# PRIMER ON SHEAVES, PRELIMINARY VERSION
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## 1. Overview

Sheaves are mostly for cohomology (or homology). If $\mathbf{F}$ is a sheaf on a topological space $X$ then $H^j(X;\mathbf{F})$ is defined. For any closed subset $Y \subset X$ we can restrict $\mathbf{F}$ to $Y$ and so $H^j(Y;\mathbf{F})$ is also defined. There are many constructions of cohomology (to be reviewed below). Conversely, if $i : Y \subset X$ is any closed subset and if $\mathbf{F}$ is a sheaf on $Y$ we can push it forward $i_*(\mathbf{F})$ into $X$ and therefore we can consider $\mathbf{F}$ to be a sheaf on $X$. For example, if we take the constant sheaf on $Y$ then we get the result that $H^*(Y;\mathbb{Z})$ is the sheaf cohomology $H^*(X;i_*(\mathbf{F}))$ of a certain sheaf on $X$. So the category of sheaves on $Y$ is the same as the category of sheaves on $X$ whose support is contained in $Y$. In Kashiwara-Schapira, the space $X$ is always a smooth manifold but the space $Y$ is singular, and so we can think about the singular space $Y$ by using sheaves on the nonsingular space $X$.

The derived category gives you a formalism for working with sheaves and it encodes various messy arguments that you otherwise need to do over and over again.

The singular support (or microsupport) of a sheaf on a manifold M is approximately the following. (A more precise formulation appears later in these notes.) Take a covector $w \in T^*_xM$. Find a smooth function $f : M \to \mathbb{R}$ so that $f(x) = 0$ and $df(x) = w$. Now, associated to $f$ at the point $x$ we can ask the Morse-theoretic question: does the homology of $M$ (with coefficients in $\mathbf{F}$) change as we pass through the critical value 0? But we need to ask this question locally near $x$ so we consider the relative homology group

$$H^j\left(B_\epsilon(x)\cap f^{-1}([-\delta,\delta]), B_\epsilon(x)\cap f^{-1}(-\delta);\mathbf{F}\right).$$

Roughly speaking, the covector $w$ is in the microsupport of $\mathbf{F}$ if this group is non-zero for some $j$. Thus, the microsupport of a sheaf is the set of covectors which have the potential to contribute to cohomology via Morse theory.

If $\epsilon$ is chosen sufficiently small, and if $\delta << \epsilon$ and if the sheaf is reasonably well behaved, then this group will be independent of $\epsilon$ and $\delta$, and it will also be independent of the function $f$ ( provided $df(x) = w$). If there exists a degree $j$ such that this group is not zero, then the covector $w$ is said to be a *characteristic* covector of the sheaf $\mathbf{F}$. The singular support,

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or microsupport $SS(\mathbf{F})$ is the closure of the set of characteristic covectors. It is a conical subset of $T^*M$.

## Part I: Background and motivations

### 2. Local systems

If $\mathbf{F}$ is a sheaf on $X$ and if $U \subset X$ is an open set then $\Gamma(U, \mathbf{F})$ denotes the group of sections of $\mathbf{F}$ over the set $U$. Sometimes a sheaf is defined in terms of its pre-sheaf of sections = a functor from the category of open sets $U \subset X$ to the category of abelian groups, plus the sheaf axiom that says if we have sections over a collection ( arbitrary collection) of open subsets and if the sections agree on the overlaps then there exists a section over the union that restricts to each of these sections.

Recall that a sheaf $\mathbf{F}$ on a topological space $X$ has stalks $\mathbf{F}_x$ at each point $x \in X$ and that the topology on $\mathbf{F}_x$ is discrete. A locally constant sheaf is a local system. This can be confusing if $\mathbf{F}$ is a local system of ( real or complex) vector spaces, (in which case $\mathbf{F}$ is also a vector bundle), because a section of $\mathbf{F}$ as a sheaf is quite different from a section of $\mathbf{F}$ as a vector bundle. The structure of a local system on a vector bundle $\mathbf{F}$ is the same as giving a flat connection on $\mathbf{F}$. If $X$ is a smooth manifold and $\mathbf{F}$ is a local system then the cohomology $H^*(X; \mathbf{F})$ can be computed using differential forms with coefficients in $\mathbf{F}$. Using a connection we get an exterior differentiation $d : \Omega^i(X, \mathbf{F}) \to \Omega^{i+1}(X, \mathbf{F})$ but $dd\theta = \omega \wedge \theta$ where $\omega$ is the Riemannian curvature of the connection. So $dd = 0$ iff the connection is flat, and in this case the cohomology makes sense.

To define the cohomology of a more complicated sheaf, or of a sheaf on a singular space, we cannot use differential forms. The simplest method is to use Cech cohomology: **Review Cech cohomology!** Take an open covering with sufficiently fine open sets (or use the covering consisting of all open sets); define $C^i(\mathbf{F})$ to be the group of Cech cochains that assign to any $i+1$-fold intersection $U = U_0 \cap \cdots \cap U_i$ the group $\Gamma(U, \mathbf{F})$. There is a way to define a differential $d : C^i(\mathbf{F}) \to C^{i+1}(\mathbf{F})$ and $H^i(X, \mathbf{F})$ is defined to be the cohomology of this complex. If the open covering is sufficiently fine then this is independent of the covering.

Another way to define the cohomology is to use an injective resolution

$$0 \longrightarrow \mathbf{F} \xrightarrow{\ i\ } I^0 \xrightarrow{\ d^0\ } I^1 \xrightarrow{\ d^1\ } \cdots$$

where $i$ is an injection whose image is equal to $\ker d^0$, where the sequence $\mathbf{I}^\bullet$ is exact at all the other spots, and where $\mathbf{I}^i$ are injective sheaves (which can be constructed by a procedure of Godement); then take the cohomology of the complex of global sections

$$\Gamma(X, \mathbf{I}^0) \xrightarrow{\ d\ } \Gamma(X, \mathbf{I}^1) \xrightarrow{\ d\ } \cdots .$$

The resulting groups are the *right derived functors* $R^i\Gamma(X,\mathbf{F}) = H^i(X,\mathbf{F})$. Other functors $T$ can be derived as well: first take an injective resolution of $\mathbf{F}$, then apply $T$, and then take the cohomology of the resulting sequence of groups, to obtain $R^iT(\mathbf{F})$.

## 3. Complexes of sheaves

It turns out that even when you think you are looking at a single sheaf, you're probably actually looking at a complex of sheaves – they arise from injective resolutions, from differential forms, from the complex of chains or cochains, etc. A complex of sheaves is a sequence

$$\mathbf{S}^0 \xrightarrow{d} \mathbf{S}^1 \xrightarrow{d} \mathbf{S}^2 \xrightarrow{d} \cdots$$

such that $d \circ d = 0$. Notation: We often write $\mathbf{S}^\bullet$ to indicate that we have a complex of sheaves, with $d \circ d = 0$.

One standard example is the complex of sheaves of differential forms (possibly with coefficients in a local system),

$$\Omega^j(X,\mathbf{F}) \xrightarrow{d} \Omega^{j+1}(X,\mathbf{F}) \xrightarrow{d} \cdots$$

Other examples include the (complex of) sheaves of (singular) chains (to be described later), and the Godement injective resolution of any sheaf $\mathbf{F}$.

If $\mathbf{S}^\bullet$ is a complex of fine sheaves, then for any open set $U \subseteq Z$ the cohomology $H^i(U,\mathbf{S}^\bullet)$ is the cohomology of the complex of sections over $U$,

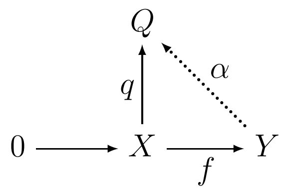
$$\to \Gamma(U,\mathbf{S}^{i-1}) \to \Gamma(U,\mathbf{S}^i) \to \Gamma(U,\mathbf{S}^{i+1}) \to$$

However if $\mathbf{S}^\bullet$ is not fine, then this procedure gives the wrong answer. (Take, for example, the constant sheaf on a smooth manifold.)

We can define the cohomology of a complex of sheaves too. There are several different ways to do this, but they each take a paragraph. You would like to say: just take the complex of global sections and take its cohomology; after all, this seems to work in the case of the complex of sheaves of differential forms. Unfortunately, this only works if the individual sheaves in the complex are flabby, fine, or injective. Otherwise, you have to replace the complex by a (flabby, fine or) injective resolution, which I will describe below. So let us pretend that we have already made this definition; then, below, we will find several alternative constructions for the cohomology of a complex of sheaves.

## 4. Interlude on Injective sheaves

An object $Q$ in an abelian category $\mathcal{C}$ is *injective* if for every injective morphism $f : X \to Y$ and for every morphism $q : X \to Q$ there exists $\alpha : Y \to Q$ so that this diagram commutes:

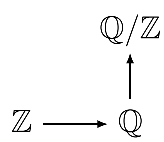


If $0 \to A \to B \to C \to 0$ is an exact sequence in $\mathcal{C}$ then for any object $Q$, the sequence

$$\mathrm{Hom(A,Q)} \xleftarrow{\beta} \mathrm{Hom(B,Q)} \longleftarrow \mathrm{Hom(C,Q)} \longleftarrow 0$$

is exact. Then $Q$ is injective if and only if the leftmost map $\beta$ is surjective, or equivalently, iff the functor $X \mapsto \mathrm{Hom}(X, Q)$ is exact. The category $\mathcal{C}$ has *enough injectives* if, for every object $A$ there is an injective object $I$ and an injective morphism $A \to I$. In this case every object has an injective resolution $0 \to A \to I^0 \to I^1 \to \cdots$ given by $I = I^0$; then take the cokernel of $A \to I^0$ and embed this into an injective $I^1$ and repeat the process.

The category of abelian groups has enough injectives, however the group $\mathbb{Z}$ is not injective; an injective resolution is



This can be used to give a *canonical* way to embed every abelian group $G$ into an injective group $\overline{G}$. In the category of vector spaces over a field, every object is injective. In the category of sheaves of abelian groups, injectivity has a topological aspect (the sheaf must be totally disconnected in some sense) and an algebraic aspect (its group of sections must be an injective group). The category of sheaves has enough injectives: if $\mathbf{A}$ is a sheaf of abelian groups on $X$, with stalks $\mathbf{A}_x$ then Godement defines the presheaf $\mathbf{I}$ by the rule

$$\Gamma(U, \mathbf{I}) := \prod_{x \in I} \bar{\mathbf{A}}_x$$

where $G \mapsto \bar{G}$ is a functorial way of embedding every abelian group $G$ into an injective one. Then $\mathbf{I}$ is actually a sheaf and there is a canonical injective morphism $\mathbf{A} \to \mathbf{I}$. (If $\mathbf{A}$ is a sheaf of vector spaces then the bars may be omitted: algebraic injectivity is automatic.) The resulting resolution $0 \to \mathbf{A} \to \mathbf{I} = \mathbf{I}^0 \to \mathbf{I}^1 \to \cdots$ is called the *Godement injective resolution.*

## 5. Quasi isomorphism

Sheaves form an abelian category. If $h : \mathbf{S}^0 \to \mathbf{S}^1$ is a morphism of sheaves on $X$, then $\ker(h)$ and $\mathrm{coker}(h)$ are defined, and they are sheaves on $X$. In particular if $\mathbf{S}^0 \to \mathbf{S}^1 \to \mathbf{S}^2 \to \cdots$ is a complex of sheaves then the cohomology $\mathbf{H}^i(\mathbf{S}^\bullet) = \ker(d)/\mathrm{Im}(d)$ is a sheaf on $X$.

Associated to any complex of sheaves $\mathcal{S}^0 \to \mathcal{S}^1 \to \cdots$ we have the complex of stalks $\mathcal{S}^0_x \to \mathcal{S}^1_x \to \cdots$. It turns out that the cohomology of this complex of stalks is equal to the stalk of the cohomology sheaf $\mathbf{H}^i(\mathbf{S}^\bullet)$. (For most of the sheaves that I know, the stalks $\mathbf{S}^i_x = \lim_{U\to x} \Gamma(U, \mathbf{S}^i)$ are germs (of differential forms, or of chains, etc.) and are only defined in the limit. However, for "reasonable" spaces, the stalk cohomology $\mathbf{H}^i_x = \lim_{U\to x} H^i(U, \mathbf{S}^\bullet)$ stabilizes as soon as the neighborhood $U$ is sufficiently small.)

**Definition.** A morphism $h : \mathbf{S}^\bullet \to \mathbf{T}^\bullet$ of complexes of sheaves (meaning that $hd = dh$ in each degree) is said to be a *quasi-isomorphism* if it induces an isomorphism of cohomology sheaves

$$h_* : \mathbf{H}^j(\mathbf{S}^\bullet) \to H^j(\mathbf{T}^\bullet)$$

for each $j$.

Since the mapping $h_*$ is automatically defined, it will be a quasi-isomorphism if and only if it induces an isomorphism on the stalk cohomology

$$h_* : H^j(\mathbf{S}^\bullet_x) \to H^j(\mathbf{T}^\bullet_x)$$

for every point $x \in X$. There are two important observations.

**Proposition.** *If $h : \mathbf{S}^\bullet \to \mathbf{T}^\bullet$ is a quasi-isomorphism then $h$ induces an isomorphism $H^i(U; \mathbf{S}^\bullet) \cong H^i(U, \mathbf{T}^\bullet)$ for every $i$ and for every open set $U \subset X$ (including $U = X$).*

The proof is just an observation that $h$ induces a mapping between the spectral sequence for $\mathbf{S}^\bullet$ and that for $\mathbf{T}^\bullet$, and the statement that $f$ is a quasi-isomorphism is exactly the statement that the mapping is an isomorphism on the $E_2$ page of this spectral sequence. (Recall that $E_2^{p,q} = H^p(X; \mathbf{H}^q(\mathbf{S}^\bullet))$.) So by the spectral sequence comparison theorem, the induced mapping on the abutment is also an isomorphism.

**Proposition.** *If each of the sheaves $\mathbf{S}^i$ is injective, or fine, or flabby, then for any open set $U \subset X$ the cohomology of the compex of sections*

$$\Gamma(U, \mathbf{S}^0) \to \Gamma(U, \mathbf{S}^1) \to \cdots$$

*is equal to the cohomology $H^j(U, \mathbf{S}^\bullet)$.*

Let $\mathbf{S}^\bullet$ be a complex of sheaves on $X$. An *injective resolution* (or flabby resolution, or fine resolution) is a complex $\mathbf{I}^\bullet$ of injective (resp. flabby, fine) sheaves together with a quasi-isomorphism $h : \mathbf{S}^\bullet \to \mathbf{I}^\bullet$. By putting the two observations together we see that *the cohomology of* $\mathbf{S}^\bullet$ *is equal to the cohomology of the complex* $\Gamma(U, \mathbf{I}^\bullet)$ *for any injective (or flabby or fine) resolution* $\mathbf{I}^\bullet$ *of* $\mathbf{S}^\bullet$.

To construct an injective resolution of a complex $\mathbf{S}^\bullet$ of sheaves, you first injectively resolve each $\mathbf{S}^j \to \mathbf{I}^{j0} \to \mathbf{I}^{j1} \to \cdots$ to obtain a double complex $\mathbf{I}^{ij}$. Then add these along the diagonals (and change the sign on 1/4 of the differentials) to obtain a single complex $T^k = \oplus_{i+j=k} I^{ij}$. This is the desired injective resolution of $\mathbf{S}^\bullet$.

A simple example is the case of a single sheaf $\mathbf{F}$ which can be expressed as a complex $\mathbf{F} \to 0 \to 0 \cdots$. In this case an injective resolution $\mathbf{I}^\bullet$ of this complex looks like this:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{F} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ \downarrow & & \downarrow{\scriptstyle i} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{I}^0 & \xrightarrow[d^0]{} & \mathbf{I}^1 & \xrightarrow[d^1]{} & \mathbf{I}^2 & \longrightarrow & \cdots \end{array}$$

To say that this is an injective resolution is to say that $i$ induces an isomorphism between the cohomology sheaf of the first sequence (which is $\mathbf{F}$) and the cohomology sheaf of the second sequence, that is, $\ker(d^0) = i(\mathbf{F})$ and the second sequence is exact at all the other spots. This is exactly the definition of an injective resolution of the sheaf $\mathbf{F}$, and in particular, the cohomology of $\mathbf{F}$ is equal to the cohomology of the complex $\Gamma(X, \mathbf{S}^\bullet)$.

## Part II: The definitions and main theorems

### 6. Derived category

The idea is that we want a category $D^b(X)$ whose objects are complexes of sheaves, but we want every quasi-isomorphism to become an isomorphism in this category (meaning that it should be invertible). If $T : \{Sheaves\} \to \mathcal{C}$ is a functor to an abelian category $\mathcal{C}$ we want to define a "derived" functor $R^iT : D^b(X) \to \mathcal{C}$ where $R^i(\mathbf{S}^\bullet)$ is defined by first replacing $\mathbf{S}^\bullet$ by a complex $\mathbf{I}^\bullet$ of injective sheaves, then applying $T$ and then taking cohomology. In fact, if we stop just before taking cohomology, we will get a functor $RT : D^b(X) \to D^b(\mathcal{C})$ by $RT(\mathbf{S}^\bullet) = T(\mathbf{I}^\bullet)$.

For example, let us consider the push forward functor: let $f : Z \to W$ be a continuous mapping, If $\mathbf{S}^\bullet$ is a complex of fine sheaves on $Z$ then the push forward $f_*(\mathbf{S}^\bullet)$ will satisfy

$$H^i(U, f_*(\mathbf{S}^\bullet)) \cong H^i(f^{-1}(U), \mathbf{S}^\bullet) \tag{1}$$

for any open set $U \subseteq W$. However if $\mathbf{S}^\bullet$ is not fine then (1) may fail, and $\mathbf{S}^\bullet$ should first be replaced by a fine (or flabby or injective) resolution $\mathbf{I}^\bullet$ before pushing forward. The resulting complex of sheaves (or rather, its quasi-isomorphism class) is denoted $Rf_*(\mathbf{S}^\bullet) = f_*(\mathbf{I}^\bullet)$. (It may appear that this complex is only defined up to quasi-isomorphism because the injective resolution $\mathbf{I}^\bullet$ is not unique. However, it is possible to make all of these constructions canonical by always taking $\mathbf{I}^\bullet$ to be the Godement injective resolution.)

**Definition 6.1.** An object in the derived category $D^b(X)$ is a complex of sheaves. A morphism $\mathbf{S}^\bullet \to \mathbf{T}^\bullet$ is a diagram

$$\mathbf{S}^\bullet \xleftarrow{\cong} \mathbf{C}^\bullet \longrightarrow \mathbf{T}^\bullet$$

where the map $\mathbf{S}^\bullet \leftarrow \mathbf{C}^\bullet$ is a quasi-isomorphism. Two such diagrams are considered equivalent if there is a diagram

$$\begin{array}{ccccc} & & \mathbf{C}_1^\bullet & & \\ & \swarrow & \uparrow & \searrow & \\ \mathbf{S}^\bullet & \longleftarrow & \mathbf{D}^\bullet & \longrightarrow & \mathbf{T}^\bullet \\ & \nwarrow & \downarrow & \nearrow & \\ & & \mathbf{C}_2^\bullet & & \end{array}$$

that is homotopy commutative, and where the three left arrows are quasi-isomorphisms.

It follows that every quasi-isomorphism is invertible in this category: If $\mathbf{S}^\bullet \to \mathbf{T}^\bullet$ is a quasi-isomorphism then its inverse is

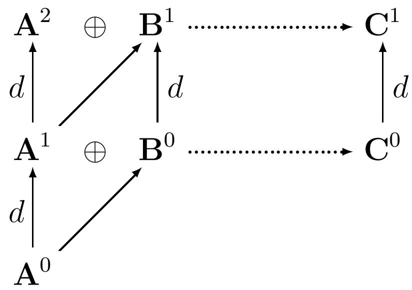
$$\mathbf{T}^\bullet \longleftarrow \mathbf{S}^\bullet \longrightarrow \mathbf{S}^\bullet.$$

## 7. Triangles

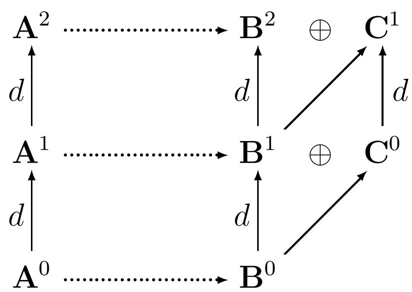
The derived category is not abelian, but it is triangulated, and triangles take the place of short exact sequences. First some motivation. If

$$0 \longrightarrow \mathbf{A}^\bullet \xrightarrow{\alpha} \mathbf{B}^\bullet \xrightarrow{\beta} \mathbf{C}^\bullet \longrightarrow 0$$

is a short exact sequence of chain complexes then it induces a long exact sequence on cohomology. Moreover, the complex $\mathbf{C}^\bullet$ can be recovered from the mapping $\alpha$ : it is the cokernel of $\alpha$. But it is also quasi-isomorphic to the following complex

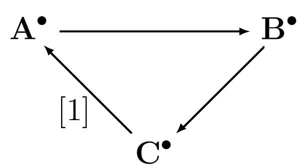


Also, $\mathbf{A}^\bullet$ is the kernel of $\mathbf{B}^\bullet \to \mathbf{C}^\bullet$, but it is also quasi-isomorphic to the following complex:



The first one is the mapping cone of $\mathbf{A}^\bullet \to \mathbf{B}^\bullet$, shifted by one; the second one is the mapping cone of $\mathbf{B}^\bullet \to \mathbf{C}^\bullet$, but not shifted.

In general, if $\mathbf{A}^\bullet \to \mathbf{B}^\bullet$ is any morphism (even if it is any morphism in the derived category) we can form the mapping cone $\mathbf{C}^\bullet$ and then there is a *distinguished triangle*



and it induces a long exact sequence on cohomology (of either kind: cohomology sheaves, or hyper cohomology)

$$\cdots \to H^i(\mathbf{A}^\bullet) \to H^i(\mathbf{B}^\bullet) \to H^i(\mathbf{C}^\bullet) \to H^{i+1}(\mathbf{A}^\bullet) \to \cdots$$

In other words, the mapping cone takes the place of both the kernel and the cokernel. Any 2 elements in such a triangle determine the third one up to quasi-isomorphism. All the standard exact sequences in topology can be interpreted in terms of distinguished triangles (see below). People usually drop the word "distinguished" and just say that we have a "triangle".

## 8. Derived functors

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between abelian categories, for example:

- global sections: sheaves $\to$ abelian groups
- Hom: sheaves $\to$ sheaves
- $f_*$: sheaves on $X \to$ sheaves on $Y$

Then $F$ gives rise to a derived functor $RF : D^b(\mathcal{C}) \to D^b(\mathcal{D})$ in a natural way: for any complex $\mathbf{S}^\bullet \in D^b(\mathcal{C})$ first take an injective resolution $\mathbf{I}^\bullet$ and then apply $F$ to get a complex

$$F(\mathbf{I}^0) \to F(\mathbf{I}^1) \to F(\mathbf{I}^2) \to \cdots$$

in $D^b(\mathcal{D})$. If you take the cohomology of this complex you get what is traditionally called $R^iF(\mathbf{S}^\bullet)$.

For example if $F = \Gamma =$ global sections then $R^i\Gamma(\mathbf{S}^\bullet) = H^i(X; \mathbf{S}^\bullet)$. However $R\Gamma(S^\bullet)$ is the complex before you take cohomology.

**Theorem 8.1.** *Let $f : X \to Y$ be a continuous map between topological spaces. Let $\mathbf{A}^\bullet$ be a complex of sheaves on $X$. Then there is a canonical isomorphism*

$$H^i(X, \mathbf{A}^\bullet) \cong H^i(Y, Rf_*(\mathbf{A}^\bullet)).$$

The proof is almost a triviality: replace $\mathbf{A}^\bullet$ by a complex of injective sheaves. Then $Rf_*(\mathbf{A}^\bullet)$ becomes the same as $f_*(\mathbf{A}^\bullet)$, whose sections over an open set $V \subset Y$ are $\Gamma(V, f_*(\mathbf{A}^\bullet) = \Gamma(f^{-1}(V), \mathbf{A}^\bullet)$. Since $f_*(\mathbf{A}^\bullet)$ is also injective, we can take $V = Y$ and $f^{-1}(V) = X$ which says that, by definition, the global sections of the two sheaves is the same. Therefore the cohomology is the same.

In many cases you don't actually need to take an injective resolution. In fact you can take any resolution by $F$-acyclic objects (that is, objects $A$ in $\mathcal{C}$ such that $R^iF(A) = 0$ for all $i$). In the case of global sections this includes fine or flabby resolutions.

Typically, here is what happens when you apply a functor (e.g. global sections, $f_*$, $Hom$, etc.) to a triangle. The functor (let's call it $T$) is probably not exact, so it does not take exact sequences to exact sequences, so it probably does not take triangles to triangles. However, often happens that the functor is exact on *injective* sheaves, i.e. if $0 \to \mathbf{A} \to \mathbf{B} \to \mathbf{C} \to 0$ is a short exact sequence of injective sheaves then $0 \to T(\mathbf{A}) \to T(\mathbf{B}) \to T(\mathbf{C}) \to 0$ is again exact. The same applies to complexes of sheaves. Therefore, if we have a triangle $\mathbf{A}^\bullet \to \mathbf{B}^\bullet \to \mathbf{C}^\bullet \to$ then we can represent the elemements in this triangle by complexes of injective sheaves, and then apply the functor $T$ and then we will again get a triangle. In summary. for such functors $T$ (that are exact on complexes of injectives), $RT$ takes triangles to triangles.

## 9. Homology as a cohomology theory

Let $X$ be a (reasonable) topological space. There is a complex of sheaves on $X$, called the *sheaf of chains* whose cohomology is the *homology* of $X$. There are many quasi-isomorphic versions of the sheaf of chains but perhaps the most intuitive one is the sheaf of piecewise linear chains. For this, we need to assume that $X$ is a topological space with a piecewise linear structure, e.g. $X$ could be a simplicial complex, or it could be of the form $X = Y - Y_1$ where $Y$ is a simplicial complex and $Y_1$ is a closed subcomplex.

We consider all possible refinements (or "triangulations") of $X$; these form some kind of directed system of simplicial complexes. For each triangulation $T$ of $X$ we have a complex of simplicial chains $C_*^T(X)$. If $T'$ is a refinement of $T$ then we get in a natural way, a homomorphism $C_i^T(X) \to C_i^{T'}(X)$ by refining the triangulation of the chain. A *piecewise linear chain* is an element of $\lim_T C_i^T(X)$. Such a chain is compact and it is simplicial with respect to some (finite) triangulation of $X$. If $X$ is not compact then we can also consider *Borel-Moore chains*, or chains with closed support. These are chains that are closed in $X$ but are not necessarily compact. The Borel-Moore chains are the direct limit over all triangulations of the complex of groups $C_i^{BM,T}(X)$ which consists of formal linear combinations of interiors of $i$-dimensional simplices in the triangulation $T$, where the boundary homomorphism is the usual one but it just ignores the terms involving those simplices $\sigma \subset Y - X$. A chain $\xi \in C_i^{BM}(X)$ in this direct limit might "wiggle" more and more as it moves off towards the "edge" $Y_1 \subset Y$ of $X = Y - Y_1$ because it may be simplicial with respect to ever finer and finer triangulations.

Now here is the main point: the Borel-Moore chains form a complex $\mathbf{C}_{BM}^{-i} = C_i^{BM}$ of pre-sheaves where $\Gamma(U, \mathbf{C}_{BM}^{-i}) = C_i^{BM}(U)$ for any open set $U \subset X$. If $V \subset U$ is an open subset and if $\xi \in C_i^{BM}(U)$ then $\xi \cap V$ can be triangulated, possibly with infinitely many simplices as we approach the boundary of $V$, but it will nevertheless define an element of $C_i^{BM}(V)$.

**Theorem 9.1.** *The complex of pre-sheaves $\mathbf{C}_{BM}^{\bullet}$ of Borel-Moore chains forms a sheaf; it is also flabby and consequently for any open set $U \subset X$ its cohomology is the homology with closed supports,*

$$H^{-i}(U, \mathbf{C}_{BM}^{\bullet}) = H_i^{BM}(U).$$

It is possible to make a similar construction with "singular" chains, and thereby avoid the requirement that $X$ should have a piecewise linear structure. In any case, the usual homology "chains" do not form a sheaf, since a chain $\xi \in C_i(U)$ is necessarily compact, but if $V \subset U$ is an open set, then $\xi \cap V$ will not be compact so it does not define a chain in $C_i(V)$.

## 10. Yes, but what is Borel-Moore homology, exactly?

First, it is a topological invariant – it only depends on the homeomorphism type of $X$. If $X$ is compact then $H_i^{BM}(X) = H_i(X)$ because ordinary homology is based on compact chains (i.e. formal sums of closed simplices). If $X$ is not compact then the Borel-Moore chains are allowed to run off to infinity. If "infinity" of $X$ is terribly complicated then there is no simple description of the Borel-Moore homology. But if $X$ can be compactified in a nice way then there is a simple description of Borel-Moore homology. Suppose that $Y$ is a compact simplicial complex, that $Y_1 \subset Y$ is a closed subcomplex and that $X = Y - Y_1$. (Or else suppose that $X$ is homeomorphic to $Y - Y_1$.) Thus, the space $X$ can be compactified by adding a simplicial complex $Y_1$ at infinity, in such a way that the compactification is also a simplicial complex. Let us say that such a space $X$ is "simplicially compactifiable". The simplicially compactifiable spaces include any space obtained from a finite union, intersection, or complementation of: (real or complex) algebraic sets, (real or complex) analytic sets, semi-algebraic, semi-analytic, and projections of semi-analytic sets (under linear projections).

**Theorem 10.1.** *For a simplicially compactifiable space $X = Y - Y_1$ there is a canonical isomorphism $H_i^{BM}(X) \cong H_i(Y, Y_1)$ for $i \geq 1$.*

Here is the reason: take any triangulation $T$ of $Y$ so that $Y_1$ is a closed subcomplex. Then $X = Y - Y_1$ is a locally closed union of interiors of finitely many simplices. (Let us call this a "Borel-Moore triangulation of $X$".) Define the complex of $T$-simplicial Borel-Moore chains $C_i^{BM,T}(X)$ to be the free abelian group with basis given by the (oriented) $i$-dimensional simplices $\sigma \in T$ such that $\sigma^o \subset X$ (meaning that the interior of $\sigma$ is contained in $X$.) Define the Borel-Moore boundary of such a simplex $\sigma$ to be the (oriented, signed) sum of those simplices $\tau < \sigma$ such that $\tau^o \subset X$, in other words, just ignore the parts of $\partial\sigma$ that are contained in $Y_1$. It is not too hard to show that the homology of this complex is the Borel-Moore homology of $X$, and yet this complex is isomorphic, on the chain level, to the complex $C_*(Y)/C_*(Y_1)$ because we are simply ignoring the simplices in $Y_1$.

**Corollary** The Borel-Moore Euler characteristic $\chi^{BM}$ is additive: Suppose that $X$ is simplicially compactifiable and that $X = X_1 \cup X_2$ is a disjoint union of two simplicially compactifiable subsets then $\chi^{BM}(X) = \chi^{BM}(X_1) + \chi^{BM}(X_2)$. The proof is simple: Compactify $X$ as above, triangulate as above, so that $X, X_1, X_2$ are each unions of interiors of finitely many simplices. Each (open) simplex in $X$ is either in $X_1$ or $X_2$ so $C_*^{BM}(X) = C_*^{BM}(X_1) \oplus C_*^{BM}(X_2)$.

**Warning** Borel-Moore homology (and cohomology with compact support) are not homotopy invariants. The open and closed unit interval $I^0$ and $I$ are homotopy equivalent but $H_1^{BM}(I^0; \mathbb{Z}) \cong \mathbb{Z}$ while $H_1^{BM}(I; \mathbb{Z}) = H_1(I; \mathbb{Z}) = 0$.

## 11. Exact sequence of a pair

Let $X$ be a (reasonable) topological space, let $j : Y \subset X$ be a closed subset and let $i : U = Y - X \to X$ be its open complement. Classically, are 4 exact sequences: $H^*(X,Y)$, $H^*(X,U)$, $H_*(X,Y)$ and $H_*(X,U)$. These have sheaf theoretic constructions.

Let $A$ be a sheaf on $U$. Consider the "direct image", $i_*(A)$. By definition, if $V \subset X$ is an open set then

$$\Gamma(V, i_*A) = \Gamma(U \cap V, A).$$

**Caution:** The sheaf $i_*(A)$ is supported on the closure of $U$. In fact, if $x \in X - U$ is a point in the closure of $U$ , the stalk of $i_*(A)$ at $x$ is

$$(i_*(A))_x = \lim \Gamma(V_j \cap U)$$

where $V_1 \supset V_2 \supset \cdots$ is a decreasing sequence of open sets whose intersection is $x$. There is another sheaf, $i_!(A)$ which is the extension by zero of $A$. It is defined so that the stalk at every $x \in X - U$ is zero. The precise definition is that it is the (sheafification of the) presheaf whose sections over an open set $V \subset X$ is

$$\Gamma(V, i_!(\mathbf{A})) = \{\xi \in \Gamma(V \cap U, \mathbf{A}) |\ \text{support}(\xi) \cap U \text{ is compact}\}.$$

Applying these constructions to complexes of injective sheaves, using the outline described above, we obtain functors

$$Ri_* : D^b(U) \to D^b(X) \quad \text{and } Ri_! : D^b(U) \to D^b(X).$$

**Theorem 11.1.** *Let $i : U \subset X$ be open and let $j : Y = X - U \to X$ be its closed complement. Let $\mathbf{A}^\bullet$ be a complex of sheaves on $X$. Then there are (distinguished) triangles,*

$$\begin{array}{ccc} Ri_!i^*\mathbf{A}^\bullet & \longrightarrow & \mathbf{A}^\bullet \\ {\scriptstyle [1]}\nwarrow & & \swarrow \\ & Rj_*j^*\mathbf{A}^\bullet & \end{array} \qquad \begin{array}{ccc} Rj_*j^!\mathbf{A}^\bullet & \longrightarrow & \mathbf{A}^\bullet \\ {\scriptstyle [1]}\nwarrow & & \swarrow \\ & Ri_*i^*\mathbf{A}^\bullet & \end{array}$$

**Important Exercise:** Verify the following: If $\mathbf{A}^\bullet$ is the constant sheaf then these are the long exact cohomology sequences for $H^*(X,Y)$ and $H^*(X,U)$ respectively. If $\mathbf{A}^\bullet$ is the sheaf of Borel-Moore chains then these are the long exact homology sequences for $H_*(X,U)$ and $H_*(X,Y)$ respectively.

The "lower shriek" construction can also be made for any mapping $f : X \to Y$. If $\mathbf{A}$ is a sheaf on $X$ then $f_!(\mathbf{A})$ is defined to be the sheaf on $Y$ whose group of sections over $U \subset Y$ consists of sections $s \in \Gamma(f^{-1}(U), \mathbf{A})$ such that $|s| \to Y$ is proper, where $|s|$ denotes the support of $s$. The sheaf $f_!(\mathbf{A})$ is called the push forward with proper supports. As above, the same construction may be applied to a complex of sheaves $\mathbf{A}^\bullet \in D^b(X)$, and if we first replace such a complex by a complex of injective sheaves then we obtain the element $Rf_!(\mathbf{A}^\bullet) \in D^b(Y)$.

## 12. Sheaf Hom

If $\mathbf{A}$ and $\mathbf{B}$ are sheaves (of abelian groups) on a topological space let $\mathrm{Hom}(\mathbf{A}, \mathbf{B})$ be the abelian group of sheaf maps $\mathbf{A} \to \mathbf{B}$. This can be interpreted in terms of presheaves: an element $f \in \mathrm{Hom}(\mathbf{A}, \mathbf{B})$ gives, for every open set $U \subset X$ a mapping $f(U) : \Gamma(U, \mathbf{A}) \to \Gamma(U, \mathbf{B})$, such that these mappings are compatible with restriction to smaller open sets $V \subset U$. The group of sheaf maps can also defined by considering $A, B$ to be spaces lying over $X$, in which case a homomorphism $f : \mathbf{A} \to \mathbf{B}$ is a continuous mapping that commutes with the projection to $X$ and which is linear on each stalk.

If $\mathbf{A}, \mathbf{B}$ are sheaves on $X$ define the sheaf $\mathbf{Hom}(\mathbf{A}, \mathbf{B})$ to be the sheafification of the presheaf whose sections over an open set $U \subset X$ is the group $\mathrm{Hom}(\mathbf{A}|U, \mathbf{B}|U)$. If $\mathbf{A}^\bullet, \mathbf{B}^\bullet$ are complexes of sheaves on $X$, define $\mathbf{Hom}^\bullet(\mathbf{A}^\bullet, \mathbf{B}^\bullet)$ to be the single complex of sheaves

$$\mathbf{Hom}^{\mathbf{t}}(\mathbf{A}^\bullet, \mathbf{B}^\bullet) = \oplus_{\mathbf{p}} \mathbf{Hom}(\mathbf{A}^{\mathbf{p}}, \mathbf{B}^{\mathbf{p+t}})$$

that is associated to the double complex $\mathbf{Hom}(\mathbf{A}^{\mathbf{p}}, \mathbf{B}^{\mathbf{q}})$ in the usual way. Finally, by replacing $\mathbf{B}^\bullet$ by an injective complex of sheaves we obtain the element

$$R\mathbf{Hom}^\bullet(\mathbf{A}^\bullet, \mathbf{B}^\bullet) \in \mathbf{D}^{\mathbf{b}}(\mathbf{X}).$$

**Theorem 12.1.** *Let $\mathbf{A}^\bullet, \mathbf{B}^\bullet \in D^b(X)$. Then there is a canonical isomorphism*

$$H^0(X; R\mathbf{Hom}^\bullet(\mathbf{A}^\bullet, \mathbf{B}^\bullet)) \cong \mathrm{Hom}_{\mathrm{D^b(X)}}(\mathbf{A}^\bullet, \mathbf{B}^\bullet).$$

This is a beautiful statement which says that the mysterious Hom in the derived category is, in fact, the cohomology of the sheaf $R\mathbf{Hom}$. The higher cohomology groups are called Ext:

$$\mathrm{Ext}^t_{D^b(X)}(\mathbf{A}^\bullet, \mathbf{B}^\bullet) = H^i(X; R\mathbf{Hom}^\bullet(\mathbf{A}^\bullet, \mathbf{B}^\bullet)).$$

In order to define $RHom$ of sheaves, it turns out that, rather than injectively resolve $\mathbf{B}^\bullet$ it also suffices to projectively resolve $\mathbf{A}^\bullet$.

## 13. Tensor product

There is a problem with defining the derived functor of $\otimes$ because the tensor product is not exact on injective complexes. However, it is exact on projective complexes. So in order to obtain a well defined object $\mathbf{A}^\bullet \otimes \mathbf{B}^\bullet$ in the derived category, it is first necessary to projectively resolve either (or both) of these sheaves. This process gives a *left derived* functor, $\mathbf{A}^\bullet \overset{L}{\otimes} \mathbf{B}^\bullet$. If the coefficient ring is a field then the resolution is not needed (since every vector space over a field is projective).

## 14. Duality

In this section, all sheaves should be taken to be cohomologically constructible. For now, let us just say that it is a "reasonableness" condition on a sheaf; it is a somewhat technical condition and it will be discussed below.

Originally it was felt that the "dual" of a sheaf (or of a complex of sheaves) should be a co-sheaf (an object similar to a sheaf, but for which the restriction arrows are reversed). However, in [BM], Borel and Moore constructed the *dual sheaf* $\mathbf{B}^\bullet$ of a complex of sheaves $\mathbf{A}^\bullet$ on $Z$. Here is some motivation that may help in thinking about this.

The model for duality is the duality between homology and cohomology. If $C_*(X;\mathbb{Z})$ is a chain complex that computes the homology of a space $X$ then the cohomology is computed by the chain complex $C^i(X;\mathbb{Z}) := \mathrm{Hom}(C_i(X;\mathbb{Z});\mathbb{Z})$ and it satisfies the "universal coefficient theorem" which says that the following sequence is (split) exact:

$$0 \to \mathrm{Ext}(H_{i-1}(X;\mathbb{Z}),\mathbb{Z}) \to H^i(X;\mathbb{Z}) \to \mathrm{Hom(H_i(X,\mathbb{Z}),\mathbb{Z})} \to 0.$$

To translate this statement into sheaf theory we need to replace the homology groups by cohomology groups. We can get a hint as to how to do this by considering the case that $X$ is a (not necessarily compact) n-dimensional oriented manifold. Then the Poincaré duality theorem says that

$$H^i_c(X;\mathbb{Z}) \cong H_{n-i}(X;\mathbb{Z}) \quad \text{and} \quad H^i(X;\mathbb{Z}) \cong H^{BM}_{n-i}(X;\mathbb{Z}).$$

so the universal coefficient theorem becomes an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}(H^{n-i+1}_c(X);\mathbb{Z}) & \longrightarrow & H^i(X;\mathbb{Z}) & \longrightarrow & \mathrm{Hom(H^{n-i}_c(X;\mathbb{Z}),\mathbb{Z})} & \longrightarrow & 0 \\ &&&& \downarrow \cong &&&& \\ &&&& H^{BM}_{n-i}(X;\mathbb{Z}) &&&& \end{array}$$

The moral of this story is that (if we ignore the Ext term for the moment), compact support cochains are dual to ordinary (= closed support) cochains (of complementary dimension). Therefore if a complex of sheaves $\mathbf{A}^\bullet$ is going to be dual to a complex of sheaves $\mathbf{B}^\bullet$ then we expect to find a short exact sequence

$$0 \longrightarrow \mathrm{Ext}(H^{i+1}_c(\mathbf{A}^\bullet),\mathbb{Z}) \longrightarrow H^{-i}(\mathbf{B}^\bullet) \longrightarrow \mathrm{Hom(H^i_c(\mathbf{A}^\bullet),\mathbb{Z})} \longrightarrow 0.$$

(There is always a shift of degree in duality; here we have used $H^{-i}$ rather than $H^{n-i}$ that occurred above.)

Given a complex of sheaves $\mathbf{A}^\bullet$, Borel and Moore constructed a complex of sheaves $\mathbf{B}^\bullet$ with this property. In hindsight it was fairly easy:

**Definition.** Given $\mathbf{A}^\bullet$, define a complex of presheaves, $\widetilde{\mathbf{B}}^\bullet$,

$$\Gamma(U, \widetilde{\mathbf{B}}^i) := \mathrm{Hom}(\Gamma_c(U, \mathbf{A}^i), \mathbb{Z}).$$

Then the dual $\mathbf{B}^\bullet$ of $\mathbf{A}^\bullet$ is defined to be the sheafification of the presheaf $\widetilde{\mathbf{B}}^\bullet$.

Borel and Moore called this the "dual" of the complex of sheaves $\mathbf{A}^\bullet$ however it was clear to them that the dual of the dual did not equal $\mathbf{A}^\bullet$.

While thinking about this question, Verdier (and perhaps Grothendieck) realized that in fact there is a complex of sheaves $\mathbf{D}^\bullet_X$ on $X$ with the amazing property that

$$\mathbf{B}^\bullet = \mathbf{RHom}^\bullet(\mathbf{A}^\bullet, \mathbf{D}^\bullet_X)$$

So what is this magic sheaf $\mathbf{D}^\bullet_X$? Let us take $\mathbf{A}^\bullet = \mathbf{Z}$. Its cohomology is the ordinary cohomology. But according to the universal coefficient theorem above, the (compact support) cohomology is dual to the *Borel-Moore homology.* In other words, the sheaf $\mathbf{D}^\bullet$ is the sheaf of Borel-Moore chains! Moreover, in the derived category, double duality works out:

**Proposition.** *For any complex of sheaves* $\mathbf{A}^\bullet$ *there is a canonical quasi-isomorphism*

$$\text{dual }(\text{dual }(\mathbf{A}^\bullet)) \to \mathbf{A}^\bullet.$$

The duality between closed and compact supports extends to maps as well. If $f : X \to Y$ is a continuous map and if $\mathbf{A}^\bullet$ is a sheaf on $X$ then we have two ways to push it forward, $Rf_*(\mathbf{A}^\bullet)$ and $Rf_!(\mathbf{A}^\bullet)$, the latter being the push forward with proper supports, i.e. the sheafification of the presheaf whose sections over an open set $U \subset Y$ consists of all sections $s$ in $f^{-1}(U)$ such that $f|(spt(s))$ is proper. Then it turns out that

$$Rf_* = \text{dual } \circ Rf_! \circ \text{dual}$$

or to be precise, there exists a natural transformation of functors between the functor $Rf_*$ and the functor

$$\mathbf{A}^\bullet \mapsto \mathbf{RHom}^\bullet(Rf_!\mathbf{RHom}^\bullet(\mathbf{A}^\bullet, \mathbf{D}^\bullet), \mathbf{D}^\bullet)).$$

If $Y$ is a point then this just says $H^{-i}(\text{dual }(\mathbf{A}^\bullet))$ is dual to $H^i_c(\mathbf{A}^\bullet)$.

So by analogy, Verdier defines

$$f^! := \text{dual } \circ f^* \circ \text{dual }.$$

For example, if $Y$ is a point and $\mathbf{A}^\bullet = \mathbb{Z}$ then the dual of $\mathbf{A}^\bullet$ is also $\mathbb{Z}$, so $f^!(\mathbb{Z}) = \mathbf{D}^\bullet$, and in general, $f^!(\mathbf{D}^\bullet_Y) = \mathbf{D}^\bullet_X$. In fact, the *Verdier duality theorem* says that for any $f : X \to Y$ and for any sheaves $\mathbf{A}^\bullet$ on $X$ and $\mathbf{B}^\bullet$ on $Y$ we have a canonical isomorphism in the derived category,

$$Rf_*\mathbf{RHom}^\bullet(\mathbf{A}^\bullet, f^!(\mathbf{B}^\bullet)) \cong \mathbf{RHom}(Rf_!(\mathbf{A}^\bullet), \mathbf{B}^\bullet).$$

(Taking $\mathbf{B}^\bullet = \mathbf{D}^\bullet_Y$ gives the preceding statements.)

There is one more tricky point: if we are considering sheaves of abelian groups then $\mathbf{RHom}(\mathbf{A}^\bullet, \mathbf{D}^\bullet)$ means that we need a model for the dualizing sheaf that is injective both topologically and algebraically. Over a point, for example, the dualizing sheaf is the complex $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$. This is one way to see why the Ext terms show up in the above sequences.

## Part III: Constructible sheaves

### 15. When Bad Sheaves Happen to Good Spaces

There are some really dangerous sheaves out there. For example, let $j : K \to \mathbb{R}$ be the inclusion of the Cantor set into the real numbers. Put the constant sheaf $\mathbb{Z}$ on $K$ and then push it into $\mathbb{R}$ to obtain $j_*(\mathbb{Z})$. Now we have a sheaf on the real numbers that is supported on the Cantor set. Not dangerous enough for you? OK, take the complement of the Alexander horned sphere $AHS$ in $\mathbb{R}^3$. This complement has a fundamental group with infinitely many generators: just go around each of the "horns". Take a local system $\mathcal{L}$ on this complement with nontrivial twisting as you go around each of the horns. (This can be done, for example, by taking $\mathcal{L}$ to be a local system of 2-dimensional vector spaces.) Push this into $\mathbb{R}^3$ by the "extension by zero" to get $i_!\mathcal{L}$ where $i : (\mathbb{R}^3 - AHS) \to \mathbb{R}^3$ is the inclusion. The result is a sheaf on $\mathbb{R}^3$ whose stalk at each point is either 2-dimensional or is zero; which vanishes on $AHS$ but which has infinitely much twisting on the complement.

For such sheaves, some of the constructions and theorems in the preceding sections of these notes are false. In the 1960's, various attempts were made to find a class of (complexes of) sheaves $\mathbf{S}^\bullet$ that are preserved under the six functors, such that for a compact space $X$, the cohomology $H^i(X; \mathbf{S}^\bullet)$ is finite dimensional. Borel and Verdier used the word "constructible" for some of these early attempts, although Verdier told us in 1980 "Il n'existe rien de paradise". He was wrong, but the solution involves choosing a geometric structure on $X$.

**Definition.** A sheaf $S$ (of Abelian groups) on a (compact) simplicial complex $X$ is *simplicially constructible* if each stalk $S_x$ is finitely generated and if the restriction $S|\sigma^o$ to the interior of each simplex is a constant sheaf. The sheaf $S$ is *piecewise-linearly constructible* if it is simplicially constructible with respect to some refinement of the triangulation of $X$. A complex of sheaves $\mathbf{S}^\bullet$ is *cohomologically* (simplicially, resp. piecewise-linearly) constructible if all of its cohomology sheaves $\mathbf{H}^i(\mathbf{S}^\bullet)$ are (simplicially, resp. piecewise linearly) constructible. If $X$ is a simplicially compactifiable space with compactification $j : X \to Y$, then a complex of sheaves $\mathbf{S}^\bullet$ on $X$ is cohomologically constructible if $Rj_*(\mathbf{S}^\bullet)$ is cohomologically constructible.

Such sheaves are preserved under sums, products, duality, direct and pre-images under simplicial (resp. piecewise linear) mappings, and if $X$ is compact then the cohomology $H^i(X, \mathbf{S}^\bullet)$ groups are finitely generated (i.e. finite dimensional, in the case of sheaves of vector spaces).

The problem with piecewise linear structures is that they never occur naturally. Moreover, given a manifold $M$, it is possible to find triangulations (even "smooth" triangulations) of $M$ that are incompatible – meaning that they belong to distinct piecewise linear structures on $M$; so that a simplex of one triangulation might intersect a simplex of the other triangulation in a Cantor set, or in an Alexander horned sphere. So we need a class of structures that occurs naturally but which still has all the advantages of a piecewise linear structure.

## 16. Subanalytic category

Perhaps the most natural is the category of subanalytic spaces. An analytic space $X$ is a subset of an analytic manifold $M$ that is cut out by analytic functions. A semi-analytic space $X$ is one that is cut out by analytic equalities and inequalities. A subanalytic space is a space that can be realized as the projection of a semi-analytic space $X \subset \mathbb{R}^N$ by a linear projection $\mathbb{R}^N \to \mathbb{R}^m$. The category of subanalytic subsets of an analytic manifold $M$ is closed under finite unions, intersection and complementation. Every subanalytic space admits a subanalytic stratification: a decomposition into locally finitely many smooth manifolds, plus further conditions: (1) the axiom of the frontier: if $B$ is a stratum then its closure is a union of strata, (and we write $A < B$ if $A \subset \overline{B}$) and (2) Whitney's conditions (a) and (b) hold. For our purposes, condition (a) is the most important one: Let $A \subset \overline{B}$ be strata. Let $b_i \in B$ be a sequence of points converging to some point $a \in A$. Suppose the sequence of tangent spaces $T_{b_i}B \subset T_{b_i}M$ converge to some limiting plane $\tau \subset T_aM$. Then $T_aA \subset \tau$.

Any two subanalytic stratifications of a subanalytic set have a subanalytic common refinement. Now we can make the same constructibility definitions:

**Definition.** A complex of sheaves $\mathbf{S}^\bullet$ on a subanalytic set $X$ is cohomologically constructible with respect to a chosen subanalytic stratification if each of the cohomology sheaves $\mathbf{H}^i(\mathbf{S}^\bullet)$ is locally constant on each of the strata of the stratification, and its stalks are finitely generated (at each point). A complex of sheaves $\mathbf{S}^\bullet$ on $X$ is cohomologically constructible if it is cohomologically constructible with respect to some subanalytic stratification.

The collection of such sheaves forms a "paradise" in that these sheaves are closed under the 6 operations of Grothendieck, and if $X$ is compact then $H^i(X, \mathbf{S}^\bullet)$ is finitely generated. **All the sheaves in the preceding section on duality should be taken to be cohomologicallyl constructible.**

## Part IV: Stratified Morse theory and the characteristic variety

Coming soon!!

## References

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