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On the Spectrum of the Equivariant Cohomology Ring

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Abstract. If an algebraic torus T acts on a complex projective algebraic variety X, then the affine scheme $\operatorname{Spec} H_T^*(X;\mathbb{C})$ associated with the equivariant cohomology is often an arrangement of linear subspaces of the vector space $H_2^T(X;\mathbb{C})$. In many situations the ordinary cohomology ring of X can be described in terms of this arrangement.

1 Introduction

1.1 Torus Actions and Equivariant Cohomology

Suppose an algebraic torus T acts on a complex projective algebraic variety X. If the cohomology $H^*(X;\mathbb{C})$ is *equivariantly formal* (see §2), then knowledge of the equivariant cohomology $H_T^*(X;\mathbb{C})$ (as a module over $H_T^*(\mathrm{pt})$) is equivalent to knowledge of the ordinary cohomology groups, viz.

$$H_T^*(X) \cong H^*(X) \otimes_{\mathbb{C}} H_T^*(\mathrm{pt}), \tag{1.1}$$

$$H^*(X) \cong H_T^*(X) \otimes_{H_T^*(\mathrm{pt})} \mathbb{C}. \tag{1.2}$$

However the equivariant cohomology is often easier to understand as a consequence of the localization theorem [3]. For example, in [16] the equivariant cohomology ring $H_T^*(X;\mathbb{C})$ of an equivariantly formal space X was described in terms of the fixed points and the one-dimensional orbits, provided there are finitely many of each. In this paper we pursue the link between the equivariant cohomology and the orbit structure of T by studying the affine scheme $\operatorname{Spec} H_T^*(X)$ that is (abstractly) associated with the equivariant cohomology ring. Under suitable hypotheses, it turns out (Theorem 3.1) that the associated reduced algebraic variety V is an "arrangement" of linear subspaces of the vector space $H_2^T(X)$. Many interesting arrangements arise this way.

The ring structure on the ordinary cohomology $H^*(X)$ can sometimes be recovered from the variety $V = \big(\operatorname{Spec} H_T^*(X)\big)_{\mathrm{red}}$. This variety comes equipped with a finite linear mapping $\pi_*\colon V \to \mathfrak{t}$ to the Lie algebra of T. For generic $t \in \mathfrak{t}$ the fiber $\pi_*^{-1}(t)$ consists of finitely many points. If the mapping π is flat and if $H^*(X)$ is generated by the part in degree ≤ 2, we show (Theorem 4.1) that the filtration by degree on the coordinate ring $A_t = \mathbb{C}(\pi_*^{-1}(t))$ induces an isomorphism of rings,

$$\operatorname{Gr} A_t \cong H^*(X).$$

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This simple observation has interesting consequences in many different circumstances.

1.2 Schubert Varieties

For (possibly singular) Schubert varieties X_w, equation (1.1) becomes (Theorem 5.1) a theorem of Akyldiz–Carrell–Lieberman [1]:

$$H^*(X_w) \cong \operatorname{Gr} \mathbb{C}(W_{\leq w} \cdot t),$$

where $t \in \mathfrak{t}$ is regular, which was originally proved using the theory of holomorphic vector fields.

1.3 Toric Varieties

For non-singular toric varieties X, we show (Theorem 6.1) that the partially ordered set of cones in the fan Σ defining X coincides with the partially ordered set of flats (multi-intersections of the component subspaces) in the arrangement of linear spaces $V = \left(\operatorname{Spec} H_T^*(X)\right)_{\mathrm{red}}$. In other words, it is possible to recover the fan Σ, in a simple way, from the equivariant cohomology $H_T^*(X)$. From this, we recover a theorem of Brion–Vergne [9]: the cohomology ring $H^*(X)$ is naturally isomorphic to the ring of piecewise polynomial functions on the fan Σ.

1.4 Springer Fibers

For Springer fibers X_a in $GL(n, \mathbb{C})$ corresponding to a nilpotent element $a \in gl(n, \mathbb{C})$, we show (in Theorem 7.1) that equation (1.1) gives a theorem of J. Carrell [10] which identifies the cohomology ring $H^*(X_a)$ with the coordinate ring of the orbit of a certain point under the Weyl group. This result in turn was used by Carrell to give a short proof of the theorem of C. DeConcini and C. Procesi [14]: the coordinate ring of the scheme-theoretic intersection $\overline{C_a} \cap \mathfrak{t}$ is isomorphic to the cohomology ring $H^*(X_a)$. (Here $\overline{C_a}$ is the closure of the conjugacy class of a.) Moreover, in Theorem 7.2 we show that the Springer representation of the Weyl group W on $H^*(X)$ lifts naturally to a representation on the equivariant cohomology. (See also [16], where a similar result was proved in the case of affine Springer fibers.) The Springer representation is somewhat mysterious because W does not act (in an algebraic way) on X, but rather it acts via deformations of X [20, 21]. However, it turns out that the action of W on the equivariant cohomology is easy to describe and that, in fact, the Weyl group can be naturally identified as the group

$$W = \operatorname{Aut}_\pi(V) = \operatorname{Aut}_{\mathbb{C}[\mathfrak{t}]} H_T^*(X),$$

of automorphisms of the arrangement V that preserve the projection π. The Springer representation on the ordinary cohomology may then be recovered from the isomorphism (1.2).

1.5 Duality

A nilpotent element $a \in \mathfrak{sl}(n, \mathbb{C})$ corresponds to a parabolic subgroup P of $G = SL(n, \mathbb{C})$ (or rather, to a class of associated parabolic subgroups). A Cartan subgroup $H \subset P$ acts on G/P, so we may consider the reduced affine scheme $V(\widehat{H}^*_H(G/P))$, which is an arrangement of linear subspaces of $H^H_2(G/P)$. (Here $\widehat{H}^*$ denotes the part of the cohomology algebra that is generated by elements of degrees $0, 2$.) In Theorem 8.1 we show that this arrangement is isomorphic to the "dual" of the arrangement $V(X_a) \subset H^T_2(X_a)$ that was considered in the preceding paragraph. We do not have a good explanation for this duality, and it would be interesting to know if there exist other examples of dual equivariant cohomology schemes.

Note added in proof: In the intervening years since this article was submitted for publication, a general setting for this duality has been developed by Braden, Licata, Phan, Proudfoot, and Webster [5].

1.6 Related Articles and Acknowledgements

Although many of the results and applications in this note can be extended to more general situations, the main object of this paper is to illustrate the use of $V = \mathrm{Spec}(H^*_T(X))$ as a unifying concept. It was implicitly considered in [2, 3], and explicitly in [6], [16, p. 27], [7, p. 83], and [8, Theorem 2]. We also wish to draw attention to the related articles [5, 11, 19, 22–26, 32] and to the unrelated article [27].

Our interest in the scheme V developed from joint work with Robert Kottwitz [16, 17] to whom we are grateful for many stimulating and rewarding discussions. We thank Tom Braden, Volker Puppe, Nick Proudfoot, Juliana Tymoczko, and an anonymous referee for their suggestions. The first author would also like to thank the Institute for Advanced Study for its support and hospitality.

2 Definitions

In this note, unless otherwise specified, all homology and cohomology groups will be taken with complex coefficients. The coordinate ring of an affine scheme W over $\mathbb{C}$ is denoted $\mathbb{C}[W]$. Throughout this paper we fix an algebraic torus $T \cong (\mathbb{C}^\times)^n$ acting algebraically on a connected complex projective algebraic variety X that is equivariantly embedded in some complex projective space. Let $ET \to BT$ be the universal principal T bundle over the classifying space and let $H^*_T(X; \mathbb{Z}) = H^*(ET \times_T X; \mathbb{Z})$ and $H^T_*(X; \mathbb{Z}) = H_*(ET \times_T X; \mathbb{Z})$ denote the equivariant cohomology and equivariant homology (respectively) of X with coefficients in the integers $\mathbb{Z}$.

Let $\chi^*(T)$ and $\chi_*(T)$ be the character and cocharacter groups, respectively, of T. The natural isomorphism

$$\chi^*(T) \cong H^2(BT; \mathbb{Z}) = H^2_T(\mathrm{pt}; \mathbb{Z})$$

associates with each character $\lambda\colon T \to \mathbb{C}^\times$ the first Chern class of the corresponding line bundle L_λ on BT. We obtain an isomorphism

$$\mathfrak{t} \cong \chi_*(T) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^T_2(\mathrm{pt}; \mathbb{C})$$

between the Lie algebra $\mathfrak{t}$ and the (second) equivariant homology of a point, as well as a degree-doubling isomorphism

$$\mathbb{C}[\mathfrak{t}] \cong H^*(BT;\mathbb{C}) = H^*_T(\mathrm{pt};\mathbb{C})$$

between the graded ring of complex valued polynomials on $\mathfrak{t}$ and the equivariant cohomology ring of a point. (This isomorphism is in fact defined over $\mathbb{Q}$.)

The inclusion $\{\mathrm{pt}\} \hookrightarrow BT$ induces an augmentation

$$\epsilon\colon \mathbb{C}[\mathfrak{t}] = H^*(BT;\mathbb{C}) \to H^*(\mathrm{pt};\mathbb{C}) = \mathbb{C}$$

whose kernel $\mathfrak{I} \subset \mathbb{C}[\mathfrak{t}]$ is the augmentation ideal consisting of all polynomials whose constant term is 0. A choice of basepoint in ET determines a mapping

$$\iota\colon X \to X \times ET \to X \times_T ET$$

with resulting homomorphism of $\mathbb{C}[\mathfrak{t}]$-modules, $\iota^*\colon H^*_T(X;\mathbb{C}) \to H^*(X;\mathbb{C})$. The mapping $p\colon X \to \{\mathrm{pt}\}$ (of X to a point) induces graded $\mathbb{C}[\mathfrak{t}]$-module homomorphisms

$$p_*\colon H^T_*(X;\mathbb{C}) \to H^T_*(\mathrm{pt};\mathbb{C}) \quad \text{and} \quad p^*\colon H^*_T(\mathrm{pt};\mathbb{C}) \to H^*_T(X;\mathbb{C}).$$

Denote by $\pi_*\colon H^T_2(X;\mathbb{C}) \to \mathfrak{t}$ the restriction of p_* to the degree 2 part.

Recall [16] that the action of T on X is *equivariantly formal* if the Leray–Serre spectral sequence

$$E_2^{pq} = H^p(BT; H^q(X;\mathbb{C})) \implies H_T^{p+q}(X) = H^{p+q}(X \times_T ET) \tag{2.1}$$

for the fibration $h\colon X \times_T ET \to BT$ collapses, where $ET \to BT$ is the universal principal T-bundle. The following results are standard but we include their proofs for completeness.

Proposition 2.1 *Suppose the action of T on X is equivariantly formal. Then the following statements hold.*

(i) *There is a (non canonical) isomorphism of graded $\mathbb{C}[\mathfrak{t}]$ modules,*

$$H^*_T(X;\mathbb{C}) \cong \mathbb{C}[\mathfrak{t}] \otimes_{\mathbb{C}} H^*(X;\mathbb{C}). \tag{2.2}$$

If the action of T on X is trivial, then this is an isomorphism of rings.

(ii) *Regarding $\mathbb{C}$ as a $\mathbb{C}[t]$-module by extending scalars $\epsilon\colon \mathbb{C}[\mathfrak{t}] \to \mathbb{C}$, $\epsilon(f) = f(0)$, induces an isomorphism of $\mathbb{C}$-algebras,*

$$H^*(X;\mathbb{C}) \cong H^*_T(X;\mathbb{C}) \otimes_{\mathbb{C}[\mathfrak{t}]} \mathbb{C} = H^*_T(X;\mathbb{C})/\mathfrak{I}H^*_T(X;\mathbb{C}). \tag{2.3}$$

(iii) *The following sequence of complex vector spaces is exact,*

$$0 \longrightarrow H_2(X;\mathbb{C}) \xrightarrow{\iota_*} H^T_2(X;\mathbb{C}) \xrightarrow{\pi_*} H^T_2(\mathrm{pt};\mathbb{C}) \cong \mathfrak{t} \longrightarrow 0. \tag{2.4}$$

If T acts with finitely many fixed points, then this sequence has a canonical splitting $\mu_0 : \mathfrak{t} \to H^T_2(X)$.

(iv) *The following sequence of $H_T^*(\mathrm{pt})$-modules is exact,*

$$0 \longrightarrow H_T^*(X) \longrightarrow H_T^*(F) \xrightarrow{\delta} H_T^*(X_1, F),$$

where $F = X^T$ is the fixed point set, and $X_1 \subset X$ denotes the union of the 0- *and* 1-*dimensional orbits of T.*

Proof The E_∞ page of the spectral sequence (2.1) is the graded group associated with the filtration

$$H^m(Y;\mathbb{C}) = F^0H^m(Y;\mathbb{C}) \supset F^1H^m(Y;\mathbb{C}) \supset \cdots \supset F^{m+1}H^m(Y;\mathbb{C}) = 0,$$

where $Y = X \times_T ET$ and where

$$F^kH^m(Y;\mathbb{C}) = \ker[H^m(Y;\mathbb{C}) \to H^m(h^{-1}(BT_{k-1};\mathbb{C}))].$$

Here, BT_k denotes the k-skeleton of BT (with respect to some triangulation). Thus, for each p, q there is a short exact sequence

$$0 \longrightarrow F^{p+1}H^{p+q}(Y;\mathbb{C}) \longrightarrow F^pH^{p+q}(Y;\mathbb{C}) \xrightarrow{\alpha} E_\infty^{p,q} \longrightarrow 0. \tag{2.5}$$

Moreover, the cup product takes

$$F^kH^m(Y;\mathbb{C}) \times F^{k'}H^{m'}(Y;\mathbb{C}) \to F^{k+k'}H^{m+m'}(Y;\mathbb{C})$$

and induces a product on E_∞ that is compatible with that on E_2 in the sense that the following diagram commutes:

$$\begin{tikzcd}
H^a(BT) \times F^pH^{p+q}(Y) \arrow[r] \arrow[d, "I\times\alpha"] & F^{p+a}H^{p+q+a}(Y) \arrow[d, "\alpha"] \\
H^a(BT) \times H^p(BT) \otimes H^q(X) \arrow[r] & H^{a+b}(BT) \otimes H^q(X)
\end{tikzcd}$$

For this standard result see, for example, [29, §9.4].

The classifying space BT is simply connected and its cohomology in odd degrees vanishes, so $E_2^{p,q} \cong H^p(BT;\mathbb{C}) \otimes H^q(X;\mathbb{C})$ and $E_2^{p,q} = 0$ if p is odd.

Now assume the action is equivariantly formal so that $E_\infty^{pq} = E_2^{pq}$. A choice of splitting of each of the sequences (2.5) determines an isomorphism (2.2), which implies (i) and (ii). (If the action of T on X is trivial, then $Y \cong X \times BT$ is a product, and the Künneth theorem implies that (2.2) is a ring isomorphism.) The filtration

$$H_T^2(X;\mathbb{C}) = F^0H^2(Y;\mathbb{C}) \supset F^1H^2(Y;\mathbb{C}) = F^2H^2(Y;\mathbb{C}) \supset F^3H^2(Y;\mathbb{C}) = 0$$

corresponds to a short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & H^2(BT) \arrow[r, "\pi^*"] \arrow[d, equal] & H^2_T(X) \arrow[r] \arrow[d, equal] & H^2(X) \arrow[r] \arrow[d, equal] & 0, \\
& F^2H^2_T(X) = E_2^{2,0} & F^0H^2_T(X) & E_2^{0,2} &
\end{tikzcd}$$

which implies (iii). If T acts with finitely many fixed points then a canonical splitting of this sequence is defined in equation (3.2).

Part (iv) is the "topological part" of the localization theorem (see [3,16,18]). ■

Let $\widehat{H}^*(X)$ (resp. $\widehat{H}^*_T(X) = \widehat{H}^*(Y)$) denote the subring of $H^*(X)$ (resp. of $H^*_T(X)$) that is generated by the degree 0 and degree 2 part, $H^0(X) \oplus H^2(X)$ (resp. $H^0_T(X) \oplus H^2_T(X)$). Similarly let $\widehat{H}^m_T(X) = \widehat{H}^*_T(X) \cap H^m_T(X)$, and $F^p\widehat{H}^a(Y) = F^pH^a(Y) \cap \widehat{H}^*(Y)$ etc., where $Y = X \times_T BT$.

Proposition 2.2 *(see also* [23, p. 14]*) Suppose* $H^*(X) = \widehat{H}^*(X)$*. Then the action of* T *is equivariantly formal and* $H^*_T(X) = \widehat{H}^*_T(X)$*.*

Proof Assume $H^*(X) = \widehat{H}^*(X)$. Then the cohomology of X vanishes in odd degrees. Since the same is true for $H^*(BT)$, the spectral sequence (2.1) collapses and $E_\infty^{p,q} \cong H^p(BT) \otimes H^q(X)$. The filtration (2.5) is then given by

$$F^pH^m(Y) = \bigoplus_{\substack{a+b=m \\ a \geq p}} H^a(BT) \otimes H^b(X) \tag{2.6}$$

for any choice of splitting of (2.5).

The mapping $\phi\colon \mathbb{C}[H_2(Y)] \to \widehat{H}^*(Y)$ associates with any monomial on $H_2(Y)$ the corresponding product of cohomology classes in $H^*(Y)$. We claim this is surjective. Let $\mathbb{C}[H_2(Y)]^m$ denote the subspace of homogeneous polynomials of degree m. A choice of splitting of (2.4) gives an isomorphism

$$\mathbb{C}[H_2(Y)]^m \cong \bigoplus_{a+b=m} \mathbb{C}[\mathrm{t}]^a \otimes \mathbb{C}[H_2(X)]^b.$$

Let

$$F^p\mathbb{C}[H_2(Y)]^m = \bigoplus_{\substack{a+b=m \\ a \geq p}} \mathbb{C}[\mathrm{t}]^a \otimes \mathbb{C}[H_2(X)]^b.$$

We will show by induction on p that the mapping ϕ takes $F^p\mathbb{C}[H_2(Y)]^m$ surjectively to $F^pH^m(Y)$. Using (2.6), we have exact sequences

$$\begin{tikzcd}
0 \arrow[r] & F^p\mathbb{C}[H_2(Y)]^m \arrow[r] \arrow[d, dotted] & \mathbb{C}[H_2(Y)]^m \arrow[r] \arrow[d] & \displaystyle\bigoplus_{\substack{a+b=m\\ a\geq p+1}} \mathbb{C}[\mathfrak{t}]^a \otimes \mathbb{C}[H_2(X)]^b \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F^pH^m(Y) \arrow[r] & H^m(Y) \arrow[r] & \displaystyle\bigoplus_{\substack{a+b=m\\ a\geq p+1}} H^a(BT)\otimes H^b(X) \arrow[r] & 0.
\end{tikzcd}$$

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which shows that ϕ takes $F^p\mathbb{C}[H_2(Y)]^m$ into $F^pH^m(Y)$. Let us assume by descending induction on p that this mapping is surjective (with the case $p = m+1$ being trivial), and consider the exact sequences,

$$\begin{tikzcd}
0 \arrow[r] & F^{p+1}\mathbb{C}[H_2(Y)]^m \arrow[r] \arrow[d] & F^p\mathbb{C}[H_2(Y)]^m \arrow[r] \arrow[d] & \mathbb{C}[\mathfrak{t}]^p \otimes \mathbb{C}[H_2(X)]^{m-p} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F^{p+1}H^m(Y) \arrow[r] & F^pH^m(Y) \arrow[r] & H^p(BT)\otimes H^{m-p}(X) \arrow[r] & 0
\end{tikzcd}$$

The right-hand vertical mapping is surjective by assumption. It follows that the middle vertical mapping is also surjective. ■

One might ask whether $\widehat{H}^*_T(X) \cong \mathbb{C}[\mathfrak{t}] \otimes \widehat{H}^*(X)$ for equivariantly formal actions, but this statement is false for the Grassmannian $G_2(\mathbb{C}^4)$.

3 Fixed Points and the Schemes V and W

As in Section 2, let T be a complex algebraic torus acting on a complex algebraic variety X that is T-equivariantly embedded in some projective space $\mathbb{P}^N$. Assume the fixed point set F consists of finitely many connected components $F_1, F_2, \ldots, F_k \in X$. For each i, choose a point $x_i \in F_i$. Each inclusion $\{x_i\} \to X$ induces a splitting of (2.4),

$$\mu_i \colon H_2^T(\{x_i\}) \cong \mathfrak{t} \to H_2^T(X;\mathbb{C}) \tag{3.1}$$

whose image is a linear subspace $\mathfrak{t}_i \subset H_2^T(X;\mathbb{C})$. The mapping μ_i is independent of the choice of point $x_i \in F_i$, however if $i \neq j$, then $\mathfrak{t}_i \neq \mathfrak{t}_j$: the subspaces $\mathfrak{t}_i \in H_2^T(X)$ are all distinct. (In fact, even the images of the $\mathfrak{t}_i$ in $H_2^T(\mathbb{P}^N;\mathbb{C})$ are disjoint, as may be seen by a direct calculation.) The average of the mappings μ_i determines a canonical splitting,

$$\mu_0 = \frac{1}{k}\sum_{i=1}^{k} \mu_i \colon \mathfrak{t} \to H_2^T(X). \tag{3.2}$$

Denote by $V = \bigcup_{i=1}^{k} \mathfrak{t}_i$ the resulting arrangement of linear subspaces of the vector space $H_2^T(X;\mathbb{C})$. (It is defined over $\mathbb{Q}$.) We may consider V to be a reduced affine scheme. Let

$$\phi_i = \mu_i^* \colon \mathbb{C}[H_2^T(X)] \to \mathbb{C}[\mathfrak{t}_i] \cong H_T^*(\{x_i\})$$

be the resulting (surjective) map on functions. Then $I(\mathfrak{t}_i) = \ker(\phi_i)$ is the ideal of functions that vanish on $\mathfrak{t}_i$ and $I(V) = \bigcap_{i=1}^{k} I(\mathfrak{t}_i)$ is the (homogeneous) ideal of functions that vanish on V.

The identifications $H_T^0(X;\mathbb{C}) \cong \mathbb{C}$ and $H_T^2(X;\mathbb{C}) \cong \mathrm{Hom}(H_2^T(X;\mathbb{C}),\mathbb{C})$ determine a degree-doubling homomorphism of graded $\mathbb{C}[\mathfrak{t}]$-algebras,

$$\mathbb{C}[H_2^T(X;\mathbb{C})] \to H_T^*(X;\mathbb{C})$$

whose image is $\widehat{H}_T^*(X)$. Denote this surjection by $\theta \colon \mathbb{C}[H_2^T(X;\mathbb{C})] \to \widehat{H}_T^*(X;\mathbb{C})$. Similarly, let ψ_i be the composition

$$\mathbb{C}[H_2^T(X;\mathbb{C})] \longrightarrow \widehat{H}_T^*(X;\mathbb{C}) \longrightarrow \widehat{H}_T^*(F_i;\mathbb{C})$$

and let R_i be the image of ψ_i. Let $W_i = \mathrm{Spec}(R_i)$. In summary we have the diagram

$$\begin{tikzcd}
I(W_i) \arrow[r] \arrow[d] & \mathbb{C}[H_2^T(X)] \arrow[r, "\psi_i"] \arrow[rd, "\phi_i"'] & \widehat{H}_T^*(F_i) \arrow[d, "\tau_i"] \\
I(\mathfrak{t}_i) \arrow[ru] & & H_T^*(\{x_i\}) .
\end{tikzcd}$$

Let $W = \bigcup_{i=1}^{k} W_i \subset \mathbb{C}[H_2^T(X)]$ be the resulting affine scheme. It is defined by the ideal $I(W)$ in the following exact sequence:

$$0 \longrightarrow I(W) \longrightarrow C[H_2^T(X;\mathbb{C})] \xrightarrow{\Sigma\psi_i} \bigoplus_{i=1}^{k} R_i .$$

The projection $\pi \colon X \to \{pt\}$ induces a mapping $\pi_* \colon W \to \mathfrak{t}$ as the composition

$$W \subset H_2^T(X) \to H_2^T(\{pt\}) \cong \mathfrak{t}.$$

For $t \in \mathfrak{t}$, denote the scheme-theoretic intersection $\pi_*^{-1}(t) \cap W$ by W_t. Let

$$A_t = \mathbb{C}[W_t] = \mathbb{C}[H_2^T(X)]/I_t \tag{3.3}$$

be the coordinate ring of W_t, where $I_t = I(W) + I(\pi_*^{-1}(t))$. If $t = 0$, then I_t is a homogeneous ideal.

Theorem 3.1 *Suppose the algebraic torus T acts on the projective variety X with a fixed point set consisting of finitely many connected components $F_1, F_2, \ldots, F_k \subset X$. Let*

$$V = \bigcup_{i=1}^{k} \mathfrak{t}_i \subset W = \bigcup_{i=1}^{k} W_i \subset H_2^T(X)$$

be the resulting affine schemes. Then the following statements hold.

(i) *The variety V is the support of W, that is, $V = (W)_{\mathrm{red}}$.*

(ii) *If the fixed points are isolated then W is reduced (so $V = W$).*

(iii) *If the action of T on X is equivariantly formal, then*

  (a) *the homomorphism θ induces an identification*

$$W \cong \operatorname{Spec} \widehat{H}_T^*(X;\mathbb{C})$$

  or equivalently, the following sequence is exact

$$\begin{tikzcd} 0 \arrow[r] & I(W) \arrow[r] & \mathbb{C}[H_2^T(X;\mathbb{C})] \arrow[r, "\theta"] & \widehat{H}_T^*(X;\mathbb{C}) \arrow[r] & 0, \end{tikzcd} \tag{3.4}$$

  (b) *the scheme W is not contained in any proper subspace of $H_2^T(X)$, and*

  (c) *there is a canonical isomorphism of rings,*

$$A_0 = \mathbb{C}[W_0] \cong \widehat{H}_T^*(X) \otimes_{\mathbb{C}[\mathfrak{t}]} \mathbb{C}. \tag{3.5}$$

(iv) *If $H^*(X) = \widehat{H}^*(X)$, then the action of T is equivariantly formal, the mapping $\pi_* \colon W \to \mathfrak{t}$ is flat, and the isomorphism (3.5) becomes $A_0 \cong H^*(X)$.*

(v) *In particular, if $H^*(X) = \widehat{H}^*(X)$ and the fixed points of T are isolated, then $V = \operatorname{Spec} H_T^*(X)$ is reduced; it is an arrangement of linear spaces, and $H_T^*(X)$ is canonically identified with the algebra of functions on this arrangement.*

Proof For part (i) it suffices to show that $\mathfrak{t}_i = (W_i)_{\mathrm{red}}$, that is, for any $f \in I(V_i) = \ker \phi_i$ there exists $N > 0$ such that $f^N \in I(W_i) = \ker \psi_i$. Given $f \in I(V_i)$, let $g = \psi_i(f)$. Since T acts trivially on F_i, we have an isomorphism of rings

$$\widehat{H}_T^*(F_i) \cong \widehat{H}^*(F_i) \otimes H_T^*\bigl(\{pt\}\bigr).$$

The element g lies in the augmentation ideal, $\ker(\tau_i)$, that is, $g = \sum_j u_j \otimes v_j$ where $u_j \in \widehat{H}^*(F_i)$ has degree ≥ 2 and $v_j \in H_T^*(\{pt\})$. Therefore $g^{\dim(F_i)+1} = 0$, which is to say that $f^{\dim(F_i)+1} \in I(W_i)$, proving (i). Part(ii) follows immediately.

Now suppose the action of T is equivariantly formal. By Proposition 2.1(iv), the restriction mapping $H_T^*(X;\mathbb{C}) \to H_T^*(F;\mathbb{C})$ is injective. It follows that $\widehat{H}_T^*(X;\mathbb{C})$ becomes canonically identified with the image of ψ in the following diagram,

$$\begin{tikzcd} 0 \arrow[r] & I(W) = \bigcap_{i=1}^{k} I(W_i) \arrow[r] & \mathbb{C}[H_2^T(X)] \arrow[r, "\psi"] & \widehat{H}_T^*(F) = \bigoplus_{i=1}^{k} \widehat{H}_T^*(F_i) \end{tikzcd}$$

which proves (3.4). If W were contained in some proper subspace $L \subset H_2^T(X)$, then there would exist a non zero linear function $\alpha\colon H_2^T(X) \to \mathbb{C}$ which vanishes on W so the (homogeneous part of degree one in the) coordinate ring $\mathbb{C}[W]$ would fail to surject to $H_T^2(X)$. This proves (iii)(b).

Observe that $I(\pi_*^{-1}(0)) = \mathfrak{I}\mathbb{C}[H_2^T(X;\mathbb{C})]$ (where $\mathfrak{I}$ denotes the augmentation ideal in $\mathbb{C}[\mathfrak{t}]$). So by equation (3.3) and (3.4),

$$A_0 \cong \widehat{H}_T^*(X)/\mathfrak{I}\widehat{H}_T^*(X) \cong \widehat{H}_T^*(X) \otimes_{\mathbb{C}[\mathfrak{t}]} \mathbb{C},$$

which proves (iii)(c). Now assume that $H^*(X) = \widehat{H}^*(X)$. By Proposition 2.2 the action is equivariantly formal and $H_T^*(X) = \widehat{H}_T^*(X)$, so $W = \operatorname{Spec} H_T^*(X)$ which, by equation (2.2), is flat over $\mathfrak{t}$. Using (2.3), this gives a canonical isomorphism $A_0 \cong H^*(X)$. ■

Remark If T acts with finitely many fixed points $x_1, \dots, x_k$ and finitely many one-dimensional orbits $E_1, \dots, E_r$, then Theorem 7.2 of [16] identifies $\operatorname{Spec}(H_T^*(X;\mathbb{C}))$ as the topological space obtained from the disjoint union $\coprod_{i=1}^k \mathfrak{t}$ by identifying, for each one-dimensional orbit E_j, the subspace $\mathfrak{t}_j$ in the two copies of $\mathfrak{t}$ that correspond to the two "end points" of this orbit, where $\mathfrak{t}_j$ is the Lie algebra of the stabilizer of any point in the orbit E_j. (See also [6, 7].) Theorem 3.1 goes one step further and identifies this union as a particular affine scheme in $H_2^T(X)$.

4 Specialization of the Coordinate Ring

As in the preceding section, suppose the algebraic torus T acts on the complex projective variety X with isolated fixed points. Let

$$W \longrightarrow H_2^T(X) \xrightarrow[\pi_*]{} \mathfrak{t} \tag{4.1}$$

be the resulting scheme, supported on the arrangement V of linear spaces, and its projection to $\mathfrak{t}$. Fix $t \in \mathfrak{t}$ and let $W_t = W \cap \pi_*^{-1}(t)$ be the scheme-theoretic intersection with the coordinate ring $A_t = \mathbb{C}[W_t]$.

Let $A_t^0 \subset A_t^{\le 1} \subset \cdots$ be the filtration of A_t by degree, that is, $A_t^{\le k}$ consists of the restrictions to V_t of the polynomials of degree $\le k$. Let $\mu\colon H_2^T(X;\mathbb{C}) \to H_2(X;\mathbb{C})$ be a choice of splitting of (2.4). (A canonical such splitting μ_0 is given in §3.)

Theorem 4.1 (see also [11] and [25, p. 131]) *Suppose the map* (4.1) *is flat. Then the splitting μ induces an isomorphism,*

$$\operatorname{Gr}(A_t) \cong A_0,$$

and different splittings give rise to the same isomorphism. If $\widehat{H}^*(X) = H^*(X)$, then Corollary 3.1 *gives an isomorphism of rings,*

$$\operatorname{Gr}(A_t) \cong H^*(X).$$

Proof (This result is probably standard but we were not able to find a simple reference.) We may assume $t \neq 0$. Because the mapping π_* is flat, by restricting to the line spanned by $t \in \mathfrak{t}$ we may reduce to the case that $\mathfrak{t}$ is one-dimensional. Choose an isomorphism of complex vector spaces $\mathfrak{t} \cong \mathbb{C}$. The splitting μ determines a splitting

$$H_2^T(X;\mathbb{C}) \cong H_2(X;\mathbb{C}) \oplus \mathbb{C}.$$

By further choosing a basis of $H_2(X;\mathbb{C})$, we may consider each polynomial function $f = f(x,b) \in \mathbb{C}[H_2^T(X;\mathbb{C})]$ to be a function of variables $x = (x_1, x_2, \ldots, x_r) \in H_2(X;\mathbb{C})$ and $b \in \mathbb{C}$, and we will sometimes write $\mathbb{C}[H_2^T(X;\mathbb{C})] = \mathbb{C}[x,b]$. Specializing $b = 0$ or $b = t$ gives two ring homomorphism $\mathbb{C}[H_2^T(X;\mathbb{C})] \to \mathbb{C}[H_2(X;\mathbb{C})]$ and we denote the image of $I(W)$ by I_0, I_t, respectively, in $\mathbb{C}[H_2(X;\mathbb{C})] = \mathbb{C}[x]$. Then $A_t = \mathbb{C}[x]/I_t$ and $A_0 = \mathbb{C}[x]/I_0$. The ideal I_0 is homogeneous and graded by degree, $I_0 = \bigoplus_j I_0^j$, while the ideal I_t is filtered by degree, $I_t^0 \subset I_t^{\leq 1} \subset \cdots$. Therefore for each k,

$$I_0^k = \{p(x) \in \mathbb{C}[x] : \deg(p) = k \text{ and } \exists q(x,b) \in \mathbb{C}[x,b] \text{ with } p + bq \in I(W)^k\}$$

$$I_t^{\leq k} = \Big\{\, p(x) \in \mathbb{C}[x] : \deg(p) \leq k \text{ and } \exists q(x,b) \in \mathbb{C}[x,b] \\ \text{with } p + (b-t)q \in I(W)^{\leq k} \Big\}.$$

Observe that if $p \in I_t^{\leq k}$, then its homogeneous part p_k of degree k satisfies $p_k \in I_0^k$. (For there exists $q \in \mathbb{C}[x,b]$ of degree $\leq k-1$ such that $p + (b-t)q \in I(W)$. But $I(W)$ is homogeneous, so the degree k homogeneous part $p_k + bq_{k-1}$ of this polynomial is also in $I(W)$, hence $p_k \in I_0^k$.) From this it follows that

$$I_0^k = \{p \in \mathbb{C}[x]^k : \exists q \in \mathbb{C}[x]^{\leq k-1} \text{ with } p + q \in I_t^{\leq k}\}. \tag{4.2}$$

We now construct the required isomorphism

$$\begin{tikzcd}
\mathrm{Gr}^k A_t \arrow[r, "\phi"] \arrow[d, equal] & \mathrm{Gr}^k A_0 = A_0^k \arrow[d, equal] \\
\mathbb{C}[x]^{\leq k}/\big(I_t^{\leq k} + \mathbb{C}[x]^{\leq k-1}\big) \arrow[r] & \mathbb{C}[x]^{\leq k}/\big(I_0^{\leq k} + \mathbb{C}[x]^{\leq k-1}\big)
\end{tikzcd}$$

as follows. The identity mapping $\mathbb{C}[x]^{\leq k} \to \mathbb{C}[x]^{\leq k}$ takes $I_t^{\leq k}$ to $I_0^{\leq k} + \mathbb{C}[x]^{\leq k-1}$, for if $p \in I_t^{\leq k}$, then $p = p_k + p_{\leq k-1}$ and (by the above observation), $p_k \in I_0^{\leq k}$. This gives a well-defined surjective homomorphism

$$\mathbb{C}[x]^{\leq k}/I_t^{\leq k} \to \mathbb{C}[x]^{\leq k}/\big(I_0^{\leq k} + \mathbb{C}[x]^{\leq k-1}\big)$$

and hence also a surjective homomorphism ϕ above. To show that ϕ is injective, suppose $p \in \mathbb{C}[x]^{\leq k}$ is in the kernel of ϕ. Then either $\deg(p) < k$ (in which case $p = 0$ in $\mathrm{Gr}^k(A_t)$) or else $p_k \in I_0^k$. From equation (4.2) it follows that there exists $q \in \mathbb{C}[x]^{\leq k-1}$ such that $p_k + q \in I_t^{\leq k}$. Thus, $p + q = (p_k + q) + p_{\leq k-1} \in I_t^{\leq k} + \mathbb{C}[x]^{\leq k-1}$ as needed. ■

Theorem 4.1 is most useful when T acts with finitely many fixed points and $t \in \mathfrak{t}$ is generic, for in this case $W_t = V_t$ is reduced and consists of k isolated points in the vector space $H_2^T(X)$. In the next few sections we will see what this says in the case of Springer fibers, Schubert varieties, and toric varieties, recovering results of [1, 10].

5 Schubert Varieties

Let G be a complex semi-simple algebraic group. Let B be a Borel subgroup, and set $X = G/B$. The Schubert variety X_w is the closure of the Bruhat cell $BwB/B \subset X$ corresponding to an element $w \in W$ of the Weyl group. It is the disjoint union of Bruhat cells $X_w = \coprod_\tau B\tau B/B$, over those $\tau \in W_{\leq w} = \{\tau \in W : \tau \leq w\}$ (with respect to the strong Bruhat order on W). Hence X_w is equivariantly formal with respect to any torus action, and $H^*(X) \to H^*(X_w)$ is surjective, so $\widehat{H}^*(X_w) = H^*(X_w)$.

The Cartan subgroup $H \subset B$ acts on X_w with a single fixed point in each of the Bruhat cells contained in X_w. Let

$$V \subset H_2^H(X_w) \xrightarrow{\pi_*} H_2^H(\mathrm{pt}) = \mathfrak{h}$$

be the resulting union of linear subspaces. Then, according to Theorem 4.1,

$$V \cong \operatorname{Spec} H_H^*(X_w)$$

is a reduced affine scheme, and for any $t \in \mathfrak{h} = \mathrm{lie}(H)$ there is a canonical isomorphism

$$\mathrm{Gr}(A_t) \cong H^*(X_w)$$

where $A_t = A(V_t)$ is the coordinate ring of the scheme theoretic intersection $V \cap \pi_*^{-1}(t)$. This gives a theorem of [1].

Theorem 5.1 *Let $t \in \mathfrak{h}$ be a generic point. Then*

$$V_t = \bigcup_{\tau \in W_{\leq w}} w \cdot t \subset H_2(X_w) \tag{5.1}$$

is a reduced affine scheme consisting of finitely many points. The filtration by degree on its coordinate ring $A_t = \mathbb{C}[H_2(X_w)]/I(V_t)$ induces an isomorphism of rings

$$\mathrm{Gr}\, A_t \cong H^*(X_w).$$

Proof The action of H on X_w is equivariantly formal, it has isolated fixed points, and $\widehat{H}^*(X_w) = H^*(X_w)$. So by Theorem 3.1 the corresponding scheme

$$V = \operatorname{Spec} H_H^*(X_w) \subset H_H^2(X_w)$$

is reduced and the projection $\pi \colon V \to \mathfrak{h}$ is flat. Therefore Theorem 7.1 will follow immediately from Theorem 4.1, provided we can establish the identification of V_t with the collection of points (5.1).

In the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H_2(X_w) \arrow[r] \arrow[d, hook] & H_2^H(X_w) \arrow[r] \arrow[d, hook] & H_2^H(\mathrm{pt}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H_2(X) \arrow[r] \arrow[d, equal] & H_2^H(X) \arrow[r] \arrow[d, equal] & H_2^H(\mathrm{pt}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathfrak{h} \arrow[r] & \mathfrak{h} \oplus \mathfrak{h} \arrow[r] & \mathfrak{h} \arrow[r] & 0
\end{tikzcd}$$

the splitting $\mu_\tau \colon H_2^H(\mathrm{pt}) \to H_2^H(X)$ corresponding to a fixed point $\tau x_0 \in X$ ($\tau \in W$) is given by

$$(\tau t, t) \leftarrow t$$

(as the result of a calculation that we omit). Therefore, as a subset of $\mathfrak{h} \oplus \mathfrak{h}$, $V = \{(\tau a, a) : a \in \mathfrak{h} \text{ and } \tau \in W_{\leq w}\}$. In fact, V is contained in the smaller vector space $H_2^H(X_w)$ and $V_t \subset H_2(X_w)$, but its coordinate ring is independent of the embedding.

■

6 Toric Varieties

As in Section 2, let $T \cong (\mathbb{C}^\times)^n$ be a complex algebraic torus with Lie algebra $\mathfrak{t}$. Let $K \cong (S^1)^n$ be the maximal compact sub-torus, with Lie algebra $\mathfrak{k}$. The natural identification of character groups

$$\chi^*(K) \cong \chi^*(T) \cong H_T^2(\mathrm{pt}; \mathbb{Z})$$

determines canonical isomorphisms

$$\mathfrak{k} = \chi_*(K) \otimes_{\mathbb{Z}} \mathbb{R} \cong H_2^T(\mathrm{pt}; \mathbb{R}).$$

As in Section 2, if T acts on a topological space X, then the mapping $\pi \colon X \to \mathrm{pt}$ induces a homomorphism

$$H_2^T(X; \mathbb{R}) \to H_2^T(\mathrm{pt}; \mathbb{R}) \cong \mathfrak{k},$$

which we denote by π_*.

Suppose X is a normal projective torus embedding or "toric variety" corresponding to a complex algebraic torus T and a (finite) rational polyhedral cone decomposition Σ of $\chi_*(T) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathfrak{k}$, as in [4, 15].

There is a simple relationship between the cone decomposition Σ and the affine variety $V = \mathrm{Spec}(H_T^*(X; \mathbb{C}))$. Using the canonical identification $\mathfrak{k} \cong \chi_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$, we obtain a rational polyhedral cone decomposition, which we also denote by Σ, of $\mathfrak{k}$. Define a section $\Phi \colon \mathfrak{k} \to H_2^T(X; \mathbb{R})$ which is linear on each cone as follows. The top dimensional (open) cones $C_1, \dots, C_k$ correspond to the fixed points

$F_1, \ldots, F_k$ of the torus action. For each of these ($1 \leq j \leq k$), there is a splitting (3.1) $\mu_j \colon \mathfrak{k} \to H_2^T(X;\mathbb{R})$ of equation (2.4), whose image we denote by $\mathfrak{k}_j$. Let

$$V_{\mathbb{R}} = \bigcup_{i=1}^{k} \mathfrak{k}_j \subset H_2^K(X;\mathbb{R}) = H_2^T(X;\mathbb{R})$$

denote the resulting real arrangement of linear subspaces. It is stratified into "flats" by the multi-intersections of the subspaces $\mathfrak{k}_i$.

Let $L_1, L_2, \ldots, L_m$ denote the one-dimensional cones in Σ, and, for each i, choose a non-zero vector $v_i \in L_i$. Define $\Phi(x) = \mu_j(x)$ for all $x \in \overline{C}_j$ (the closure of the cone C_j).

Theorem 6.1 *The mapping $\Phi \colon \mathfrak{k} \to H_2^T(X;\mathbb{R})$ is continuous and linear on each cone. For each maximal cone C_i the linear span of $\Phi(C_i)$ is the linear space $\mathfrak{k}_i$. If X is rationally nonsingular, then the vectors $\Phi(v_1), \ldots, \Phi(v_m)$ form a basis of $H_2^T(X;\mathbb{R})$, and the partially ordered set of flats in the arrangement $V_{\mathbb{R}}$ coincides with the partially ordered set of cones in Σ.*

Proof First we verify the continuity of Φ. We will show, for any codimension one cone τ in Σ, that

$$\phi_a|\tau = \phi_b|\tau,$$

where C_a and C_b are the two open cones such that $\overline{\tau} = \overline{C}_a \cap \overline{C}_b$. Such a cone τ corresponds to a one-dimensional orbit $E_\tau \subset X$ whose endpoints $F_a, F_b \in F$ are the fixed points corresponding to the open cones C_a and C_b. Let $\mathfrak{k}_\tau \subset \mathfrak{k}$ be the Lie algebra of the stabilizer in K of any point in E_τ. Define the mapping

$$\psi_\tau \colon \mathfrak{k}_\tau \to H_2(F;\mathbb{R}) \cong \bigoplus_{i=1}^{k} \mathfrak{k}$$

by $\psi_\tau(h) = (0, \ldots, h, \ldots, -h, \ldots, 0)$, where the $h, -h$ occur at the coordinates a and b corresponding to the fixed points $F_a, F_b \in F$ at the two ends of the orbit E_τ. (Reversing the order of F_a, F_b changes ψ_t by a sign, but it does not affect the argument which follows.)

After translating it to homology by dualizing, Proposition 2.1(iv) says (among other things) that the following sequence is exact:

$$\bigoplus_{\tau} \mathfrak{k}_\tau \xrightarrow{\Sigma_\tau \psi_\tau} H_2^T(F;\mathbb{R}) \cong \bigoplus_{j=1}^{k} \mathfrak{k} \xrightarrow{\sum_j \phi_j} H_2^T(X) \longrightarrow 0.$$

In the first sum, τ varies over the one-dimensional orbits in X or equivalently, over the codimension one cones in Σ and the second sum is indexed by the fixed points $F_j \in F$. If $\overline{\tau} = \overline{C}_a \cap \overline{C}_b$ is a codimension one cone and if $h \in \mathfrak{k}_\tau$, then exactness of this sequence implies that $\phi_a(h) + \phi_b(-h) = 0$ as claimed.

Now suppose X is rationally non-singular. Then X is equivariantly formal, $\widehat{H}^*(C) = H^*(X)$, and each cone C in Σ is simplicial. Moreover, the number of one-dimensional cones in Σ coincides with the dimension β_2 of $H_2^T(X)$. In fact the Poincaré polynomial of X is given [12] by

$$P(q) = \sum_{j=1}^{2n} \beta_j \sqrt{q}^j = \sum_{j=1}^{n} c_{n-j}(q-1)^j,$$

where c_{n-j} denotes the number of $n-j$-dimensional cones in Σ. By Poincaré duality, $\beta_2 = \beta_{2n-2} = c_1 - n$ as claimed.

On the other hand, the vectors $\Phi(v_1), \ldots, \Phi(v_m) \subset H_2^T(X;\mathbb{R})$ cannot lie in a proper subspace, otherwise $\Phi(C)$ would also lie in this subspace, for every cone C in Σ, which implies that $V_{\mathbb{R}}$ is also contained in this subspace. This contradicts Theorem 3.1. So we have shown that the vectors $\Phi(v_1), \ldots, \Phi(v_m)$ form a basis of $H_2^T(X;\mathbb{R})$.

Let Σ_1 denote the collection of all the one-dimensional cones in Σ and let $\mathcal{S}$ denote the collection of all subsets $S \subset \Sigma_1$ with the following property: if $S = \{\sigma_1, \sigma_2, \ldots, \sigma_r\}$, then there exists an n-dimensional cone $C \in \Sigma_n$ such that $\overline{C}$ contains the one-dimensional cones $\sigma_1, \ldots, \sigma_r$. Because the fan Σ is simplicial, every collection $S \in \mathcal{S}$ of one-dimensional cones spans (the closure of) a unique cone $C = \mathrm{span}(S) \in \Sigma$. The mapping span: $\mathcal{S} \to \Sigma$ is a bijection of partially ordered sets.

There is also a mapping Θ from $\mathcal{S}$ to the set of all subspaces of $H_2^T(X;\mathbb{R})$ which associates with any collection $S = \{\sigma_1, \ldots, \sigma_r\}$ of one-dimensional cones, the subspace spanned by $\Phi(v_1), \ldots, \Phi(v_r)$. Because these vectors are linearly independent, this mapping commutes with intersection, $\Theta(S_1 \cap S_2) = \Theta(S_1) \cap \Theta(S_2)$, and it is compatible with the partial ordering defined by containment. We claim that the image of Θ is precisely the partially ordered set of flats in the arrangement V. For if $S \in \mathcal{S}$ is such a collection of one-dimensional cones, then

$$\mathrm{span}(S) = \bigcap \{C_i \in \Sigma_n : \mathrm{span}(S) \subset \overline{C}_i\},$$

where Σ_n is the collection of top-dimensional cones in Σ. (In other words, each cone is the intersection of the maximal cones containing it.) Therefore, the subspace $\Theta(S)$ is the intersection of the corresponding maximal subspaces $\mathfrak{k}_i$. ■

We remark that together with Theorem 3.1(v), this gives an alternate proof of the theorem of Brion–Vergne [9] that the equivariant cohomology $H_T^*(X;\mathbb{R})$ of any rationally smooth toric variety is naturally isomorphic to the algebra of continuous, piecewise polynomial (real-valued) functions on the corresponding fan Σ. (For, each equivariant cohomology class corresponds to a polynomial function on the arrangement V, which gives a continuous piecewise polynomial function on Σ by composing with $\Phi\colon \mathfrak{t} \to V \subset H_2^T(X;\mathbb{R})$. Conversely, every continuous piecewise polynomial function on Σ is determined by its values on the one-dimensional rays $\Phi(L_1), \ldots, \Phi(L_m)$.)

7 Springer Fibers for $SL(n,\mathbb{C})$

7.1 Recollections on Springer fibers

Let $G = SL(n,\mathbb{C})$. Let $B \subset G$ be a Borel subgroup. Each element $a \in \mathfrak{g}$ of the Lie algebra of G determines a vector field V_a on the flag manifold $X = G/B$ whose zero set we denote by X_a. If $a \in \mathfrak{g}$ is nilpotent, then X_a is referred to as a “Springer fiber”. T. A. Springer [30, 31] constructed a surprising representation of the Weyl group W of G on the cohomology $H^*(X_a)$ of each Springer fiber. This representation has since been constructed by other means [20, 21] and it has been thoroughly analyzed. Each Springer fiber X_a admits an action of a certain algebraic torus T. In this section we will show that Springer's representation has a canonical lift to a representation of W on the equivariant cohomology $H^*_T(X_a)$. (See [16] for analogous results in the case of affine Springer fibers.)

Let us first recall some standard facts concerning the flag manifold and Springer fibers for $SL(n,\mathbb{C})$. If $H \subset B$ is a Cartan subalgebra with Lie algebra $\mathfrak{h}$, then there is a natural identification $\mathfrak{h} \cong H_2(X;\mathbb{C})$ whose dual $\mathfrak{h}^* \cong H^2(X;\mathbb{C})$ associates with each character $\chi\colon T \to \mathbb{C}^\times$ the first Chern class $c^1(\mathcal{L}_\chi) \in H^2(X;\mathbb{C})$ of the corresponding line bundle $\mathcal{L}_\chi$ on $X = G/B$.

For any nilpotent $a \in \mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$, there exists [13, 28] a paving of the flag manifold X by affine spaces so that the Springer fiber X_a is a closed union of affines. It follows that $H_i(X_a) \to H_i(X)$ is injective and $H^i(X) \to H^i(X_a)$ is surjective for all i. In particular, $\widehat{H}^*(X_a) = H^*(X_a)$ (since the same is true for X). Moreover, provided a is not regular, the mapping induced by inclusion

$$H_2(X_a) \to H_2(X) \cong \mathfrak{h}$$

is an isomorphism. This is because the mapping is injective and W-equivariant (with respect to Springer's action on $H_2(X_a)$), but W acts irreducibly on $\mathfrak{h}$.

Every nilpotent element $a \in \mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$ is a *Richardson nilpotent*. In other words, there exists a Levi subgroup L of a parabolic subgroup $P \subset G$ such that $a \in \mathfrak{l} = \mathrm{Lie}(L)$ is a regular nilpotent element, *i.e., a* lies in the largest nilpotent conjugacy class in $\mathfrak{l}$. Then the center T of L is a torus in G which acts on the Springer fiber X_a with finitely many fixed points. In fact (see [10]), the fixed point set X^T (of the action of T on the flag manifold X) consists of $[W : W_L]$ copies of the flag manifold X_L, each of which intersects the Springer fiber X_a in a single point. (Here W_L is the Weyl group for L and, if B is chosen so that $T \subset B \subset P$, then $X_L = L/L \cap B$.)

Now consider the arrangement of linear spaces $V \subset H_2^T(X) \to \mathfrak{t}$. Using the canonical splitting (3.2), we obtain a commutative diagram,

$$\begin{tikzcd}
0 \arrow[r] & H_2(X_a) \arrow[r] \arrow[d, equal] & H_2^T(X_a) \arrow[r] \arrow[d, equal] & H_2^T(\mathrm{pt}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathfrak{h} \arrow[r] & \mathfrak{h} \oplus \mathfrak{t} \arrow[r] & \mathfrak{t} \arrow[r] & 0.
\end{tikzcd} \tag{7.1}$$

In this case, Theorem 4.1 is a theorem of Carrell [10].

Theorem 7.1 *Let $a \in \mathfrak{g}$ be a non-regular nilpotent element with Springer fiber X_a. Let $T \subset L$ be the associated Richardson data. Let $t \in \mathfrak{t}$ be a generic element. Then the scheme $V_t \subset H_2(X)$ is reduced and it consists of the $[W : W_L]$ points $W \cdot t$. The filtration* *by degree on the coordinate ring $A_t = \mathbb{C}[W \cdot t]$ of this Weyl group orbit induces an isomorphism of rings,*

$$\mathrm{Gr}A_t \cong H^*(X_a;\mathbb{C}).$$

Proof The action of T on X_a is equivariantly formal, has isolated fixed points, and $\widehat{H}^*(X_a) = H^*(X_a)$. So by Theorem 3.1, the corresponding scheme

$$V = \mathrm{Spec}\, H_T^*(X_a) \subset H_T^2(X_a)$$

is reduced and the projection $\pi\colon V \to \mathfrak{t}$ is flat. Therefore, Theorem 7.1 will follow immediately from Theorem 4.1, provided that we can establish the identification $V_t = W \cdot t$. This is most directly accomplished by first considering the case of $X = G/B$. Let us take B to be the group of upper triangular matrices and H to be the Cartan subgroup of diagonal matrices of determinant 1. The Weyl group $W \cong S_n$ may be realized as the subgroup of permutation matrices in G and also as the group of permutations of the set $\{1, 2, \ldots, n\}$. It acts linearly on $\mathbb{C}^n$ by permuting the coordinate axes so we obtain an induced action on the flag manifold X. For any $\lambda \in H$ we have

$$w \cdot (\lambda \cdot x) = (w\lambda w^{-1}) \cdot (w \cdot x)$$

for any $x \in X$. But $w\lambda w^{-1}$ is the reflection action of W on H. So induced action on $H_*(X)$ is trivial, but the induced action on $H_*^H(X)$ is non-trivial and the projection $\pi_*\colon H_2^H(X) \to H_2^H(\mathrm{pt}) \cong \mathfrak{h}$ is W-equivariant with respect to the reflection representation on $\mathfrak{h}$.

A second action (the Springer action) of W on X may be defined by first choosing a Hermitian metric on $\mathbb{C}^n$ that is invariant under W and also under the maximal compact (connected) subtorus $K \subset H$. Then each flag

$$0 = P_0 \subset P_1 \subset \cdots \subset P_n = \mathbb{C}^n$$

gives rise to an orthogonal decomposition $\mathbb{C}^n = M_1 \oplus \cdots \oplus M_n$ such that $P_k = \bigoplus_{j=1}^k M_k$. Let W act by permuting the M's. This determines an action from the right on X by $P_k \cdot w = \bigoplus_{j=1}^k M_{wk}$ for each permutation w. This action is not algebraic, but it commutes with the action of H so it also determines an action (from the right) on $H_2(X) \cong \mathfrak{h}$ which turns out to be the right-reflection representation, that is, $a \cdot w = w^{-1}a$ for $a \in \mathfrak{h}$, $w \in W$.

The fixed points of the action of H on X are the coordinate flags. If $x_0 \in X$ denotes the standard coordinate flag $0 \subset \mathbb{C}^1 \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^n$ then the other fixed points are τx_0 for $\tau \in W$. Using the canonical splitting (3.2), we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H_2(X) \arrow[r] \arrow[d, equal] & H_2^H(X) \arrow[r, "\pi_*"] \arrow[d, equal] & H_2^H(\mathrm{pt}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathfrak{h} \arrow[r] & \mathfrak{h} \oplus \mathfrak{h} \arrow[r] & \mathfrak{h} \arrow[r] & 0
\end{tikzcd}$$

such that W acts on the first factor (and on $H_2(X)$) via the right action, and W acts on the second factor (and on $H_2^H(\mathrm{pt})$ via the left action. If $x_1 \in G/B$ denotes the base point (that is, the standard flag in $\mathbb{C}^n$), then (following a calculation which we omit) the splitting corresponding to this point turns out to be

$$(b, b) \leftarrow b.$$

The arrangement of linear spaces $V \subset H_2^H(X)$ is clearly preserved under both actions of W, so it follows that

$$V = \bigcup_{\tau \in W} \{(\tau b, b) : b \in \mathfrak{h}\}. \tag{7.2}$$

Consequently the fiber of π_* is

$$\pi_*^{-1}(t) = \{(\tau t, t) : \tau \in W\}. \tag{7.3}$$

For the Springer fiber $X_a \subset X$ the diagram (7.1) consists of subspaces of the diagram 7.1, so if $t \in \mathfrak{t}$, then $V_t = \pi_*^{-1}(t)$ is still described by (7.3). (The only change is that the subgroup W_L acts trivially on $\mathfrak{t}$.) ■

We see from the preceding proof that, in fact, we have a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} H_H^*(X) \arrow[d, hook] \arrow[dr, phantom, "\times"] & \supset & \operatorname{Spec} H_T^*(X) \arrow[d, hook] & \supset & \operatorname{Spec} H_T^*(X_a) \arrow[d] \\
H_2^H(X) \arrow[d] \arrow[dr, phantom, "\times"] & & H_2^T(X) \arrow[ll, hook'] \arrow[d] & = & H_2^T(X_a) \arrow[d] \\
H_2^H(\mathrm{pt}) & & H_2^T(\mathrm{pt}) \arrow[ll, hook'] & = & H_2^T(\mathrm{pt}),
\end{tikzcd}$$

where the squares marked × are fiber squares. In other words, $\operatorname{Spec} H_T^*(G/B)$ *is the scheme-theoretic pullback of the scheme* $\operatorname{Spec} H_H^*(G/B)$ *under the inclusion* $\mathfrak{t} \to \mathfrak{h}$. The scheme $\operatorname{Spec} H_T^*(G/B)$ is not necessarily reduced.

Theorem 7.2 *Let X denote the variety of complete flags in $\mathbb{C}^n$. Let $a \in \mathfrak{sl}(n, \mathbb{C})$ be a non-zero nilpotent, let $T \subset L$ be the associated Richardson torus in the centralizer of a, and let X_a be the associated Springer fiber. (It determines the element a up to conjugacy.) Then*

$$\operatorname{Spec}(H_T^*(X_a)) = \left(\operatorname{Spec} H_T^*(X)\right)_{\mathrm{red}} \tag{7.4}$$

as sub-schemes of $H_2^T(X)$. Moreover, the Springer action on $H^*(X) \to H^*(X_a)$ lifts canonically to an action on the equivariant cohomology $H_T^*(X) \to H_T^*(X_a)$, and it determines an identification of the Weyl group,

$$W \cong \operatorname{Aut}_{\mathbb{C}[\mathfrak{t}]} H_T^*(X_a).$$

Proof The fixed points of the action of T on X determine an arrangement $V(X) \subset \mathfrak{h} \oplus \mathfrak{h} = H_2^T(X) \cong H_2^T(X)$. This coincides with the arrangement $V(X)$ because each connected component of X^T intersects X_a in a single fixed point. Consequently the scheme $\operatorname{Spec} H_T^*(X_a)$ is reduced and this proves (7.4).

The action of W on $H_2^T(X)$ preserves the reduced variety $V = \operatorname{Spec} H_T^*(X_a)$ and the projection $\pi_*\colon H_2^T(X) \to \mathfrak{t}$ is W-invariant. Hence this action passes to an action on $H_T^*(X_a)$ which is compatible with the Springer action on $H_T^*(X)$ according to the above diagram. In particular, $W \subset \operatorname{Aut}_{\pi_*}(V)$. Since W acts transitively on the components $\mathfrak{t}_i$ of V, it also accounts for the full group of automorphisms. ■

8 An Unusual Duality

8.1 Dual Arrangements

If V is a complex vector space let $V^\vee = \operatorname{Hom}(V, \mathbb{C})$ be the dual vector space. If $L_1, L_2, \dots, L_k \subset V$ is a finite collection of linear subspaces, define the *dual arrangement* $(V^\vee, L_1^\perp, L_2^\perp, \dots, L_k^\perp)$ by $L_i^\perp = \ker(V^\vee \to L_i^\vee)$.

Let us say that an arrangement $L_1, L_2, \dots, L_k \subset V$ is *fibered* by a surjective linear map $\pi\colon V \to T$ if each L_i is a section of π. Let $K = \ker(\pi)$. The arrangement $L_1, \dots, L_k \subset V$ is fibered by π if and only if K is a complement to each L_i in V. This holds if and only if $K^\perp \subset V^\vee$ is a complement to each $L_i^\perp \subset V^\vee$. Therefore, the dual arrangement $L_1^\perp, L_2^\perp, \dots, L_k^\perp \subset V^\vee$ is also fibered: use the linear surjection $\pi^*\colon V^\vee \to T^* = V^\vee/K^\perp$.

An *isomorphism* of arrangements $(V, L_1, L_2, \dots, L_k) \cong (V', L_1', L_2', \dots, L_k')$ is a linear isomorphism $\phi\colon V \to V'$ which takes L_i isomorphically to L_i' (possibly after a permutation of the indices). If these arrangements are fibered by linear surjections $\pi\colon V \to T$ and $\pi'\colon V' \to T'$, then the isomorphism is *fibration preserving* if it also takes $K = \ker(\pi)$ isomorphically to $K' = \ker(\pi')$.

8.2 Partial Flag Manifold

As in Section 7.1, let $H \subset B \subset G = SL(n, \mathbb{C})$ be the standard Borel pair of diagonal and upper triangular matrices of determinant 1. Let $n = n_1 + n_2 + \cdots + n_k$ be an ordered partition. It corresponds to a partial flag $E_1 \subset E_2 \subset \cdots \subset \mathbb{C}^n$ of dimensions $n_1, n_1 + n_2, \dots, n$. Let $P \subset G$ be the parabolic subgroup of G that fixes this flag and let $L \subset G$ be its Levi subgroup consisting of block-diagonal matrices with block sizes $n_1, n_2, \dots, n_k$. Let $T = H \cap L$ be its center. Let $K = SO(n, \mathbb{C})$ be the (standard) maximal compact subgroup. The diffeomorphism $G/P \cong (K \cap L)/(K \cap T)$ induces an isomorphism $H_2(G/P) \cong \mathfrak{t}$, so we obtain an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & H_2(G/P) \arrow[r] \arrow[d, equal] & H_2^H(G/P) \arrow[r] \arrow[d, equal] & H_2^H(\mathrm{pt}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathfrak{t} \arrow[r] & \mathfrak{t} \oplus \mathfrak{h} \arrow[r] & \mathfrak{h} \arrow[r] & 0.
\end{tikzcd}$$

The projection $q\colon G/B \to G/P$ is H equivariant. The induced mapping

$$q_*\colon H_2(G/B) \to H_2(G/P)$$

is given by orthogonal projection $\mathfrak{h} \to \mathfrak{t}$ with respect to the Killing form. The resulting mapping,

$$H_2^T(G/B) \cong \mathfrak{h} \oplus \mathfrak{h} \to H_2^T(B/P) \cong \mathfrak{t} \oplus \mathfrak{h}$$

takes the arrangement $V(G/B)$ to the arrangement $V(G/P)$ which can therefore be described as follows,

$$V(G/P) = \bigcup_{\tau \in W} \{(q_*(\tau b), b) \in \mathfrak{t} \oplus \mathfrak{h} : b \in \mathfrak{h}\}. \tag{8.1}$$

The fixed point set $(G/P)^H$ is the image under q of the fixed point set $(G/B)^H$, so the fixed points are isolated. Hence (Theorem 3.1) the scheme $W(G/P) = \operatorname{Spec} \widehat{H}_H^*(G/P)$ is reduced. (However the projection $V(G/P) \to \mathfrak{h}$ is not flat, because $H^*(G/P) \neq \widehat{H}^*(G/P)$.)

Theorem 8.1 *Let $a \in \mathfrak{sl}(n, \mathbb{C})$ be the standard nilpotent element with Jordan blocks of sizes $n_1, n_2, \ldots, n_k$. Let X_a be the corresponding Springer fiber. Then the fibered arrangement*

$$\operatorname{Spec} H_T^*(X_a) = V(X_a) \subset \mathfrak{t} \oplus \mathfrak{h} \to \mathfrak{t}$$

is isomorphic to the dual of the fibered arrangement

$$\operatorname{Spec} \widehat{H}_H^*(G/P) = V(G/P) \subset \mathfrak{h} \oplus \mathfrak{t} \to \mathfrak{h}.$$

Proof Let $E_1 = \mathfrak{t} \oplus \mathfrak{h}$ and $E_2 = \mathfrak{h} \oplus \mathfrak{t}$. There is a dual pairing $E_1 \times E_2 \to \mathbb{C}$ given by

$$\langle (t_1, h_1), (h_2, t_2) \rangle_E = \langle t_1, t_2 \rangle - \langle h_1, h_2 \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the Killing form on $\mathfrak{h}$ (and its restriction to $\mathfrak{t}$). With respect to this pairing, the dual to $\mathfrak{t} \oplus 0 \subset E_1$ is $\mathfrak{h} \oplus 0 \subset E_2$, and the dual to $0 \oplus \mathfrak{h} \subset E_1$ is $0 \oplus \mathfrak{t} \subset E_2$. We claim that the dual to $V(X_a) \subset E_1$ is $V(G/B) \subset E_2$. The two arrangements contain the same number of subspaces and they have complementary dimensions. We must show that each subspace in the first arrangement is the dual of a particular subspace in the second arrangement. From equations (7.2) and (8.1) it suffices to show that for any $w \in W$, the dual of the subspace

$$L = \{(q_*(wb), b) : b \in \mathfrak{h}\} \subset E_1$$

is the subspace $L' = \{(w^{-1}t, t) : t \in \mathfrak{t}\} \subset E_2$. Since q_* is the orthogonal projection to $\mathfrak{t}$, for any $t \in \mathfrak{t}$ and any $x \in \mathfrak{h}$ we have $\langle q_*(x), t \rangle = \langle x, t \rangle$. Therefore a point $(h, t) \in E_2$ is in $\{(q_*(wb), b) : b \in \mathfrak{h}\}^\perp$ if and only if

$$0 = \langle (q_*(wb), b), (h, t) \rangle_E = \langle q_*(wb), t \rangle - \langle b, h \rangle = \langle wb, t \rangle - \langle b, h \rangle$$

for all $b \in \mathfrak{h}$. Since the action of W preserves the Killing form, this is equivalent to $\langle b, w^{-1}t \rangle = \langle b, h \rangle$ or $h = w^{-1}t$, which is the subspace $L' \subset E_2$. ■

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