions of One Variable II, Lect. Notes Math. **349**, Springer Verlag, New York 1973.
- [L2] *R. Langlands*, On the functional equation satisfied by Eisenstein series, Lect. Notes Math. **544**, Springer Verlag, New York 1976.
- [Le1] *E. Leuzinger*, An exhaustion of locally symmetric spaces by compact manifolds with corners, Inv. Math. **121** (1995), 389–410.
- [Le2] *E. Leuzinger*, On the Gauss-Bonnet formula for locally symmetric spaces of noncompact type, Enseign. Math. **42** (1996), 201–214.
- [Lo] *E. Looijenga*, L^2 -cohomology of locally symmetric varieties, Comp. Math. **67** (1988), 3–20.
- [Mac] *R. MacPherson*, Chern classes for singular varieties, Ann. Math. **100** (1974), 423–432.
- [Mat] *J. Mather*, Notes on Topological Stability, mimeographed notes, Harvard University, 1970.
- [Mo] *H. Moscovici*, Lefschetz formula for Hecke operators, Lie Groups and Representations III, Lect. Notes Math. **1077**, Springer Verlag, New York (1984), 321–258.
- [N] *A. Nair*, Weighted cohomology of locally symmetric spaces, Ann. Math. **150** (1999), 1–31.
- [No1] *K. Nomizu*, On the cohomology of compact homogeneous spaces of nilpotent Lie groups, Ann. Math. **59** (1954), 531–538.
- [No2] *K. Nomizu*, Invariant affine connections on homogeneous spaces, Amer. J. Math. **76** (1954), 33–65.
- [R] *M. Rapoport*, On the shape of the contribution of a fixed point on the boundary: the case of \mathbb{Q} -rank one, in: The Zeta Functions of Picard Modular Surfaces, R. Langlands and D. Ramakrishnan, eds., Publ. C. R. M., Univ. de Montréal (1992), 465–478.
- [Sa] *L. Saper*, Tilings and finite energy retractions of locally symmetric spaces, Comm. Math. Helv. **72** (1997), 167–202.

- [SaS] *L. Saper and M. Stern*, L_2 cohomology of arithmetic varieties, *Ann. Math.* **132** (1990), 1–69.
- [Sat] *I. Satake*, On compactifications of the quotient spaces for arithmetically defined discontinuous groups, *Ann. Math.* **72** (1960), 555–580.
- [Sh] *S. Shokranian*, Lefschetz numbers on weighted cohomology for twisted groups, preprint, Dept. Matemática, Univ. de Brasília, 1999.
- [Sp] *T. A. Springer*, Linear Algebraic Groups, *Progr. Math.* **9**, Birkhäuser, Boston, MA, 1981.
- [St] *M. Stern*, Lefschetz formulae for arithmetic varieties, *Inv. Math.* **115** (1994), 241–296.
- [Ve] *J. L. Verdier*, The Lefschetz fixed point formula in étale cohomology, *Proceedings of Conference on Local Fields*, T. A. Springer, ed., *Lect. Notes Math.* **192**, Springer Verlag, Heidelberg (1971), 165–168.
- [Vo] *D. Vogan*, Representations of Real Reductive Lie Groups, *Progr. Math.* **15**, Birkhäuser, Boston, MA, 1981.
- [Wa] *G. Warner*, Harmonic Analysis on Semisimple Lie Groups I, *Grundle Math. Wiss.* **188**, Springer Verlag, Heidelberg 1972.
- [We] *U. Weselmann*, thesis, University of Bonn, 1996, <http://www.mathi.uni-heidelberg.de/~weselman/>.
- [Z1] *S. Zucker*, L_2 cohomology of warped products and arithmetic groups, *Inv. Math.* **70** (1982), 169–218.
- [Z2] *S. Zucker*, L_2 cohomology and intersection homology of locally symmetric spaces II, *Comp. Math.* **59** (1986), 339–398.
- [Z3] *S. Zucker*, Satake compactifications, *Comm. Math. Helv.* **58** (1983), 312–343.

School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540, USA
e-mail: goresky@ias.edu

Eingegangen 25. Juni 2002